International Journal of Computational Geometry
& Applications
Vol. 23, No. 3 (2013) 213–230

DOI: 10.1142/S0218195913500088

PROXIMITY DRAWINGS OF HIGH-DEGREE TREES

FERRAN HURTADO*
Departament de Matemàtica Aplicada II
Universitat Politècnica de Catalunya
Barcelona, Spain
ferran.hurtado@upc.edu

GIUSEPPE LIOTTA†
Dipartimento di Ingegneria Elettronica e dell'Informazione
Università di Perugia
Perugia, Italy
liotta@diei.unipg.it

DAVID R. WOOD‡
School of Mathematical Sciences
Monash University, Melbourne, Australia
david.wood@monash.edu

Received 16 November 2011
Revised 2 October 2013
Communicated by Godfried Toussaint

ABSTRACT

A drawing of a given (abstract) tree that is a minimum spanning tree of the vertex set is considered aesthetically pleasing. However, such a drawing can only exist if the tree has maximum degree at most 6. What can be said for trees of higher degree? We approach this question by supposing that a partition or covering of the tree by subtrees of bounded degree is given. Then we show that if the partition or covering satisfies some natural properties, then there is a drawing of the entire tree such that each of the given subtrees is drawn as a minimum spanning tree of its vertex set.

Keywords: Graph drawing; tree; proximity graph; minimum spanning tree; relative neighbourhood graph.

*Research partially supported by projects MTM2012-30951, Gen. Cat. DGR 2009SGR1040, and ESF EUROCORES programme EuroGIGA, CRP ComPoSe: MICINN Project EUI-EURC-2011-4306.
†Research partially supported by MIUR.
‡Research supported by the Australian Research Council. Research initiated at Universitat Politècnica de Catalunya, where supported by the Marie Curie Fellowship MEIF-CT-2006-023865, and by the projects MEC MTM2006-01267 and DURSI 2005SGR00692.

1. Introduction

The field of graph drawing studies aesthetically pleasing drawings of graphs.[a] There are a number of recognised criteria for measuring the quality of a drawing of a given graph. These include:

- no two edges should cross in drawings of planar graphs;
- the edges should be drawn as straight line-segments; and
- the drawing should have large *angular resolution* (defined to be the minimum angle determined by two consecutive edges incident to a vertex).

These three criteria are adopted in the present paper. More formally, a (*straight-line general position*) *drawing* of graph G is an injective function $\phi : V(G) \to \mathbb{R}^2$ such that the points $\phi(u), \phi(v), \phi(w)$ are not collinear for all distinct vertices $u, v, w \in V(G)$. The *image* of an edge $vw \in E(G)$ under ϕ is the line segment $\overline{\phi(v)\phi(w)}$. Where no confusion is caused, we henceforth do not distinguish between a graph element and its image in a drawing. Two edges *cross* if they intersect at a point other than a common endpoint.

Our focus is on drawings of trees. Here a number of other criteria have been studied that will not be considered in this paper. These include: small bounding box area [7–9, 11, 24, 27, 33], small aspect ratio [8, 24], few bends in the edges [28], few distinct edge-slopes [14], few distinct edge-lengths [6], layered vertices [34], upwardness in rooted trees [7, 11, 28, 36], and maximising symmetry [25].

A *minimum spanning tree* of a finite set $P \subset \mathbb{R}^2$, denoted by $\mathsf{MST}(P)$, is a straight-line drawing of a tree with vertex set P and with minimum total edge length; see Fig. 1 for an example. A drawing of a given (abstract) tree that is a minimum spanning tree of its vertex set is considered to be particularly aesthetically pleasing. In particular, every minimum spanning tree is crossing-free and has angular resolution at least $\frac{\pi}{3}$. Drawings defined in this way are called 'proximity drawings'; see Sec. 2 and [1, 2, 4, 12, 30–32] for more on proximity drawings.

