

CYCLES OF GIVEN SIZE IN A DENSE GRAPH*

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Abstract. We generalize a result of Corrádi and Hajnal and show that every graph with average degree at least $\frac{4}{3}kr$ contains k vertex disjoint cycles, each of order at least r , as long as $k \geq 6$. This bound is sharp when $r = 3$.

Key words. cycles, graph minors, average degree

AMS subject classifications. 05C38, 05C35, 05C83

DOI. 10.1137/15M100852X

1. Introduction. A well-known result by Corrádi and Hajnal [6] states that every graph G with at least $3k$ vertices and minimum degree $2k$ contains k (vertex) disjoint cycles. Since every graph G contains a subgraph with minimum degree greater than half the average degree of G , this implies that every graph with average degree at least $4k - 2$ (and as such at least $4k - 1 \geq 3k$ vertices) contains k disjoint cycles. This result is asymptotically sharp since the complete bipartite graph $K_{n,2k-1}$ does not contain k disjoint cycles (since each cycle contains at least two vertices from the small part) but has average degree tending to $4k - 2$ as $n \rightarrow \infty$.

Many extensions of the result of Corrádi and Hajnal [6] have been established. Justesen [12] proved a density version of the theorem of Corrádi and Hajnal, showing a graph with $n \geq 3k$ vertices and more than $\max\{(2k-1)(n-k), \binom{3k-1}{2} + (n-3k+1)\}$ edges contains k disjoint cycles. Verstraëte [28] proved that for each integer k there exists an n_k such that every graph with minimum degree at least $2k$ and at least n_k vertices contains k disjoint cycles of the same order. Egawa et al. [8] proved that given $k \geq 2$ and $d \geq 2k$ and $n \geq 3k$, every graph with order n and minimum degree at least d contains k disjoint cycles containing at least $\min\{2d, n\}$ vertices. Chiba et al. [4] proved that every graph with minimum degree at least $2k$ and a sufficiently large number of vertices contains k disjoint even cycles except in a handful of exceptional cases.

Other extensions find specified 2-factors of G (where a 2-factor is a set of disjoint cycles that span $V(G)$), for example [1, 2, 3, 9, 14, 19, 31, 32], or replace the minimum degree requirement by alternatives like Ore-type degree conditions,¹ for example [3, 8, 9, 10, 11, 13, 19].

In this paper, we consider a different direction, in which a lower bound on the size of the desired cycles is specified. This direction has previously been investigated by Wang [33], who proved that every graph with minimum degree at least $2k$ and at least $4k$ vertices (where $k \geq 2$) contains k disjoint cycles of order at least four, except in three exceptional cases. Hence, since every graph with average degree at least $4k - 1$ contains a subgraph with average degree $4k - 1$ (and thus $4k$ vertices) and minimum

*Received by the editors February 13, 2015; accepted for publication September 18, 2015; published electronically November 24, 2015. This research was supported by the Australian Research Council.

<http://www.siam.org/journals/sidma/29-4/M100852.html>

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¹Recall Ore [24] showed that an n -vertex graph G contains a Hamiltonian cycle if $\deg v + \deg w \geq n$ for every pair of nonadjacent vertices v, w .

degree at least $2k$, it follows that a graph with average degree at least $4k - 1$ contains k disjoint cycles of order at least four, when $k \geq 2$. (The lower bound on the average degree precludes the subgraph from being one of the exceptional cases; we omit the proof of this fact.) Wang [29, 30] previously showed that if G is a balanced bipartite graph with average degree at least $(s - 1)k + 1$ and at least sk vertices in each part (where $s \geq 2$), then G contains k disjoint cycles of length at least $2s$. We prove the following theorem.

THEOREM 1.1. *For integers $k \geq 6$ and $r \geq 3$, every graph with average degree at least $\frac{4}{3}kr$ contains k disjoint cycles, each containing at least r vertices.*

If $r = 3$, then this gives the above corollary of the theorem of Corrádi and Hajnal (albeit with a restriction on k and without the -2). However, Theorem 1.1 allows us to instead ensure the existence of larger cycles. Our approach to proving this result relies heavily on the concept of minors.

2. A minor approach. Recall a graph H is a minor of a graph G if a graph isomorphic to H can be constructed from G by a series of vertex deletions, edge deletions and edge contractions. Let H be the graph consisting of k disjoint cycles of order r . If G contains k disjoint cycles, each with at least r vertices, then clearly G contains H as a minor. Alternatively, if G contains H as a minor, then by “uncontracting” each vertex of H we obtain a subgraph of G consisting of k cycles, each with at least r vertices. Hence the following theorem is equivalent to Theorem 1.1.

THEOREM 2.1. *For integers $k \geq 6$ and $r \geq 3$, every graph with average degree at least $\frac{4}{3}kr$ contains H as a minor, where H is the graph consisting of k disjoint cycles of order r .*

A well-known result by Mader [21] shows that sufficiently large average degree ($\geq 2^{t-2}$) is sufficient to force the existence of a complete minor K_t . Much work was done improving this required lower bound on the average degree, until Thomason [26] and Kostochka [15, 16] showed that average degree at least $\Theta(t\sqrt{\ln t})$ is sufficient and best possible. Thomason [27] later determined the asymptotic constant for this bound. Similarly, we may consider what average degree is required to force an arbitrary given graph H as a minor. Myers and Thomason [23] answered this question when H is dense. Similarly, sparse graphs of the form $K_{s,t}$ (where $s \ll t$) have been well studied. Myers [22] showed that every n -vertex graph with more than $\frac{1}{2}(t+1)(n-1)$ edges contains a $K_{2,t}$ minor, as long as t is sufficiently large. More recently Chudnovsky, Reed, and Seymour [5] proved this result for all t . Kostochka and Prince [18] proved that every n -vertex graph with more than $\frac{1}{2}(t+3)(n-2) + 1$ edges contains a $K_{3,t}$ minor. Myers [22] conjectured that the average degree required to force a $K_{s,t}$ minor is linear in t ; this was proven independently by Kostochka and Prince [17] and Kühn and Osthus [20]. In a recent paper on sparse graphs H , Reed and Wood [25] conjectured that average degree $\frac{4}{3}t - 2$ was sufficient to force an arbitrary 2-regular t -vertex graph H as a minor. Theorem 2.1 essentially proves this conjecture when each component of H has the same order (apart from the -2). Thus we can interpret our result in both the theme of Corrádi and Hajnal [6] and the theme of Mader [21].

Call a graph G *minimal* if every vertex deletion or edge contraction lowers the average degree $d(G)$. Let $\delta(G)$ be the minimum degree of G , and let $\tau(G)$ be the minimum number of common neighbors for any pair of adjacent vertices in G . If G is minimal, then $\delta(G) > \frac{1}{2}d(G)$ and $\tau(G) > \frac{1}{2}d(G) - 1$; otherwise deleting a minimum degree vertex or contracting an edge with at most $\tau(G)$ common neighbors does not lower the average degree. This lower bound on $\tau(G)$ is the extra information gained when given a lower bound on the average degree instead of a lower bound on the minimum degree, and it is crucial to proving our result.

We prove the following theorem.

THEOREM 2.2. *For integers $k \geq 6$ and $r \geq 3$, every minimal graph with average degree at least $\frac{4}{3}kr$ contains a set of k disjoint cycles, each with at least r vertices.*

Theorem 2.1 follows from Theorem 2.2 since every graph G contains a minor G' such that G' is minimal and $d(G') \geq d(G)$. The proof of Theorem 2.2 is presented in the following section. It depends on a few technical lemmas, which we present in section 4.

3. Proof of Theorem 2.2. Assume for the sake of a contradiction that there is a minimal graph G with $d(G) \geq \frac{4}{3}kr$ that does not contain k disjoint cycles of order at least r . For the sake of simplicity, we allow K_2 to be thought of as a 2-vertex cycle and K_1 as a 1-vertex cycle. Let \mathcal{C} be a collection of disjoint cycles in G each with order at most r . Let $\mathcal{C}(i)$ denote the cycles of order i in \mathcal{C} for $1 \leq i \leq r$.

