Fault-Tolerant Metric Dimension of Graphs*

Carmen Hernando [a], Merce Mora [b], Peter J. Slater [c], David R. Wood [b]

[a] Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya, Barcelona. Spain
carmen.hernando@upc.edu

[b] Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya, Barcelona. Spain
merce.mora@upc.edu, david.wood@upc.edu

[c] Mathematical Sciences Dept. and Computer Sciences Dept.
University of Alabama in Huntsville. Huntsville. AL, USA 35899
slater@math.uah.edu, pslater@cs.uah.edu

Abstract

An ordered set S of vertices in a graph G is said to *resolve* G if every vertex in G is uniquely determined by its vector of distances to the vertices in S. The *metric dimension* of G is the minimum cardinality of a resolving set of G. In this paper we introduce the study of the fault-tolerant metric dimension of a graph. A resolving set S for G is *fault-tolerant* if $S \setminus \{v\}$ is also a resolving set, for each v in S, and the *fault-tolerant metric dimension* of G is the minimum cardinality of such a set. In this paper we characterize the fault-tolerant resolving sets in a tree T. We show that the fault-tolerant metric dimension values are bounded by a function of the metric dimension values independent of any graphs.

1 Introduction

For a graph G with vertex set $V(G)$ and edge set $E(G)$, the distance between two vertices u and v in $V(G)$ is the minimum number of edges in a $u-v$ path and is denoted by $d_G(u,v)$ or simply $d(u,v)$ if the graph G is clear. A vertex x *resolves* two vertices u and v if $d(x,u) \neq d(x,v)$. A vertex set $S \subseteq V(G)$ is said to be *resolving* for G if for every two distinct vertices u and v in $V(G)$ there is a vertex x in S that resolves u and v. The minimum cardinality of a resolving set of G is called the *metric dimension* of G and is denoted by $\beta(G)$. A resolving set of order $\beta(G)$ is called a *metric basis* of G.

*Research supported by the projects MCYT-FEDER-BFM2003-00368, Gen-Cat-2001SGR00224, MCYT-HU2002-0010, MEC-SB2003-0270, MCYT-FEDER BFM2003-00368. The research of David Wood is supported by a Marie Curie Fellowship of the European Community under contract 023865.

Equivalently, for an ordered subset $S = (v_1, v_2, \ldots, v_k)$ of vertices in $V(G)$ the S-coordinates of a vertex x in $V(G)$ are $f_S(x) = (d(x, v_1), d(x, v_2), \ldots, d(x, v_k))$. Then S is a resolving set if for every two vertices x and y in $V(G)$ we have $f_S(x) \neq f_S(y)$.

These concepts were introduced for general graphs independently by Slater [6] and by Harary and Melter [4]. Resolving sets have since been widely investigated. (See the bibliographies of [1] and [3].)

As described in Slater [6], each v_i in S can be thought of as the site for a sonar or loran station, and each vertex location must be uniquely determined by its distances to the sites in S. In this paper we consider (single) fault-tolerant resolving sets S for which the failure of any single station at vertex location v in S leaves us with a set that still is a resolving set.

A resolving set S for a graph G is *fault-tolerant* if $S \setminus \{v\}$ is also resolving for each v in S. The *fault-tolerant metric dimension* of G is the minimum cardinality of a fault-tolerant resolving set, and it will be denoted by $\beta'(G)$. A fault-tolerant resolving set of order $\beta'(G)$ is called a *fault-tolerant metric basis*.

We consider fault-tolerant resolving sets for trees in section 2. In section 3 we show that the fault-tolerant metric dimension values are bounded by a function of the metric dimension values independent of any given graph.

2 Fault-tolerant resolving sets for trees

Consider the tree T' in Figure 1. It will be seen below that $\beta(T') = 10$ and that $S = (1, 2, \ldots, 10)$ is a metric basis. We have, for example, that

$$\begin{aligned} f_S(x) &= (11,\, 11,\, 11,\, 11,\, 10,\, 10,\, 10,\, 10,\, 1,\, 4)\,, \\ f_S(y) &= (11,\, 11,\, 11,\, 11,\, 10,\, 10,\, 10,\, 10,\, 3,\, 4)\,, \\ f_S(v) &= (8,\, 8,\, 8,\, 8,\, 3,\, 3,\, 7,\, 7,\, 8,\, 9)\,, \\ f_S(t) &= (8,\, 8,\, 8,\, 8,\, 1,\, 3,\, 7,\, 7,\, 8,\, 9)\,. \end{aligned}$$

Note that $f_S(x)$ and $f_S(y)$ agree everywhere except for the ninth component, that is, the only vertex in S that resolves vertices x and y is vertex 9. Likewise, only vertex 5 resolves vertices v and t. We will see that $\beta'(T') = 14$ and that $S \cup \{y, v, r, s\}$ is one example of a fault-tolerant metric basis of T'.