Monma and Suri [31] proved that every degree-5 tree can be drawn as a minimum spanning tree of its vertex set, and they provided a linear time (real RAM) algorithm to compute the drawing. In any drawing of a vertex v with degree at least 7, some angle at v is greater than $\frac{\pi}{3}$, and the same is true for a degree-6 vertex if the points are required to be in general position. Thus a tree that contains a vertex with degree at least 7 cannot be drawn as a minimum spanning tree, and the same is true for a degree-6 vertex if the points are in general position. If collinear vertices are allowed, then Eades and Whitesides [18] showed that it is NP-hard to decide whether a given

[a]We consider graphs G that are simple and finite. Let G be an (undirected) graph. The *degree* of a vertex v of G, denoted by $\deg_G(v)$, is the number of edges of G incident with v. The maximum degree of G is denoted by $\Delta(G)$. We say G is *degree-d* if $\Delta(G) \leq d$. Now let G be a directed graph. Let v be a vertex of G. The *indegree* of v, denoted by $\mathrm{indeg}_G(v)$, is the number of incoming edges incident to v. The *outdegree* of v, denoted by $\mathrm{outdeg}_G(v)$, is the number of outgoing edges incident to v. The maximum outdegree of G is denoted by $\Delta^+(G)$. We say G is *outdegree-d* if $\Delta^+(G) \leq d$.

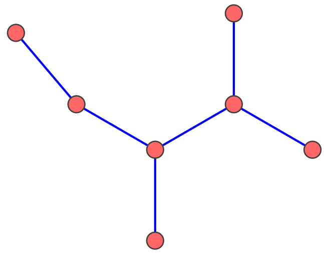

Fig. 1. Example of a minimum spanning tree.

degree-6 tree can be drawn as a minimum spanning tree. In this sense, the problem of testing whether a tree can be drawn as a minimum spanning tree is essentially solved. (In related work, Liotta and Meijer [29] characterised those trees that have drawings that are Voronoi diagrams of their vertex set.)

What can be said about drawings of a high degree tree T that 'approximate' the minimum spanning tree of the vertex set? We prove the following solutions to this question based on partitions of T into subtrees of bounded degree. A *partition* of a graph G is a set of subgraphs of G such that every edge of G is in exactly one subgraph. We emphasise that 'trees' and 'subtrees' are necessarily connected.

Theorem 1. *Let $\mathcal{P}$ be a partition of a tree T into degree-5 subtrees. Then there is a drawing of T such that each subtree in $\mathcal{P}$ is drawn as the minimum spanning tree of its vertex set.*

The drawing of T produced by Theorem 1 possibly has crossings, which are undesirable. The next result eliminates the crossings, at the expense of a slightly stronger assumption about the partition, which is expressed in terms of rooted trees. A *rooted tree* is a directed tree such that exactly one vertex, called the *root*, has indegree-0. It follows that every vertex except r has indegree-1, and every edge vw of T is oriented 'away' from r; that is, if v is closer to r than w, then vw is directed from v to w. If r is a vertex of a tree T, then the pair (T, r) denotes the rooted tree obtained by orienting every edge of T away from r.

Theorem 2. *Let $\mathcal{P}$ be a partition of a rooted tree T into outdegree-4 subtrees. Then there is a non-crossing drawing of T such that each subtree in $\mathcal{P}$ is drawn as the minimum spanning tree of its vertex set.*

By further restricting the partition we introduce large angular resolution as an additional property of the drawing, again at the expense of a slightly stronger assumption about the partition.

Theorem 3. *Let $\mathcal{P}$ be a partition of a rooted tree T into outdegree-3 subtrees. Then there is a non-crossing drawing of T with angular resolution at least*

$$\frac{\pi}{\max\{\Delta^+(T)-1,4\}}$$

such that each subtree in $\mathcal{P}$ is drawn as the minimum spanning tree of its vertex set.

Since every drawing of T has angular resolution at most $\frac{2\pi}{\Delta(T)}$, the bound on the angular resolution in Theorem 3 is within a constant factor of optimal.

Our final drawing theorem concerns a given covering of a tree by two bounded degree subtrees. A *covering* of a graph G is a set of connected subgraphs of G such that every edge of G is in at least one subgraph.