Choose \mathcal{C} such that $|\mathcal{C}(r)|$ is maximized, then $|\mathcal{C}(r-1)|$ is maximized, and so on until $|\mathcal{C}(1)|$ is maximized. Let \mathcal{U} denote $\mathcal{C}(1) \cup \mathcal{C}(2) \cup \dots \cup \mathcal{C}(r-1)$. If $|\mathcal{C}(r)| \geq k$, then G contains at least k disjoint cycles of order r , giving our desired contradiction. Hence we may assume that $|\mathcal{C}(r)| \leq k-1$.

Let $V(\mathcal{U}) := \bigcup_{C \in \mathcal{U}} V(C)$ and $V(\mathcal{C}(r)) := \bigcup_{C \in \mathcal{C}(r)} V(C)$. Also, let $|C| := |V(C)|$ for a cycle C , and let $|P| := |V(P)|$ for a path P .

CLAIM 1. *Every vertex of G is in a cycle of \mathcal{C} .*

Proof. Assume there is some $v \in V(G)$ that is not in a cycle of \mathcal{C} . Add v to \mathcal{C} as a cycle of length 1. This clearly gives a better choice of \mathcal{C} , contradicting our initial choice. \square

CLAIM 2. *If C is a cycle of \mathcal{U} and vw is an edge of C , then v and w do not have a common neighbor in any other cycle of \mathcal{U} with order at most $|C|$.*

Proof. Assume otherwise, and let v, w have a common neighbor x in a cycle $D \in \mathcal{U}$, where $|D| \leq |C|$. Replace the edge vw in C with the path v, x, w , and remove D from \mathcal{C} . Now \mathcal{C} has lost a cycle of order $|C|$ and a cycle of order $|D|$ but has gained a cycle of order $|C| + 1$. This gives a better choice of \mathcal{C} since $|C| \leq r-1$. \square

CLAIM 3. *If C' is a cycle of \mathcal{U} , vw is an edge of C' , and C is a cycle of $\mathcal{U} - \{C'\}$ with $|C| \geq |C'|$, then v and w have at most $\frac{1}{3}|C|$ common neighbors in C .*

Proof. Assume otherwise. Then there exist two vertices u_1, u_2 of C such that u_2 is either one or two vertices clockwise from u_1 . In the first case, v is a common neighbor of u_1 and u_2 , which contradicts Claim 2. In the second case, let u_3 be the vertex between u_1 and u_2 . We construct a new cycle D from C by removing u_3 and adding the path u_1, w, v, u_2 between u_1, u_2 . The cycle D has order $|C| - 1 + 2 = |C| + 1$. Replacing C and C' in \mathcal{C} with D gives a better choice of \mathcal{C} since $|D| \leq r$. \square

CLAIM 4. *If x, y are 1-vertex cycles of \mathcal{U} , then there is no edge xy .*

Proof. If the edge xy exists, then replace x and y in \mathcal{C} with xy , giving a better choice of \mathcal{C} . \square

Given an edge vw of a cycle in \mathcal{U} , let $W(vw) := N(v) \cap N(w) \cap V(\mathcal{C}(r))$. Given a vertex v of a cycle in \mathcal{U} , let $W(v) := N(v) \cap V(\mathcal{C}(r))$.

We now present some key claims that use lemmas from section 4.

CLAIM 5. *Let $C_1, C_2 \in \mathcal{U}$ be cycles such that $|C_1| \geq |C_2| \geq 2$, and let vw, xy be edges of C_1, C_2 , respectively. At least one of $W(vw), W(xy)$ contains at most $\frac{2}{3}|V(\mathcal{C}(r))|$ vertices.*

Proof. Assume for the sake of a contradiction that $|W(vw)| > \frac{2}{3}|V(\mathcal{C}(r))|$ and $|W(xy)| > \frac{2}{3}|V(\mathcal{C}(r))|$. Apply Lemma 4.1 with $\mathcal{F} := \mathcal{C}(r)$, $q_i := r - |C_2|$ for all $i = 1, \dots, t$, $S := W(xy)$, and $T := W(vw)$. Thus there exists a path P in some F_i

with $r - |C_2|$ vertices and both end vertices in $W(xy)$, together with a vertex u of $W(vw)$ in $V(F_i) - V(P)$.

Construct cycle C'_1 from C_1 by removing the edge vw and adding vertex u and edges vu, uw . Construct cycle C'_2 from C_2 by removing the edge xy and adding the path P together with an edge from x to one end vertex of P and an edge from y to the other end vertex. Thus $|C'_1| = |C_1| + 1$ and $|C'_2| = |C_2| + r - |C_2| = r$. Hence if we remove C_1, C_2, F_i from \mathcal{C} and add C'_1, C'_2 we lose cycles of order $|C_1|, |C_2|$ and r but gain cycles of order r and $|C_1| + 1$. Since $r - 1 \geq |C_1| \geq |C_2|$, this contradicts our initial choice of \mathcal{C} . \square

This technique also works when one or both of the cycles is just a single vertex. The proofs of the next two claims are almost identical to the proof of Claim 5, so we omit them.

CLAIM 6. *Let $C_1, C_2 \in \mathcal{U}$ be cycles such that $|C_1| \geq 2$ and $|C_2| = 1$, let vw be an edge of C_1 , and let $V(C_2) = \{x\}$. At least one of $W(vw), W(x)$ contains at most $\frac{2}{3}|V(\mathcal{C}(r))|$ vertices.*

CLAIM 7. *Let $C_1, C_2 \in \mathcal{U}$ be cycles such that $V(C_1) = \{v\}$ and $V(C_2) = \{x\}$. At least one of $W(v), W(x)$ contains at most $\frac{2}{3}|V(\mathcal{C}(r))|$ vertices.*

Say a cycle of \mathcal{U} is *big* if it contains greater than $\frac{2}{3}r$ vertices; otherwise it is *small*.

CLAIM 8. *The set \mathcal{U} contains at least two big cycles or two small cycles. In particular $|\mathcal{U}| \geq 2$.*

Proof. Assume for the sake of a contradiction that \mathcal{U} contains at most one big cycle and one small cycle. By Claim 1, $V(G) = V(\mathcal{U}) \cup V(\mathcal{C}(r))$. The cycles of $\mathcal{C}(r)$ contain at most $(k-1)r$ vertices in total. Every big cycle of \mathcal{U} contains at most $r-1$ vertices, and every small cycle contains at most $\frac{2}{3}r$ vertices. Thus $|V(G)| \leq (k-1)r + r - 1 + \frac{2}{3}r = (k + \frac{2}{3})r - 1$. Hence $\frac{4}{3}kr \leq d(G) \leq (k + \frac{2}{3})r - 2$, implying $\frac{1}{3}kr \leq \frac{2}{3}r - 2$. This is a contradiction as $k \geq 2$. \square

CLAIM 9. $|\mathcal{C}(r)| = k - 1$.

Proof. Assume $|\mathcal{C}(r)| \leq k - 2$ for the sake of a contradiction. Thus $|V(\mathcal{C}(r))| \leq (k-2)r$. Let C_1, C_2 be the two largest cycles of \mathcal{U} such that $|C_1| \geq |C_2|$. If $|C_1| \geq 2$, let vw be an edge of C_1 and let $A := W(vw)$. If $|C_1| = 1$, then let v be the only vertex of C_1 and let $A := W(v)$. Similarly, let xy be an edge of C_2 and set $B := W(xy)$, unless C_2 contains only one vertex x and $B := W(x)$. We now show $|A|, |B| > \frac{2}{3}|V(\mathcal{C}(r))|$, which contradicts one of Claims 5, 6, or 7.

If $|C_1| = 1$, then every cycle in \mathcal{U} is a 1-cycle, and by Claim 4, $V(\mathcal{U})$ is an independent set, implying $A = N(v)$ and $|A| \geq \delta(G) > \frac{1}{2}d(G) > \frac{2}{3}|V(\mathcal{C}(r))|$, as desired. Otherwise, since C_1 is the largest cycle in \mathcal{U} , Claim 2 implies A contains all common neighbors of v and w except those in C_1 itself. Hence $|A| \geq \tau(G) - |C_1| + 2 > \frac{1}{2}d(G) - 1 - (r-1) + 2 = \frac{1}{2}d(G) - r + 2 \geq \frac{2}{3}kr - r + 2 = \frac{2}{3}(k-2)r + \frac{4}{3}r - r + 2 > \frac{2}{3}|V(\mathcal{C}(r))|$, as desired.