It is easy to see that for the path P_n on $n \geq 2$ vertices we have $\beta(P_n) = 1$ and $\beta'(P_n) = 2$. Observe that the unique fault-tolerant metric basis of P_n is formed by the two endpoints.

Henceforth we will only be concerned with trees T that have maximum degree $\Delta(T) \geq 3$. The degree of vertex v is denoted by $\deg(v)$.

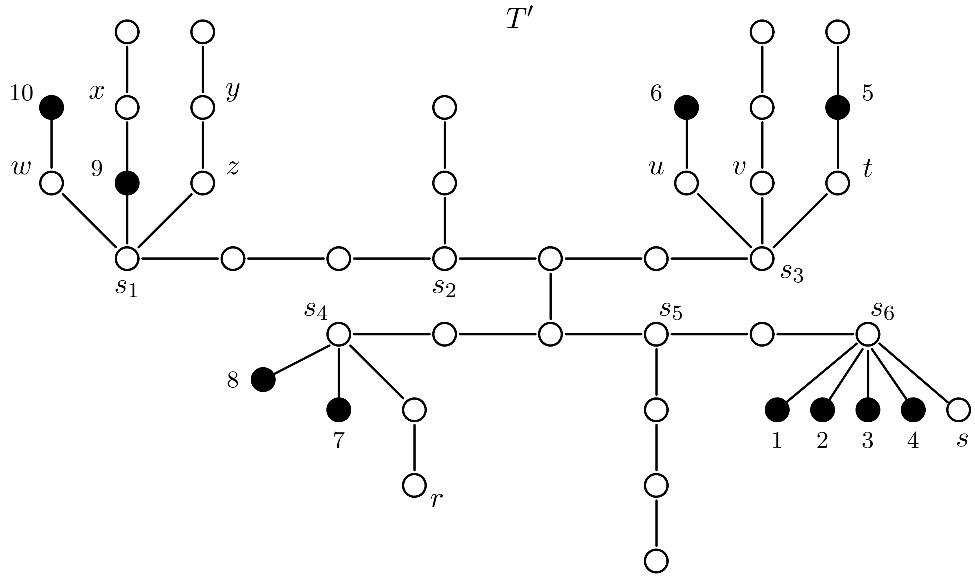

Figure 1: Tree T' with $\beta(T')$=10 and $\beta'(T') = 14$

A *branch* of a tree T at a vertex v is the subgraph induced by v and one of the components of $T \setminus \{v\}$. Note that each $v \in V(T)$ has $\deg(v)$ branches. A branch B of T at v which is a path will be called a *branch path* when $\deg(v) \geq 3$. Tree T' (see Figure 1) has sixteen branch paths. The vertex v in a branch path with $\deg(v) \geq 3$ will be called a *stem* of the branch path. Tree T' has six stems: $s_1, \ldots, s_6$. We let $L_1, \ldots, L_k$ be the components of the subtree induced by the set of all branch paths. Thus k is the number of stems, and each L_i can be obtained by subdividing edges starting with a star. For T' we have $k = 6$ and subdivisions: one $K_{1,5}$, three $K_{1,3}$ and two $K_{1,1}$.

Theorem 2.1 *([6]) Let T be a tree of order $n \geq 3$. Vertex set S is a resolving set if and only if for each vertex u there are vertices from S on at least* $\deg(u) - 1$ *of the* $\deg(u)$ *components of $T \setminus \{u\}$.*

Theorem 2.2 *([6]) Let T be a tree with set L of endpoints with $|L| \geq 3$. Let $L_1, \ldots, L_k$ be the components of the subtree induced by the set of all branch paths, and let e_i be the number of branch paths in T that are in L_i. Then $\beta(T) = |L| - k$, and S is a metric basis if and only if it consists of exactly one vertex from each of exactly $e_i - 1$ of the branch paths of L_i, for each L_i, $1 \leq i \leq k$.*

Assume that S is a fault-tolerant resolving set for tree T. For each v in S, the set $S \setminus \{v\}$ is a resolving set. By Theorem 2.1, for any $v \in S \cap L_i$, the set $S \setminus \{v\}$ must contain a vertex from each of $e_i - 1$ of the branch paths of L_i. Thus, $|S \cap Li| \geq e_i$ when $e_i \geq 2$. If L is the set of endpoints and E_1 is the set of endpoints to branch paths where $e_i = 1$, then, by Theorem 2.1 , $L \setminus E_1$ is a fault-tolerant resolving set. This implies the next theorem.