Theorem 4. *Let $\{T_1, T_2\}$ be a covering of a tree T by two degree-5 subtrees. Then there is a non-crossing drawing of T such that each T_i is drawn as a minimum spanning tree of its vertex set.*

A number of notes about Theorems 1–4 are in order:

- Each of Theorems 1, 2 and 4 imply and generalise the above-mentioned result by Monma and Suri [31] that every degree-5 tree T can be drawn as a minimum spanning tree of its vertex set. (Take $k = 1$ in Theorem 1; root T at a leaf in Theorem 2; and take $T_1 = T$ and $T_2 = \emptyset$ in Theorem 4.)
- For each of Theorems 1–4, we actually prove a stronger result in terms of the relative neighbourhood graph, which is introduced in Sec. 2. The idea, first introduced by Bose *et al.* [4], is to construct point sets for which the relative neighbourhood graph is a tree, in which case it is a minimum spanning tree.
- Theorem 4 cannot be generalised for coverings by three or more subtrees, as proved in Sec. 5.
- The above theorems are loosely related to the notion of geometric thickness. The *geometric thickness* of a graph G is the minimum integer k such that there is a straight-line drawing of G and an edge k-colouring such that monochromatic edges do not cross; see [3, 13, 15–17, 19, 20, 26]. Thus in the drawing of G, the subgraph induced by each colour class is crossing-free. The above theorems also produce drawings in which the edges are partitioned into non-crossing subgraphs, but with additional proximity properties. Moreover, each subgraph of the partition is connected, which is a desirable property in visualisation applications.
- All our proofs are constructive, and lead to polynomial time algorithms (in the real RAM model). These algorithmic details are omitted.

2. Relative Neighbourhood Graphs

To aid in the proofs of Theorems 1–4, we now introduce some notation and a number of geometric objects. Let x and y be points in the plane. Let $|xy|$ be the Euclidean distance between x and y. Let $\mathsf{circle}(x, \delta)$ be the circle of radius δ centred at x. Let

$\mathsf{disc}(x, \delta)$ be the open disc of radius δ centred at x. Let $\overline{\mathsf{disc}}(x, \delta)$ be the closed disc of radius δ centred at x. As illustrated in Fig. 2, for every real number δ such that $0 < \delta < |xy|$, let

$$\mathsf{lune}(x, y, \delta) := (\mathsf{disc}(y, \delta) - \overline{\mathsf{disc}}(x, |xy|)) \cup \{y\} \ .$$

The *relative neighbourhood lens*[b] of x and y is

$$\mathsf{lens}(x, y) := \mathsf{disc}(x, |xy|) \cap \mathsf{disc}(y, |xy|) \ .$$

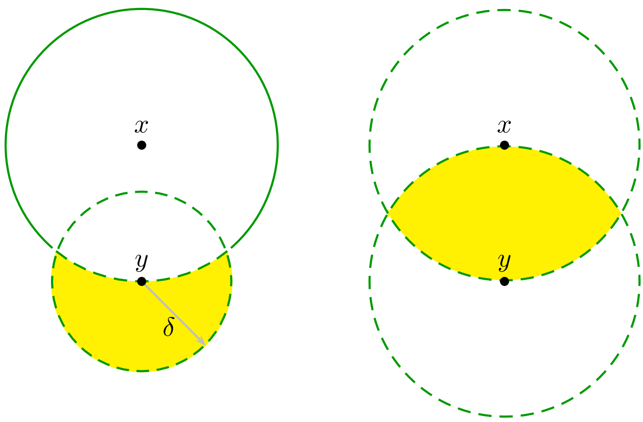

Fig. 2. The regions $\mathsf{lune}(x, y, \delta)$ and $\mathsf{lens}(x, y)$.

Let $P \subset \mathbb{R}^2$ be a finite set of points in the plane. Toussaint [35] defined the *relative neighbourhood graph* of P, denoted by $\mathsf{RNG}(P)$, to be the graph with vertex set P, where two vertices $v, w \in P$ are adjacent if and only if $\mathsf{lens}(x, y) \cap P = \emptyset$. That is v and w are adjacent whenever no vertex is simultaneously closer to v than w and closer to w than v. Toussaint [35] proved that $\mathsf{MST}(P) \subseteq \mathsf{RNG}(P)$. Hence if $\mathsf{RNG}(P)$ is a tree, then $\mathsf{RNG}(P) = \mathsf{MST}(P)$. The result of Monma and Suri [31] mentioned in Sec. 1 was strengthened by Bose *et al.* [4] as follows.