If $|C_2| = 1$, then by Claim 4, B contains all neighbors of x except those in C_1 . If x is adjacent to more than $\frac{1}{2}|C_1|$ vertices of C_1 , then two of those neighbors are themselves adjacent, contradicting Claim 2. Hence $|B| \geq \delta(G) - \frac{1}{2}|C_1| > \frac{2}{3}kr - \frac{1}{2}r = \frac{2}{3}(k-2)r + \frac{4}{3}r - \frac{1}{2}r > \frac{2}{3}|V(\mathcal{C}(r))|$, as desired. Alternatively, if $|C_2| \geq 2$, then B contains all common neighbors of x and y except those in C_1 and C_2 (by Claim 2). The cycle C_1 contains at most $\frac{1}{3}|C_1|$ common neighbors of x and y by Claim 3, and C_2 contains at most $|C_2| - 2$ common neighbors. Thus $|B| \geq \tau(G) - \frac{1}{3}|C_1| - |C_2| + 2 > \frac{2}{3}kr - \frac{4}{3}r + 2 \geq \frac{2}{3}(k-2)r + 2 > \frac{2}{3}|V(\mathcal{C}(r))|$, as desired. This gives our desired contradiction. \square

Claim 5, 6, or 7 shows how we apply Lemma 4.1; we now show how we use Lemma 4.3. Note that since $\frac{2}{3}r \geq 2$, each big cycle contains an edge.

CLAIM 10. *Let $C_1, C_2 \in \mathcal{U}$ be big cycles such that $|C_1| \geq |C_2|$, and let vw, xy be edges of C_1, C_2 , respectively. If $r \geq 4$, then at least one of $W(vw)$ and $W(xy)$ contains at most $\frac{2}{3}(k-1)r - \frac{1}{3}r$ vertices.*

Proof. Assume for the sake of a contradiction that $r \geq 4$, $|W(vw)| > \frac{2}{3}(k-1)r - \frac{1}{3}r$, and $|W(xy)| > \frac{2}{3}(k-1)r - \frac{1}{3}r$. Apply Lemma 4.3 with $\mathcal{F} := \mathcal{C}(r)$, $q_i := r - |C_2|$ for $i = 1, \dots, k-1$, $S := W(xy)$, and $T := W(vw)$. We must ensure that $1 \leq q_i < \frac{1}{3}r$ for $i = 1, \dots, k-1$ to be able to apply this. Since $|C_2| \leq r-1$, it follows $q_i \geq 1$. Since C_2 is big, $|C_2| > \frac{2}{3}r$, and thus $q_i < \frac{1}{3}r$, as required.

The first outcome of Lemma 4.3 is identical to the result of Lemma 4.1, and as such we get a contradiction by the same argument as in Claim 5. Thus we consider only the second outcome. Let F_i be the cycle of $\mathcal{C}(r)$ containing P and Q . Construct the cycle C'_1 by removing the edge vw from C_1 and adding the path Q , together with an edge from v to one end vertex of Q and an edge from w to the other end vertex. Similarly, construct C'_2 from C_2 by removing the edge xy and adding the path P , an edge from x to one end vertex of P , and an edge from y to the other end vertex of P . Thus $|C'_1|, |C'_2| > \frac{2}{3}r + \frac{1}{3}r = r$. Hence $(\mathcal{C}(r) - \{F_i\}) \cup \{C'_1, C'_2\}$ is a set of k cycles, where $k-2$ have order exactly r and two have order $> r$. This contradicts our initial assumption about G . \square

The following claim will also be helpful.

CLAIM 11. *Let $C_1, C_2 \in \mathcal{U}$ be cycles such that $|C_1| \geq |C_2| \geq 2$ and let vw be an edge of C_2 . If v, w have $q \geq 2$ common neighbors in $V(C_1)$, then $|C_2| \leq \frac{|C_1|}{q} - 1$.*

Proof. Label the common neighbors of v, w in $V(C_1)$ clockwise by x_1, \dots, x_q . Let P_i be the clockwise path from x_i to x_{i+1} inclusive (where $i+1$ is taken modulo q). Hence $\sum_{i=1}^q |P_i| = |C_1| + q$, and thus we may fix i so that P_i contains at most $\frac{|C_1|+q}{q}$ vertices. Let Q denote the path between v and w in C_2 that contains all vertices of C_2 .

Construct a cycle C'_1 from C_1 by removing the interior vertices of P_i and adding the path Q and the edges vx_i and wx_{i+1} . If $|C'_1| > r$, then C'_1 together with the $k-1$ cycles of $\mathcal{C}(r)$ contradicts our initial assumption about G . Now assume $|C'_1| \leq r$. If $|C'_1| > |C_1|$, then replacing C_1, C_2 in \mathcal{C} with C'_1 gives a better choice of \mathcal{C} . Otherwise $|C'_1| \leq |C_1|$. Hence $|C_1| - (|P_i| - 2) + |C_2| \leq |C_1|$, implying $|C_2| \leq |P_i| - 2 \leq \frac{|C_1|}{q} - 1$. \square

CLAIM 12. *There is at most one big cycle in \mathcal{U} .*

Proof. First suppose that $r \geq 4$. Let C_1 and C_2 be the two largest cycles in \mathcal{U} such that $|C_1| \geq |C_2|$. Assume for the sake of a contradiction that C_1 and C_2 are both big. Recall both must contain an edge, and so let $A := W(vw)$ and $B := W(xy)$, where vw, xy are edges of C_1, C_2 , respectively. We now show that $|A|, |B| > \frac{2}{3}(k-1)r - \frac{1}{3}r$, which contradicts Claim 10.

By Claim 2, A contains every common neighbor of v and w except those in C_1 itself, of which there are at most $r-3$. Hence $|A| \geq \tau(G) - r + 3 > \frac{2}{3}kr - 1 - r + 3 = \frac{2}{3}(k-1)r - \frac{1}{3}r + 2$.

By Claim 2, B contains every common neighbor of x and y except those in C_1 and in C_2 . If there are at least two common neighbors of x and y in C_1 , then by Claim 11, $|C_2| \leq \frac{|C_1|}{2} - 1 < \frac{r}{2} - 1$. However, $|C_2| > \frac{2}{3}r$, so this gives a contradiction. Hence x and y have at most one common neighbor in C_1 and at most $r-3$ common neighbors in C_2 . Thus $|B| \geq \tau(G) - 1 - r + 3 > \frac{2}{3}(k-1)r - \frac{1}{3}r + 1$, which is sufficient.

Finally, when $r = 3$, every cycle in \mathcal{U} has at most $r - 1 = 2$ vertices, but every big cycle in \mathcal{U} contains greater than $\frac{2}{3}r = 2$ vertices, and thus there are no big cycles in \mathcal{U} . \square

Claims 8 and 12 together imply that \mathcal{U} contains at most one big cycle and at least two small cycles.

CLAIM 13. *Let C_1 be the largest small cycle. Suppose $|C_1| \geq 2$, and let vw be an edge of C_1 . Then $|W(vw)| > \frac{2}{3}(k-1)r$.*

Proof. If there is a big cycle, then, by Claim 12, there is only one; label such a cycle C_0 . By Claim 2, $W(vw)$ contains all common neighbors of v and w except those in C_0 and C_1 .

If $|C_1| \leq \frac{1}{3}r + 1$, then there are at most $\frac{1}{3}|C_0|$ common neighbors of v and w in C_0 (by Claim 3) and at most $\frac{1}{3}r - 1$ common neighbors in C_1 . Thus $|W(vw)| > \tau(G) - \frac{1}{3}|C_0| - (\frac{1}{3}r - 1) \geq \frac{2}{3}kr - 1 - \frac{1}{3}|C_0| - \frac{1}{3}r + 1 > \frac{2}{3}(k-1)r + \frac{2}{3}r - \frac{2}{3}r$, which is sufficient.