Theorem 2.3 *Let T be a tree with set L of endpoints with $|L| \geq 3$. Let $L_1, \ldots, L_k$ be the components of the subtree induced by the set of all branch paths, and let e_i be the number of branch paths in T that are in L_i. Let E_1 be the set of endpoints corresponding to branch paths where $e_i = 1$. Then $\beta'(T) = |L \setminus E_1|$ and $L \setminus E_1$ is a fault-tolerant metric basis.*

Remark 2.4 *If S is a fault-tolerant metric basis for T and s_i is the stem vertex of L_i, then $e_i \geq 3$ implies that $S \cap L_i$ will consist of exactly one vertex from each path of $L_i \setminus \{s_i\}$, and, if $e_i = 2$, then $S \cap L_i$ can be any two vertices of $L_i \setminus \{s_i\}$.*

In particular, for the tree T' of Figure 1, $\beta'(T') = 14$ and (1, 2, 3, …, 10, y, v, r, s) is one example of a fault-tolerant metric basis.

3 Relation between $\beta'(G)$ and $\beta(G)$.

In this section we prove that fault-tolerant metric dimension is bounded by a function of metric dimension (independent of the graph). As usual, $N(v)$ and $N[v]$ denote vertex v's open and closed neighborhoods, respectively.

Lemma 3.1 *Let S be a resolving set of G. For each vertex $v \in S$, let $T(v) := \{x \in V(G) : N(v) \subseteq N(x)\}$. Then $S' := \cup_{v \in S}(N[v] \cup T(v))$ is a fault-tolerant resolving set of G.*

Proof. Consider a vertex $v \in S'$. If $v \notin S$ then $S' \setminus \{v\}$ resolves G since $S \subseteq S' \setminus \{v\}$. Now assume that $v \in S$.

Let p and q be distinct vertices of G. We must show that p and q are resolved by some vertex in $S' \setminus \{v\}$. If not, then v must resolve p and q since S resolves p and q. Without loss of generality $d(v,p) \leq d(v,q) - 1$.

First suppose that $p \neq v$. Let w be the neighbour of v on a shortest path between v and p. Then $w \in S' \setminus \{v\}$ and $d(v,p) = d(w,p) + 1$. Thus $d(w,p) + 1 \leq d(v,q) - 1$. Now $d(v,q) - 1 \leq d(w,q)$. Hence $d(w,p) + 1 \leq d(w,q)$. Thus w resolves p and q.

Now assume that $p = v$. If $q \in S'$ then $q \in S' \setminus \{v\}$ and q resolves p and q. Otherwise $d(v,q) \geq 2$ and q is not adjacent to some neighbour w of v. Then $d(v,w) = 1$ and $d(q,w) \geq 2$. Thus $w \in S' \setminus \{v\}$ resolves v $(= p)$ and q.

Hence S' is a fault-tolerant resolving set of G. □

The following lemma is implicit in [2, 5]. We include the proof for completeness.

Lemma 3.2 *Let S be a resolving set in a graph G. Then for each vertex $v \in S$, the number of vertices of G at distance at most k from v is at most $1 + k(2k+1)^{|S|-1}$.*

Proof. Say $1 \leq d(v,w) \leq k$. For every vertex $u \in S$ with $u \neq v$, we have $|d(w,u) - d(w,v)| \leq k$. Thus there are $2k+1$ possible values for $d(w,u)$, and there are at most k possible values for $d(w,v)$. Thus the vector of distances from w to S has $k(2k+1)^{|S|-1}$ possible values. The result follows, since the vertices at distance at most k are resolved by S. □

Theorem 3.3 *Fault-tolerant metric dimension is bounded by a function of the metric dimension (independent of the graph). In particular, $\beta'(G) \leq \beta(G)(1 + 2 \cdot 5^{\beta(G)-1})$ for every graph G.*

Proof. Let S be a metric basis for a graph G. Lemma 3.2 with $k = 2$ implies that $|N[v] \cup T(v)| \leq 1 + 2 \cdot 5^{\beta(G)-1}$ for each vertex $v \in S$. Thus $|S'| \leq \beta(G)(1 + 2 \cdot 5^{\beta(G)-1})$. □

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