Lemma 1 (Bose *et al.* [4]). *Every degree-5 tree has a drawing that is the relative neighbourhood graph of its vertex set.*

Analogous to Lemma 1, for all of the theorems introduced in Sec. 1, we in fact prove stronger results about relative neighbourhood graphs.

[b]Unfortunately the computational geometry literature, and especially the literature on relative neighbourhood graphs, often refers incorrectly to a 'lens' as a 'lune'.

3. Drawings Based on a Partition

Theorem 1 is implied by the following result, since a relative neighbourhood graph that is a tree is a minimum spanning tree.

Theorem 5. *Let $\{T_1, \ldots, T_k\}$ be a partition of a tree T into degree-5 subtrees. Then there is a drawing of T in which each T_i is drawn as the relative neighbourhood graph of its vertex set.*

Proof. Let D be the maximum distance between any two vertices in T (the *diameter* of T). Let Q be the complete 5-ary tree of height D. That is, every non-leaf vertex in Q has degree-5, and for some vertex r, the distance between r and every leaf equals D.

By Lemma 1, there is a drawing of Q that is the relative neighbourhood graph of its vertex set. Since the vertices of Q are in general position, for some $\varepsilon > 0$, for all distinct vertices $x, y \in V(Q)$, the discs $\mathsf{disc}(x, \varepsilon)$ and $\mathsf{disc}(y, \varepsilon)$ are disjoint, and if P is a point set that contains exactly one point from each disc $\mathsf{disc}(x, \varepsilon)$ (where $x \in V(Q)$), then $Q \cong \mathsf{RNG}(P)$. (Here $\mathsf{disc}(x, \varepsilon)$ means the disc centred at the point where x is drawn.)

Define a homomorphism[c] f from T to Q as follows. Choose an arbitrary starting vertex v of T, let $f(v) = r$, and recursively construct a function f such that $f(v)f(w)$ is an edge of Q for every edge vw of T, and if $f(v)f(w) = f(v')f(w')$ for distinct edges $vw \in E(T_i)$ and $v'w' \in E(T_j)$, then $i \neq j$. That is, edges in the same subtree are mapped to distinct edges of Q. Hence for each subtree T_i of T, no two vertices in T_i are mapped to the same vertex in Q (otherwise the image of the path in T_i between the two vertices would form a cycle in Q). Moreover, if Q_i is the subgraph of Q induced by $\{f(v) : v \in V(T_i)\}$ then $Q_i \cong T_i$. Draw each vertex $v \in V(T)$ at a distinct point $\phi(v) \in \mathsf{disc}(f(v), \varepsilon)$ so that $\{\phi(v) : v \in V(T)\}$ is in general position. Thus $P_i := \{\phi(v) : v \in V(T_i)\}$ contains exactly one point from each disc $\mathsf{disc}(x, \varepsilon)$ where $x \in V(Q_i)$. Hence $T_i \cong Q_i \cong \mathsf{RNG}(P_i)$ as desired. $\square$

Theorem 2 is implied by the following stronger result.

Theorem 6. *Let $\{T_1, \ldots, T_k\}$ be a partition of a rooted tree T into outdegree-4 subtrees. Then there is a non-crossing drawing of T such that each T_i is drawn as the relative neighbourhood graph of its vertex set.*

Theorem 6 is proved by induction with the following hypothesis. This proof method generalises that of Bose *et al.* [4].

Lemma 2. *Let $\{T_1, \ldots, T_k\}$ be a partition of a rooted tree T into outdegree-4 subtrees. Let r be the root of T. Let p and q be distinct points in the plane. Let δ be a*

[c]A *homomorphism* from a graph G to a graph H is a function $f : V(G) \to V(H)$ such that if $vw \in E(G)$ then $f(v)f(w) \in E(H)$.