Otherwise $\frac{1}{3}r + 1 < |C_1|$. Let q be the number of common neighbors of v and w in C_0 . If $q \geq 3$, then by Claim 11, $|C_1| \leq \frac{|C_0|}{q} - 1 = \frac{1}{3}r - 1$, which is a contradiction. Hence $q \leq 2$. If $q = 2$, then (by Claim 11 again) $|C_1| \leq \frac{1}{2}r - 1$, and hence $|W(vw)| \geq \tau(G) - q - (|C_1| - 2) > \frac{2}{3}kr - 1 - q + 2 - |C_1| \geq \frac{2}{3}(k-1)r + \frac{2}{3}r - 1 - \frac{1}{2}r + 1 > \frac{2}{3}(k-1)r$, as desired. Thus $q \leq 1$, and $|W(vw)| \geq \tau(G) - q - (|C_1| - 2) > \frac{2}{3}kr - 1 - q + 2 - |C_1| \geq \frac{2}{3}(k-1)r + \frac{2}{3}r - |C_1| \geq \frac{2}{3}(k-1)r$, since C_1 is small and $|C_1| \leq \frac{2}{3}r$. Hence our lower bound on $|W(vw)|$ holds. \square

Our goal now is to show that \mathcal{U} contains few vertices; this shall give the final contradiction.

CLAIM 14. *If $C_1, C_2 \in \mathcal{U}$ are the two largest small cycles such that $|C_1| \geq |C_2|$, then $|C_2| = 1$.*

Proof. Assume for the sake of a contradiction that $|C_1| \geq |C_2| \geq 2$. Pick edges vw in C_1 and xy in C_2 , and let $A := W(vw)$ and $B := W(xy)$. As we did in Claim 9, we show that $|A|, |B| > \frac{2}{3}(k-1)r$, which contradicts one of Claims 5, 6, or 7. By Claim 12, it is possible that one big cycle exists in \mathcal{U} ; we denote this cycle by C_0 .

By Claim 13, $|A| > \frac{2}{3}(k-1)r$. Now consider the lower bound on $|B|$. Let $p := \frac{2}{3}r - |C_2|$ and note $p \geq 0$ (since C_2 is small). The set B contains all common neighbors of x and y except those in C_0 (the number of which we denote by q_0), those in C_1 (the number of which we denote by q_1), and at most $|C_2| - 2$ in C_2 . Thus $|B| \geq \tau(G) - q_0 - q_1 - |C_2| + 2 > \frac{2}{3}(k-1)r + \frac{2}{3}r - q_0 - q_1 - |C_2| + 1$. If $\frac{2}{3}r - q_0 - q_1 - |C_2| + 1 \geq 0$, then we are done. So assume $p < q_0 + q_1 - 1$. Since $p \geq 0$ we have $q_0 + q_1 \geq 2$.

If $q_0 \geq 2$, then by Claim 11, $|C_2| \leq \frac{|C_0|}{q_0} - 1$. Since $|C_2| \geq 2$ and $|C_0| \leq r - 1$, it follows that $3q_0 + 1 \leq r$. Also $\frac{2}{3}r - p = |C_2| \leq \frac{|C_0|}{q_0} - 1$. Hence $\frac{2}{3}r - p + 1 \leq \frac{r-1}{q_0}$ and thus $2q_0^2 + \frac{2}{3}q_0 - 3q_0 - pq_0 + q_0 = (\frac{2}{3}q_0 - 1)(3q_0 + 1) - pq_0 + q_0 + 1 \leq (\frac{2}{3}q_0 - 1)r - pq_0 + q_0 + 1 \leq 0$. Thus $2q_0 + \frac{2}{3} - 3 - p + 1 \leq 0$ and $q_0 \leq \frac{1}{2}p + \frac{2}{3}$.

Similarly, if $q_1 \geq 2$, by Claim 11, $|C_2| \leq \frac{|C_1|}{q_1} - 1$. Since $|C_2| \geq 2$ and $|C_1| \leq \frac{2}{3}r$, it follows $\frac{9}{2}q_1 \leq r$. Also $\frac{2}{3}r - p = |C_2| \leq \frac{|C_1|}{q_1} - 1$. Thus $\frac{2}{3}r - p + 1 \leq \frac{2r}{3q_1}$, and so $\frac{2}{3}(q_1 - 1)(\frac{9}{2}q_1) - pq_1 + q_1 \leq \frac{2}{3}(q_1 - 1)r - pq_1 + q_1 \leq 0$. Hence $3(q_1 - 1) - p + 1 \leq 0$ and $q_1 \leq \frac{1}{3}p + \frac{2}{3}$.

We now show that $p < 2$. If $q_0 \geq 2$ and $q_1 \geq 2$, then $p < q_0 + q_1 - 1 \leq \frac{1}{2}p + \frac{2}{3} + \frac{1}{3}p + \frac{2}{3} - 1 = \frac{5}{6}p + \frac{1}{3}$, and thus $p < 2$. If $q_0 \geq 2$ and $q_1 \leq 1$, then $p < q_0 \leq \frac{1}{2}p + \frac{2}{3}$, and $p < \frac{4}{3} < 2$. If $q_1 \geq 2$ and $q_0 \leq 1$, then $p < q_1 \leq \frac{1}{3}p + \frac{2}{3}$ and $p < 1$. Finally, if $q_0 \leq 1$ and $q_1 \leq 1$, then $p < 1$. Hence $p < 2$.

If $q_0 \geq 2$ or $q_1 \geq 2$, then $q_0 < \frac{5}{3} < 2$ or $q_1 < \frac{4}{3} < 2$, respectively, and thus $q_0 \leq 1$ and $q_1 \leq 1$. Since $q_0 + q_1 \geq 2$, it follows $q_0 = q_1 = 1$, and $p < 1$. Hence $|C_2| > \frac{2}{3}r - 1$. Since $|C_2| \leq \frac{2}{3}r$ and $|C_2|$ is an integer, $|C_2| \leq \lfloor \frac{2}{3}r \rfloor$, and together with our lower bound, $|C_2| = \lfloor \frac{2}{3}r \rfloor$. Since $|C_1| \geq |C_2|$ and C_1 is also small, $|C_1| = |C_2|$. However, since $q_1 = 1$, this contradicts Claim 2. \square

We now know that \mathcal{U} contains

- at most one big cycle (by Claim 12)
- at least two small cycles (by Claims 12 and 8), and
- every small cycle, except possibly the largest, is a 1-cycle (by Claim 14).

Let \mathcal{I} be the set of all small cycles of \mathcal{U} other than the largest. Since these are all 1-cycles, we interpret \mathcal{I} as a set of vertices; \mathcal{I} is an independent set by Claim 4. Let C_1 be the largest small cycle in \mathcal{U} .

CLAIM 15.

- (a) *There exists a big cycle in \mathcal{U} , denoted C_0 ,*
- (b) *Every $u \in \mathcal{I}$ has more than $\frac{1}{3}|C_0|$ neighbors in C_0 ,*
- (c) *Every $u \in \mathcal{I}$ has at least two neighbors in C_1 , and*
- (d) *$|C_1| > \frac{1}{3}r$.*

Proof. Assume, for the sake of a contradiction, that at least one of the following holds:

- (a) \mathcal{U} contains no big cycle,
- (b) there exists a big cycle C_0 and some $u \in \mathcal{I}$ has at most $\frac{1}{3}|C_0|$ neighbors in C_0 ,
- (c) there exists a big cycle C_0 and some $u \in \mathcal{I}$ has at most one neighbor in C_1 ,
or
- (d) $|C_1| \leq \frac{1}{3}r$.

If C_1 contains an edge vw , let $A := W(vw)$; otherwise let $A := W(v)$, where $V(C_1) = \{v\}$. Let $B := W(u)$. Once again we show that $|A|, |B| > \frac{2}{3}(k-1)r$, which contradicts either Claim 6 or Claim 7.

If $|C_1| \geq 2$, then by Claim 13, $|A|$ is sufficiently large. If $|C_1| = 1$, then A contains all neighbors of v except those in C_0 (by Claim 4). By Claim 2, there are at most $\frac{1}{2}|C_0| \leq \frac{1}{2}r$ of these. Hence $|A| \geq \delta(G) - \frac{1}{2}r > \frac{2}{3}kr - \frac{1}{2}r = \frac{2}{3}(k-1)r + \frac{1}{6}r$, as desired.

Now consider $|B|$. The set B contains all vertices adjacent to u except those in C_0 and those in C_1 (since \mathcal{I} is an independent set); say there are q_0 neighbors of u in C_0 and q_1 neighbors of u in C_1 . Thus $|B| \geq \delta(G) - q_0 - q_1 > \frac{2}{3}kr - q_0 - q_1 = \frac{2}{3}(k-1)r + \frac{2}{3}r - q_0 - q_1$. If $\frac{2}{3}r - q_0 - q_1 \geq 0$ we are done. So assume $q_0 + q_1 > \frac{2}{3}r$.