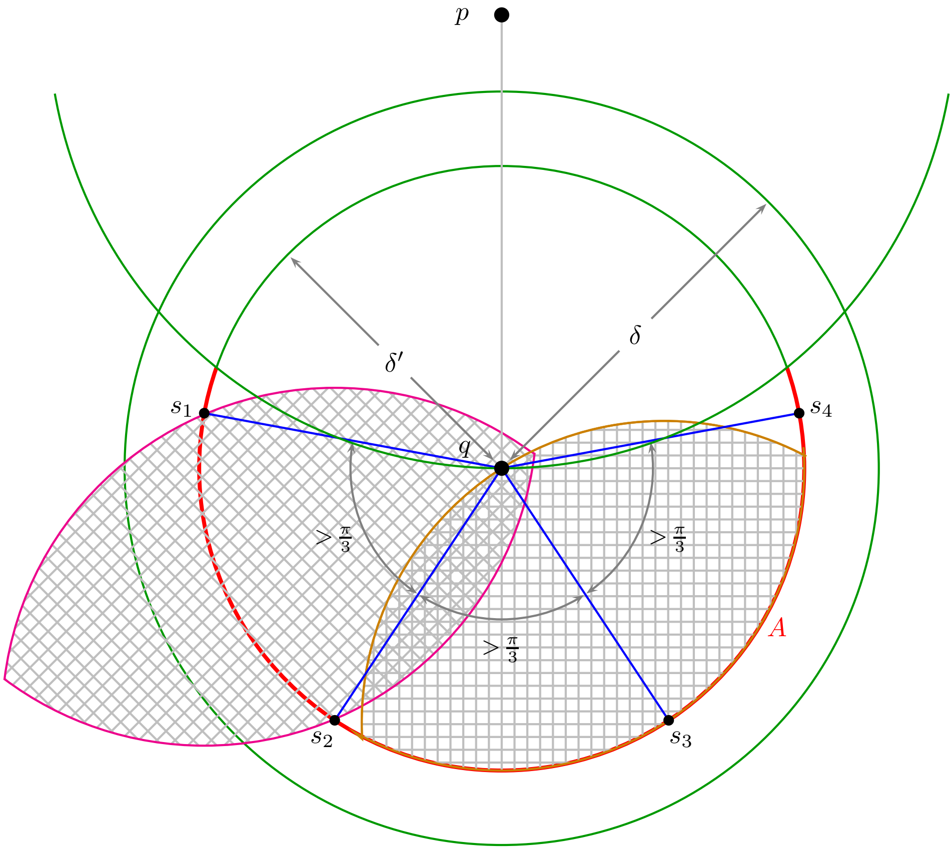

Fig. 3. The points s_1, s_2, s_3, s_4, showing that $q \in \mathsf{lens}(s_1, s_2)$ and $\mathsf{lens}(q, s_3) \cap \{s_1, s_2, s_3, s_4\} = \emptyset$.

real number with $0 < \delta < |pq|$*. Then there is a non-crossing drawing of* T *contained in* $\mathsf{lune}(p, q, \delta)$ *such that:*

- r*, which is drawn at* q*, is in* $\mathsf{lens}(x, p)$ *for every vertex* x *of* $T - r$*, and*
- *for all* $i \in \{1, \ldots, k\}$*, the subtree* T_i *is drawn as the relative neighbourhood graph of its vertex set.*

Proof. We proceed by induction on $|V(T)|$. The result is trivial if $|V(T)| = 1$. Now assume that $|V(T)| \geq 2$. Let δ' be a real number with $0 < \delta' < \delta$. The circular arc $A := \mathsf{circle}(q, \delta') - \mathsf{disc}(p, |pq|)$ has an angle (measured from q) greater than π. Thus, as illustrated in Fig. 3, there are four points s_1, s_2, s_3, s_4 in the interior of A, such that the angle (measured from q) between distinct points s_i and s_j is greater than $\frac{\pi}{3}$, implying $|s_i q| = |s_j q| < |s_i s_j|$ and $q \in \mathsf{lens}(s_i, s_j)$, and $\mathsf{lens}(q, s_i) \cap \{s_1, s_2, s_3, s_4\} = \emptyset$.