By Claim 2, it follows that $q_0 \leq \frac{1}{2}|C_0|$ and $q_1 \leq \frac{1}{2}|C_1|$. In (a) or (b), $\frac{2}{3}r < q_0 + q_1 \leq \frac{1}{3}(r-1) + \frac{1}{2}(\frac{2}{3}r) = \frac{2}{3}r - \frac{1}{3}$, since C_1 is small; this is a contradiction. In (c), $\frac{2}{3}r < q_0 + q_1 \leq \frac{1}{2}(r-1) + 1$, which implies $r < 3$, again a contradiction. In (d), $\frac{2}{3}r < q_0 + q_1 \leq \frac{1}{2}(r-1) + \frac{1}{6}r = \frac{2}{3}r - \frac{1}{2}$, yet again a contradiction. Thus we have proven our desired lower bounds on $|A|$ and $|B|$ in all cases. \square

By Claims 8 and 12, $\mathcal{I} \neq \emptyset$. Our final step is to bound $|\mathcal{I}|$.

CLAIM 16. $|\mathcal{I}| = 1$.

Proof. Assume $|\mathcal{I}| \geq 2$ for the sake of a contradiction. Let $x, x' \in \mathcal{I}$. Let S, T be the neighbors of x, x' in C_0 , respectively. By Claim 15, $|S|, |T| > \frac{1}{3}|C_0|$. Given that $|C_0| > \frac{2}{3}r \geq 2$, it follows from Lemma 4.4 that there exists a path P in C_0 with one end vertex in S , the other in T , and $2 \leq |P| \leq \frac{1}{6}|C_0| + 4 < \frac{1}{6}r + 4$. Let v be the end vertex of P in S and w the end vertex of P in T . Since x, x' both have two or more neighbors in C_1 (again by Claim 15), there is a neighbor a of x and a neighbor $b \neq a$ of x' . Let Q be the longer path in C_1 between a and b ; since, by Claim 15, $|C_1| > \frac{1}{3}r$, it follows $|Q| > \frac{1}{6}r$.

Create the cycle C in the following way. Start with C_0 and remove the interior vertices of $V(P)$. Add edges vx and wx' and then the edges xa and $x'b$. Finally add the path Q between a and b . Thus $|C| = |C_0| - (|P| - 2) + 2 + |Q| = |C_0| - |P| + 4 + |Q| > |C_0| - (\frac{1}{6}r + 4) + 4 + \frac{1}{6}r = |C_0|$. If $|C| \leq r$, then removing $C_0, C_1, \{x\}$, and $\{x'\}$ from \mathcal{C} and adding C gives a better choice of \mathcal{C} . Otherwise, $\mathcal{C}(r) \cup \{C\}$ is a set of k cycles, where $k - 1$ have order exactly r and one has order $> r$; this contradicts our initial assumption about G . \square

Now we know that \mathcal{U} contains three cycles: big cycle C_0 , small cycle C_1 , and a single 1-cycle in \mathcal{I} . Hence $|V(\mathcal{U})| = |C_0| + |C_1| + 1 \leq (r - 1) + \frac{2}{3}r + 1 = \frac{4}{3}r$. Given that $|V(\mathcal{C}(r))| = (k - 1)r$, it follows by Claim 1 that $|V(G)| \leq (k - 1)r + \frac{4}{3}r = kr + \frac{1}{3}r$, and $d(G) \leq kr + \frac{1}{3}r - 1$. Thus $\frac{4}{3}kr \leq d(G) \leq kr + \frac{1}{3}r - 1$, implying $0 \leq \frac{1}{3}(k - 1)r \leq -1$. This is the final contradiction, and Theorems 1.1, 2.1, and 2.2 are proved. Note that we have actually proved the following strengthening of Theorem 2.2: for integers $k \geq 6$ and $r \geq 3$, every minimal graph with average degree at least $\frac{4}{3}kr$ contains a set of k disjoint cycles, $k - 2$ with exactly r vertices, and two with at least r vertices.

4. Ancillary lemmas. This section presents some technical lemmas that were used in section 3. Note that none of these results depend on the choice of G . Recall if C is a cycle, then $|C| := |V(C)|$, and if P is a path, then $|P| := |V(P)|$. Also if \mathcal{F} is a collection of disjoint cycles, then let $V(\mathcal{F}) := \bigcup_{F \in \mathcal{F}} V(F)$. In this section, all cycles contain at least three vertices; that is, we no longer treat K_2 and K_1 as cycles. (The lemmas in this section are applied to either $\mathcal{C}(r)$, which contains only cycles of length $r \geq 3$, or to big cycles of \mathcal{U} , which contain strictly more than $\frac{2}{3}r \geq 2$ vertices, so this is acceptable.)

LEMMA 4.1. *Let $\mathcal{F} = \{F_1, \dots, F_t\}$ be a collection of disjoint cycles and let q_1, \dots, q_t be integers such that $1 \leq q_i \leq |F_i| - 1$. If $S, T \subseteq V(\mathcal{F})$ and $|S|, |T| > \frac{2}{3}|V(\mathcal{F})|$, then there exists a path P in some F_i with exactly q_i vertices, such that both end vertices of P are in S and there exists a vertex of T in $V(F_i) - V(P)$.*

Proof. Assume otherwise for the sake of a contradiction. Fix an orientation on every cycle in F_1, \dots, F_t . For each vertex $v \in V(F_i)$, let $f(v)$ denote the vertex $q_i - 1$ vertices clockwise from v in F_i , and let P_v be the q_i -vertex path from v clockwise to $f(v)$. Let $f(S) := \{f(v) : v \in S\}$. Since $f : V(\mathcal{F}) \rightarrow V(\mathcal{F})$ is a 1 - 1 correspondence, $|f(S)| = |S|$. Let $\mathcal{P} := \{P_v : v, f(v) \in S\}$. If $P_v \in \mathcal{P}$, then $f(v) \in S \cap f(S)$. Conversely, if $w \in S \cap f(S)$, then $w = f(v)$ for some $v \in S$, and so $P_v \in \mathcal{P}$. Hence $|\mathcal{P}| = |S \cap f(S)|$. Thus,

$$(4.1) \quad |V(\mathcal{F})| \geq |S \cup f(S)| = |S| + |f(S)| - |S \cap f(S)| = 2|S| - |\mathcal{P}|.$$

Let $x(v)$ denote the vertex counterclockwise from v . Since $|P_v| = q_i \leq |F_i| - 1$, the vertex $x(v)$ is not in P_v . Let $\mathcal{X} := \{x(v) : P_v \in \mathcal{P}\}$. As x is 1 - 1, $|\mathcal{X}| = |\mathcal{P}|$. If $\mathcal{X} \cap T \neq \emptyset$, then there is a path in \mathcal{P} , which has both end vertices in S , together with a vertex of T in the same cycle as P but not intersecting P itself. Hence $\mathcal{X} \cap T = \emptyset$. Thus,

$$(4.2) \quad |V(\mathcal{F})| \geq |\mathcal{X} \cup T| = |\mathcal{X}| + |T| - |\mathcal{X} \cap T| = |\mathcal{P}| + |T|.$$

Equations (4.1) and (4.2) imply it follows $2|V(\mathcal{F})| \geq 2|S| + |T| > \frac{4}{3}|V(\mathcal{F})| + \frac{2}{3}|V(\mathcal{F})| = 2|V(\mathcal{F})|$, which is a contradiction. \square

LEMMA 4.2. *Let C be a cycle such that $|C| \geq 6$ and let $p := \lfloor \frac{|C|}{3} \rfloor$. If $S, T \subseteq V(C)$ such that $|S| \geq p + 1$ and $|T| \geq 2p + 2$, then there exist disjoint paths P, Q such that the end vertices of P are both in S , the end vertices of Q are both in T , and $|V(P)|, |V(Q)| \geq p + 1$.*

Proof. Note $|C| \in \{3p, 3p+1, 3p+2\}$. First, we shall show there is a path A in C with both end vertices in S such that $p+1 \leq |A| \leq |C|+1-2p$. Label the vertices of C clockwise $1, \dots, |C|$ so that, if possible, $3p+1, 3p+2 \notin S$. Let $X_i := \{i, p+i, 2p+i\}$ for $i = 1, \dots, p$. If $|X_i \cap S| \geq 2$ for some i , then the shorter path between two vertices of $X_i \cap S$ has at least $p+1$ vertices and at most $p+1 + (|C| - 3p) = |C| + 1 - 2p$, as desired. Otherwise $|X_i \cap S| \leq 1$ for all i . If $3p+1, 3p+2 \notin S$, then $|S| \leq p$, which is a contradiction.

Hence we may assume that at least one of $3p+1$ and $3p+2$ is in S . If $|C| = 3p$, no vertex is labeled $3p+1$ or $3p+2$, and this case cannot occur. If $|C| = 3p+1$, then $3p+1 \in S$ and, by our choice of labeling, all vertices of C are in S . But then it is trivial to construct a $(p+1)$ -vertex path A with both end vertices in S . If $|C| = 3p+2$, then by our choice of labeling, there do not exist two adjacent vertices such that neither is in S . Let v be a vertex of S , and let w, w' be the two vertices $p, p+1$ vertices clockwise from v . At least one of w, w' is in S , and together with v this gives a path with both end vertices in S of either $p+1$ or $p+2$ vertices, as required.

Thus there is a path A with $p+1 \leq |A| \leq |C|+1-2p$ and both end vertices in S . If there are multiple choices for A , choose one of shortest length. If $|T - V(A)| \geq p+1$, set $P = A$ and let Q be the longest path in $C - V(A)$ with both end vertices in T ; such a path must contain all the vertices of $T - V(A)$, and thus $|Q| \geq p+1$, as desired. Hence we may assume that $|T - V(A)| \leq p$. Since $|T| \geq 2p+2$, we have $|V(A) \cap T| \geq p+2$ and $|A| \geq p+2$.

If $|C| = 3p$, then $|V(A) \cap T| \leq |A| = p+1$, which is a contradiction. Hence $|C| \geq 3p+1$. Let v be the counterclockwise end vertex of A and w the clockwise end vertex. Label the two vertices clockwise from v by v', v'' and the two vertices counterclockwise from w by w', w'' . These vertices are all in A since $|A| \geq p+2 \geq 4$. If $|A| = p+2$, then $v', w' \notin S$, or else we could replace A by a smaller path that still contains $p+1$ vertices. Thus $|V(A) \cap S| \leq p$ (since $|A| \geq 4$, v' and w' are distinct vertices). Alternatively, if $|A| = p+3$, then by the same argument $v', w', v'', w'' \notin S$, and $|V(A) \cap S| \leq p$ (since $|A| = p+3 \geq 5$ and thus $|\{v', w', v'', w''\}| \geq 3$). In either case, since $|S| \geq p+1$, at least one vertex of S is not in A ; label one such vertex x . If the clockwise path from w to x and the clockwise path from x to v both contain at most p vertices, then the clockwise v to w path contains at most $2p-1$ vertices, and hence $|C| - (2p-1) + 2 \leq |A| \leq |C| + 1 - 2p$, which is a contradiction. Without loss of generality, say the clockwise path from w to x contains at least $p+1$ vertices, and let this be P . Since $|V(A) \cap T| \geq p+2$, it follows $|(V(A) - \{w\}) \cap T| \geq p+1$, so the longest path with both end vertices in T contained within $V(A) - \{w\}$ contains at least $p+1$ vertices, and we set it as Q . This proves the lemma. \square

LEMMA 4.3. Let $\mathcal{F} = \{F_1, \dots, F_{k-1}\}$ be a collection of $k-1 \geq 5$ disjoint cycles of order $r \geq 4$, and let q_1, \dots, q_{k-1} be integers such that $1 \leq q_i < \frac{1}{3}r$. If $S, T \subseteq V(\mathcal{F})$ and $|S|, |T| > \frac{2}{3}(k-1)r - \frac{1}{3}r$, then at least one of the following cases holds:

- There exists a path P in some F_i with exactly q_i vertices, such that both end vertices of P are in S , and there exists a vertex of T in $V(F_i) - V(P)$.
- There exist disjoint paths P, Q in some F_i such that the end vertices of P are in S , the end vertices of Q are in T , and $|P|, |Q| > \frac{1}{3}r$.

Proof. Note that the first part of this proof is very similar to Lemma 4.1. For the sake of a contradiction, assume neither outcome occurs. For each $v \in V(F_i)$, let $f(v)$ denote the vertex $q_i - 1$ vertices clockwise from v , and let P_v be the q_i -vertex path from v clockwise to $f(v)$. Let $f(S) := \{f(v) : v \in S\}$, and note $|f(S)| = |S|$ since f is a 1-1 correspondence. Let $\mathcal{P} := \{P_v : v, f(v) \in S\}$. If $w \in S \cap f(S)$, then

$w = f(v)$ such that $P_v \in \mathcal{P}$. Conversely, if $P_v \in \mathcal{P}$, then $f(v) \in S \cap f(S)$. Hence $|\mathcal{P}| = |S \cap f(S)|$. Thus,

$$(4.3) \quad (k-1)r \geq |S \cup f(S)| = |S| + |f(S)| - |S \cap f(S)| = 2|S| - |\mathcal{P}|.$$

Let $x(v)$ denote the vertex counterclockwise from v . Then $x(v) \notin V(P_v)$ since $q_i < \frac{1}{3}r < r$. Let $\mathcal{X} := \{x(v) : P_v \in \mathcal{P}\}$. As x is 1-1, $|\mathcal{P}| = |\mathcal{X}|$. Let $X_i := \mathcal{X} \cap V(F_i)$ and let $\mathcal{P}_i := \{P_v \in \mathcal{P} : v \in V(F_i)\}$. We classify each of F_1, \dots, F_{k-1} as one of the four following types. If $\mathcal{P}_i \neq \emptyset$ and there exists at least one vertex $w \in V(F_i)$ such that $w \in V(P_v)$ for all $P_v \in \mathcal{P}_i$, then F_i has *type 1*. If $\mathcal{P}_i \neq \emptyset$ but there is no such w , then F_i has *type 2*. Otherwise $\mathcal{P}_i = \emptyset$; if $|S \cap V(F_i)| \leq \frac{1}{3}r$, then F_i has *type 3*, otherwise it has *type 4*.

We now define a set Y_i for each F_i . If F_i has type 2, 3, or 4, let $Y_i := X_i$. (Note if F_i has type 3 or 4, then $X_i = \emptyset$.) If F_i has type 1, label the vertices of F_i clockwise by $1, 2, \dots, r$, starting one vertex clockwise from w , where $w \in V(P_v)$ for all $P_v \in \mathcal{P}_i$. Let v be the vertex of minimum index such that $P_v \in \mathcal{P}_i$. Let $J := V(F_i) - P_v$. The set J starts at $x(v)$ and travels counterclockwise; it may reach as far as vertex 1, but does not include vertex r by our choice of labeling (since $w = r$ is in P_v). Also, $J \cap X_i = \{x(v)\}$, by our choice of v . As $q_i < \frac{1}{3}r$, we have $|J| > \frac{2}{3}r$. Let $Y_i = X_i \cup J$, and note $|Y_i| > |X_i| + \frac{2}{3}r - 1$ when F_i is type 1.

Define $\mathcal{Y} := \cup_{i=1}^{k-1} Y_i$. If $\mathcal{Y} \cap T \neq \emptyset$, then there exists a vertex of T in a cycle of type 1 or 2 that avoids a path of \mathcal{P} in the same cycle; together this satisfies the first outcome. Now assume that $\mathcal{Y} \cap T = \emptyset$. Let θ_1 denote the number of type 1 cycles. First suppose that $\theta_1 \geq 2$. Thus $|\mathcal{Y}| > |\mathcal{X}| + 2(\frac{2}{3}r - 1)$. Hence $(k-1)r \geq |\mathcal{Y} \cup T| = |\mathcal{Y}| + |T| > |\mathcal{X}| + 2(\frac{2}{3}r - 1) + |T| = |\mathcal{P}| + |T| + \frac{4}{3}r - 2$.