For small enough discs around the s_i, these properties are extended to every point in the disc. More precisely, there is a real number $\varepsilon \in (0, \delta')$ such that:

(a) $\mathsf{disc}(s_i, \varepsilon) \subset \mathsf{lune}(p, q, \delta)$ for all $i \in \{1, 2, 3, 4\}$;
(b) $q \in \mathsf{lens}(x, y)$ for all points $x \in \mathsf{disc}(s_i, \varepsilon)$ and $y \in \mathsf{disc}(s_j, \varepsilon)$ for all distinct $i, j \in \{1, 2, 3, 4\}$;
(c) $q \notin \mathsf{lens}(x, y)$ for all points $x, y \in \mathsf{disc}(s_i, \varepsilon)$ for all $i \in \{1, 2, 3, 4\}$; and
(d) $\mathsf{lens}(x, y) \cap \mathsf{disc}(s_j, \varepsilon) = \emptyset$ for all points $x, y \in \mathsf{disc}(s_i, \varepsilon)$ and for all distinct $i, j \in \{1, 2, 3, 4\}$.

For $j \in \{1, 2, 3, 4\}$, since $\mathsf{disc}(s_j, \varepsilon)$ has diameter 2ε, there are points $t_{j,1}, \ldots, t_{j,k}$ on the arc $A \cap \mathsf{disc}(s_j, \varepsilon)$ such that discs of radius $\frac{\varepsilon}{k}$ centred at $t_{j,1}, \ldots, t_{j,k}$ are pairwise disjoint, as illustrated in Fig. 4.

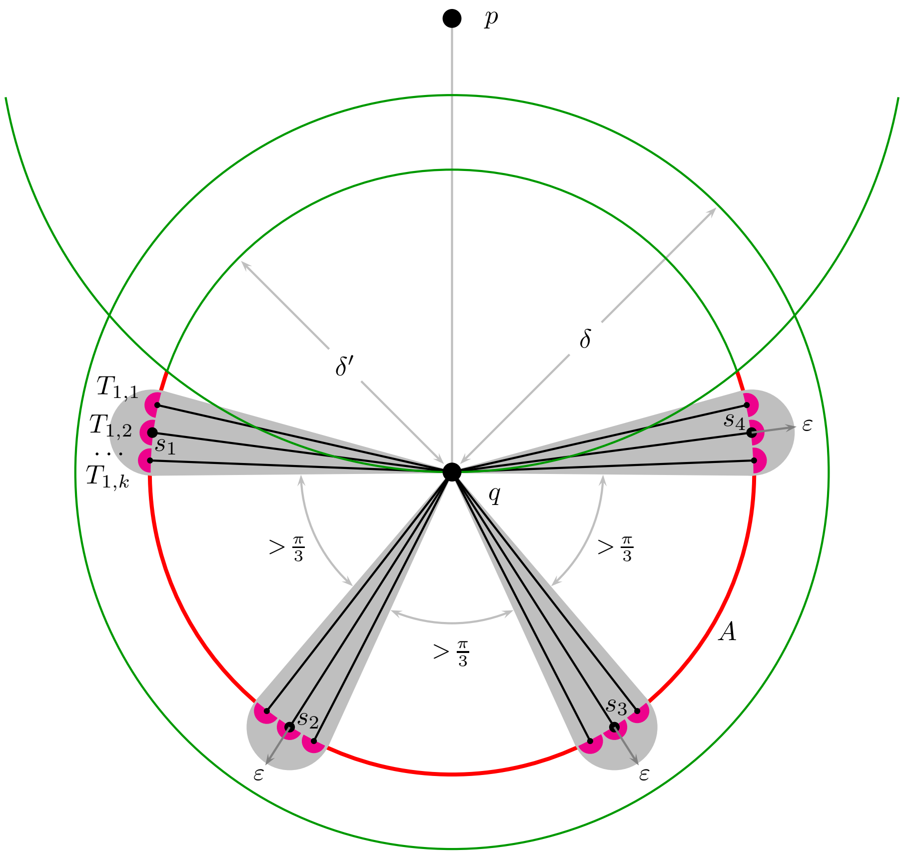

Fig. 4. Construction in the proof of Lemma 2.