Let $r = 3a + b$ where $a, b \in \mathbb{Z}$ and $a \geq 1$ and $b \in \{0, 1, 2\}$. Thus $(k-1)r > |\mathcal{P}| + |T| + 4a + \frac{4}{3}b - 2$, and so

$$(4.4) \quad (k-1)r \geq |\mathcal{P}| + |T| + 4a + b + \lfloor \frac{1}{3}b \rfloor + 1 - 2 = |\mathcal{P}| + |T| + 4a + b - 1.$$

Inequalities (4.3) and (4.4) imply

$$2(k-1)r \geq 2|S| - |\mathcal{P}| + |\mathcal{P}| + |T| + 4a + b - 1 > 2(k-1)r - r + 4a + b - 1.$$

Thus $3a + b = r \geq 4a + b$, contradicting $a \geq 1$. Hence we may assume $\theta_1 \leq 1$.

Define $\theta_2, \theta_3, \theta_4$ to be the number of type 2, 3, and 4 cycles, respectively. Note $\theta_1 + \theta_2 + \theta_3 + \theta_4 = k - 1$. Let $S_i := S \cap V(F_i)$ and $T_i := T \cap V(F_i)$. We now prove some bounds on $|S_i|$ and $|T_i|$.

- If F_i has type 1, then $|S_i| \leq r$ trivially, and since $T_i \cap Y_i = \emptyset$ and $Y_i = X_i \cup J$ and $|J| > \frac{2}{3}r$, it follows that $|T_i| < \frac{1}{3}r$.
- If F_i has type 2, then $|S_i| \leq r$ trivially. Since for every $w \in V(F_i)$ there exists some $P_v \in \mathcal{P}_i$ such that $w \notin V(P_v)$, if $w \in T_i$, then w together with P_v satisfies the first outcome. Thus we may assume that $T_i = \emptyset$.
- If F_i has type 3, then $|S_i| \leq \lfloor \frac{1}{3}r \rfloor$ by definition, and $|T_i| \leq r$ trivially.
- If F_i has type 4, then $|S_i| > \frac{1}{3}r$ by definition. If $r \leq 6$, then $q_i < 2$, and hence $q_i = 1$. But then $\mathcal{P}_i = S_i \neq \emptyset$, contradicting the definition of type 4. Hence $r > 6$. Let $p := \lfloor \frac{r}{3} \rfloor$, and note $|S_i| \geq p + 1$. If $|T_i| \geq 2p + 2$, then by Lemma 4.2 there exist disjoint paths P, Q in F_i such that the end vertices of P are in $S_i \subseteq S$ and the end vertices of Q are in $T_i \subseteq T$ and $|P|, |Q| \geq p + 1 > \frac{r}{3}$. This would satisfy the second outcome, and hence we may assume that $|T_i| \leq 2p + 1$.

If $r \geq 9$, then $|T_i| \leq \frac{2}{3}r + 1 = \frac{7}{9}r - \frac{1}{9}r + 1 \leq \frac{7}{9}r$. Alternatively, $7 \leq r \leq 8$, and $|T_i| \leq 2p + 1 = 2\lfloor \frac{8}{9} \rfloor + 1 = 5 < \frac{7}{9}(7) \leq \frac{7}{9}r$. Thus in all cases $|T_i| \leq \frac{7}{9}r$. Finally, for each $v \in S_i$, recall $f(v)$ is the vertex $q_i - 1$ vertices clockwise from v and let $f(S_i) := \{f(v) : v \in S_i\}$. Since $\mathcal{P}_i = \emptyset$ we have $S_i \cap f(S_i) = \emptyset$. Since f is 1-1 it follows that $r \geq |S_i \cup f(S_i)| = |S_i| + |f(S_i)| = 2|S_i|$, and thus $|S_i| \leq \frac{1}{2}r$.

Thus

$$(4.5) \quad \frac{2}{3}(k-1) - \frac{1}{3} < \frac{1}{r}|S| = \frac{1}{r} \sum_{i=1}^{k-1} |S_i| \leq \theta_1 + \theta_2 + \frac{1}{3}\theta_3 + \frac{1}{2}\theta_4 \text{ and}$$

$$(4.6) \quad \frac{2}{3}(k-1) - \frac{1}{3} < \frac{1}{r}|T| = \frac{1}{r} \sum_{i=1}^{k-1} |T_i| \leq \frac{1}{3}\theta_1 + \theta_3 + \frac{7}{9}\theta_4.$$

Inequalities (4.5) and (4.6), together with $k-1 = \theta_1 + \theta_2 + \theta_3 + \theta_4$ and $\theta_1 \leq 1$, form a set of integer linear inequalities. If $k-1 \geq 6$, then it is easily shown by hand or by computer that this set of inequalities has no feasible solution.²

Finally, assume that $k-1 = 5$. In this case, by the above upper bounds on $|S_i|$ and $|T_i|$, we may replace (4.5) and (4.6) with the following more precise bounds:

$$\begin{aligned} \frac{2}{3}(5)r - \frac{1}{3}r < |S| &\leq r\theta_1 + r\theta_2 + \lfloor \frac{1}{3}r \rfloor \theta_3 + \lfloor \frac{1}{2}r \rfloor \theta_4 && \text{and} \\ \frac{2}{3}(5)r - \frac{1}{3}r < |T| &\leq (\lceil \frac{1}{3}r \rceil - 1)\theta_1 + r\theta_3 + (2\lfloor \frac{1}{3}r \rfloor + 1)\theta_4. \end{aligned}$$

Let $r = 3a + b$, where $a \geq 1$ and $b \in \{0, 1, 2\}$. That is,

$$(4.7) \quad 9a + 3b + 1 \leq (3a + b)(\theta_1 + \theta_2) + a\theta_3 + \frac{1}{2}(3a + b)\theta_4,$$

$$(4.8) \quad 9a + 3b + 1 \leq (\lceil a + \frac{1}{3}b \rceil - 1)\theta_1 + (3a + b)\theta_3 + (2a + 1)\theta_4.$$

Inequalities (4.7) and (4.8), together with $\theta_1 \in \{0, 1\}$ and $b \in \{0, 1, 2\}$ and $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 5$, form a set of integer quadratic inequalities. This set has no solution since there are finitely many choices for $\theta_1, \theta_2, \theta_3, \theta_4$, and b , and for each such choice, (4.7) and (4.8) reduce to linear inequalities in a , which have no solution.²

Note that if $k-1 \in \{1, 2, 3, 4\}$, then there are choices of $\theta_1, \theta_2, \theta_3, \theta_4$, and r that satisfy the integer linear inequalities implied by our bounds on $|S_i|$ and $|T_i|$. \square

LEMMA 4.4. *Let C be a cycle and $S, T \subseteq V(C)$ such that $|S|, |T| > \frac{1}{3}|C|$. Then there exists a path P with one end vertex in S and the other in T such that $2 \leq |P| \leq \frac{1}{6}|C| + 4$.*

Proof. Suppose there exist vertices $v, w \in S$ such that there are at least $\frac{1}{3}|C| + 2$ vertices between v and w (noninclusive) in both directions around C . If there exists a vertex $u \in T$ within the $\lceil \frac{1}{6}|C| \rceil + 1$ vertices counterclockwise from v , then the path from u clockwise to v has at least two vertices and at most $\lceil \frac{1}{6}|C| \rceil + 2 \leq \frac{1}{6}|C| + 3$ vertices, as required. A similar result holds for the $\lceil \frac{1}{6}|C| \rceil + 1$ vertices clockwise from v , the $\lceil \frac{1}{6}|C| \rceil + 1$ vertices counterclockwise from w , and the $\lceil \frac{1}{6}|C| \rceil + 1$ vertices clockwise from w . Label these sets Z_1, Z_2, Z_3, Z_4 , respectively. If $Z_1 \cap Z_4 = \emptyset$, then $|Z_1 \cup Z_4| = 2(\lceil \frac{1}{6}|C| \rceil + 1) \geq \frac{1}{3}|C| + 2$; otherwise $Z_1 \cup Z_4$ covers all vertices from v counterclockwise to w and so $|Z_1 \cup Z_4| \geq \frac{1}{3}|C| + 2$. A similar result holds for $Z_2 \cup Z_3$, and hence $|Z_1 \cup Z_2 \cup Z_3 \cup Z_4| \geq \frac{2}{3}|C| + 4$. Given that $|T| > \frac{1}{3}|C|$ it follows $(Z_1 \cup Z_2 \cup Z_3 \cup Z_4) \cap T \neq \emptyset$ and our desired path can be constructed.

²This was computed using AMPL and Couenne 0.4.3, an open-source solver for mixed integer nonlinear optimization, available at <http://www.neos-server.org>.

Hence we may assume that any two vertices in S have fewer than $\frac{1}{3}|C| + 2$ vertices of C between them. Suppose $u \in T$ is an interior vertex on the short path D between two vertices $v, w \in S$. Let $A \subset D$ be the path from v to u , and let $B \subset D$ be the path from u to w . Clearly $|A|, |B| \geq 2$. If $|A|, |B| > \frac{1}{6}|C| + 4$, then $\frac{1}{3}|C| + 4 > |D| = |A| + |B| - 1 > \frac{1}{3}|C| + 7$, which is a contradiction. Hence either A or B satisfies our lemma, and as such we may suppose no such u exists.

Fix a vertex $v \in S$. Let $L \subseteq S$ be the set of vertices w where a short path from v to w travels counterclockwise from v ; let R be the equivalent clockwise set. If $L = \emptyset$ and $R = \emptyset$, then $S = \{v\}$, but $|S| > \frac{1}{3}|C| \geq 1$, which is a contradiction. Thus, without loss of generality, $L \neq \emptyset$. Let $x \in L$ be the vertex of greatest counterclockwise distance from v , and define y to be the vertex of greatest clockwise distance from v in R (if $R \neq \emptyset$) or let $y := v$ if $R = \emptyset$. Let A be the path from x clockwise to v and B the path from v clockwise to y . The interior vertices of A and B are not in T as they are both short paths. Let $Q := A \cup B$, and note Q contains all vertices of S . Construct Q' from Q by adding the $\lfloor \frac{1}{6}|C| \rfloor + 3$ vertices counterclockwise from x and the $\lfloor \frac{1}{6}|C| \rfloor + 3$ vertices clockwise from y ; none of these added vertices may be in T without creating a satisfactory path. Thus $(Q' - \{x, y, v\}) \cap T = \emptyset$ and either $|Q'| > \frac{1}{3}|C| + 2(\lfloor \frac{1}{6}|C| \rfloor + 3) \geq \frac{1}{3}|C| + 2(\frac{1}{6}|C| + 2) = \frac{2}{3}|C| + 4$ (if these added vertices do not overlap) or $Q' = C$ (if they do). In the first case, $|Q' - \{x, y, v\}| > \frac{2}{3}|C| + 1$, and as such $Q' - \{x, y, v\}$ intersects T , thus creating a satisfactory path. In the second case, $T \subseteq \{x, y, v\}$. Given $|T| > \frac{1}{3}|C|$, it follows $|C| \leq 8$.

Given that $|S|, |T| \geq 2$, choose a vertex $a \in S$ and $b \in T - \{a\}$. Since $|C| \leq 8$, there is a path from a to b containing at most $\frac{|C|+2}{2} = \frac{1}{6}|C| + 4 + \frac{1}{3}|C| - 3 < \frac{1}{6}|C| + 4$ vertices, as required. \square

5. Open problems. We conclude by noting some possible extensions of our results. We mentioned the following conjecture previously.

CONJECTURE 5.1 (Reed and Wood [25]). *Every graph with average degree at least $\frac{4}{3}t - 2$ contains every 2-regular t -vertex graph as a minor.*

Conjecture 5.1 essentially extends Theorem 2.1 by removing the requirement that all cycles in the desired minor have the same order.

Recall that the original result of Corrádi and Hajnal [6] was a stronger result in terms of minimum degree (rather than average degree). The following conjecture would imply our result and be a more direct generalization of the Corrádi and Hajnal theorem.

CONJECTURE 5.2. *Every graph with minimum degree at least $\frac{2}{3}kr$ and at least kr vertices contains k disjoint cycles, each containing at least r vertices.*

Wang [33] recently made the following two conjectures.

CONJECTURE 5.3 (Wang [33]). *For every integer $k \geq 2$ and odd integer $r \geq 3$, every graph G with at least rk vertices and minimum degree at least $\frac{r+1}{2}k$ contains k disjoint cycles, each containing at least r vertices.*

CONJECTURE 5.4 (Wang [33]). *For every integer $k \geq 3$ and even integer $r \geq 6$, every graph G with at least rk vertices and minimum degree at least $\frac{r}{2}k$ contains k disjoint cycles, each containing at least r vertices, unless k is odd and $rk + 1 \leq |V(G)| \leq rk + r - 2$.*

These conjectures follow from the best known lower bounds on the required minimum degree. Consider the split graph $\mathcal{S}_{k,r,n}$ with a clique of order $\lceil \frac{r}{2} \rceil k - 1$, an n -vertex independent set, and all edges between the two. Such a graph does not contain k disjoint cycles of order at least r since any such cycle must contain at least $\lceil \frac{r}{2} \rceil$ vertices of the clique. However, the minimum degree of $\mathcal{S}_{k,r,n}$ is $\lceil \frac{r}{2} \rceil k - 1$ when $n \geq 1$.

Due to this example the lower bounds on the minimum degree in Conjectures 5.3 and 5.4 are necessary.

As Wang shows, the exception when k is odd and $rk + 1 \leq |G| \leq rk + r - 2$ is due to the disjoint union $K_{(k-1)p+p+i} \cup K_{(k-1)p+p+j}$, where $0 \leq i \leq j \leq \frac{1}{2}r - 1$ and $j \neq 0$ and $p = \frac{1}{2}r$. Due to the limited number of vertices, each component contains at most $\frac{k-1}{2}$ disjoint cycles of order at least r , and hence the graph does not contain k such disjoint cycles in total. However, if $i \geq 1$, then the minimum degree is at least $\frac{k-1}{2}r + \frac{1}{2}r + i - 1 \geq \frac{r}{2}k$. If $i = 0$, we can achieve the same minimum degree by adding edges from one vertex of the second clique to all vertices of the first—this does not increase the number of disjoint cycles.

We conjecture the following weakening of Conjectures 5.3 and 5.4 by considering the average degree instead of the minimum degree. Note that while the average degree of $\mathcal{S}_{k,r,n}$ tends to $2\lceil \frac{r}{2} \rceil k - 2$ as $n \rightarrow \infty$, the average degree of the disjoint union of complete graphs (or its modification) is small enough that we no longer need to consider it. Thus, the lower bound follows only from the existence of $\mathcal{S}_{k,r,n}$.

CONJECTURE 5.5. *For every integer $k \geq 2$ and odd integer $r \geq 3$, every graph G with average degree at least $(r+1)k - 2$ contains k disjoint cycles, each containing at least r vertices.*

CONJECTURE 5.6. *For every integer $k \geq 3$ and even integer $r \geq 6$, every graph G with average degree at least $rk - 2$ and at least rk vertices contains k disjoint cycles, each containing at least r vertices.*

Conjecture 5.3 or 5.4 would imply Conjecture 5.5 or 5.6, respectively. The fact that considering the average degree rather than the minimum degree eliminates a class of possible counterexamples is promising and suggests that results of this kind may be simpler and cleaner.

Finally, we present the following extension of Conjectures 5.5 and 5.6 to 2-regular t -vertex graphs. This is also a strengthening of Conjecture 5.1.

CONJECTURE 5.7. *Let H be 2-regular t -vertex graph with c odd order components such that H is not a t -vertex even cycle or $\frac{t}{4}$ cycles of order 4. Every graph with average degree at least $t + c - 2$ and at least t vertices contains H as a minor.*

We require that H not be a single even cycle since the graph consisting of $n \geq 2$ disjoint copies of K_{t-1} has average degree $t - 2$ and at least t vertices but contains no t -vertex cycle. With regard to the other exception in Conjecture 5.7, consider the graph $\mathcal{S}'_{k,r,n}$ constructed from $\mathcal{S}_{k,r,n}$ by adding a perfect matching on the independent set of order n . The graph $\mathcal{S}'_{k,r,n}$ does not contain k disjoint cycles of order at least four since any such cycle still requires two vertices of the clique. However, the average degree of \mathcal{S}' tends toward $4k - 1 = t - 1 > t - 2$ as $n \rightarrow \infty$. The graph $\mathcal{S}'_{k,r,n}$ is one of the three exceptions presented in [33]; again, promisingly, the other two exceptions presented there are not exceptions when considering the average degree. Hence Conjecture 5.7 seems plausible, if strong.

Note added in proof. Conjecture 5.5 was recently proved by Cs6ka et al. [7].

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