For $i \in \{1, \ldots, k\}$, let d_i be the outdegree of r in T_i. So $d_i \in \{0, 1, 2, 3, 4\}$. Let $v_{i,1}, \ldots, v_{i,d_i}$ be the neighbours of r in T_i. For $j \in \{1, \ldots, d_i\}$, let $T_{i,j}$ be the component of $T - r$ that contains $v_{i,j}$. So $T_{i,j}$ is rooted at $v_{i,j}$, and $\{T_1 \cap T_{i,j}, \ldots, T_k \cap T_{i,j}\}$ is a partition of $T_{i,j}$ into outdegree-4 subtrees. By induction, there is a non-crossing drawing of each $T_{i,j}$ contained in $\mathsf{lune}(q, t_{j,i}, \frac{\varepsilon}{k})$ such that:

(e) $v_{i,j}$, which is drawn at $t_{j,i}$, is in $\mathsf{lens}(x, q)$ for every vertex x of $T_{i,j} - v_{i,j}$, and
(f) for all $\ell \in \{1, \ldots, k\}$, the subtree $T_\ell \cap T_{i,j}$ is drawn as the relative neighbourhood graph of its vertex set.

Draw r at q, and draw a straight-line edge from r to each neighbour $v_{i,j}$ of r. Each subtree $T_{i,j}$ is drawn outside of $\mathsf{disc}(q, \delta')$, while the edges incident to r are contained within $\mathsf{disc}(q, \delta')$, and therefore do not cross any other edge. Hence the drawing of T is non-crossing. By (a), $T_{i,j}$ is drawn within $\mathsf{lune}(q, t_{j,i}, \frac{\varepsilon}{k}) \subset \mathsf{disc}(t_j, \varepsilon) \subset \mathsf{lune}(p, q, \delta)$. The edges incident to r are drawn within $\mathsf{lune}(p, q, \delta)$. Hence all of T is drawn within $\mathsf{lune}(p, q, \delta)$.

Now consider a vertex x of $T - r$. Then x is in $T_{i,j}$ for some $i \in \{1, 2 \ldots, k\}$ and $j \in \{1, \ldots, d_i\}$. Thus x is drawn in $\mathsf{disc}(q, \delta) - \mathsf{disc}(p, |pq|)$, implying $|xq| < \delta < |xp|$ and $|pq| < |px|$. Hence $q \in \mathsf{lens}(x, p)$, implying $r \in \mathsf{lens}(x, p)$. This proves the first claim of the induction hypothesis.

It remains to prove that each subtree T_i is drawn as the relative neighbourhood graph of its vertex set. Consider distinct vertices v and w in T_i. We must show that $\mathsf{lens}(v, w) \cap V(T_i) = \emptyset$ if and only if $vw \in E(T_i)$. Without loss of generality, $w \neq r$.

Case 1. $v = r$ and $vw \in E(T_i)$: So $w = v_{i,j}$ for some $j \in \{1, 2, 3, 4\}$. Then v is drawn at q, and w is drawn at $t_{j,i}$. Now $\mathsf{lens}(q, t_{j,i}) \subset \mathsf{disc}(q, \delta')$, which contains no vertex except r (at q). Thus $\mathsf{lens}(v, w) \cap V(T) = \emptyset$, as desired.

Case 2. $v = r$ and $vw \notin E(T_i)$: Then w is in $T_{i,j}$ for some $j \in \{1, 2, 3, 4\}$. Since v is drawn at q, by induction, the vertex $t_{j,i}$, which is in T_i, is in $\mathsf{lens}(v, w)$, as desired.

Now assume that $v \neq r$ and $w \neq r$.

Case 3. v and w are in the same component $T_{\ell,j}$ of $T - r$: Then v and w are drawn within $\mathsf{disc}(t_\ell, \varepsilon)$. Each vertex in T_i is r, is in $T_{\ell,j}$, or is in $T_{i,j'}$ for some $j' \neq j$. Since r is drawn at q, (c) implies that $r \notin \mathsf{lens}(v, w)$. Since $T_{i,j'}$ is drawn within $\mathsf{disc}(t_{j',i}, \varepsilon)$, by (d), $\mathsf{lens}(v, w) \cap V(T_{i,j'}) = \emptyset$. Hence $\mathsf{lens}(v, w) \cap V(T_i) = \emptyset$ if and only if $\mathsf{lens}(v, w) \cap V(T_{\ell,j}) \cap T_i = \emptyset$. By induction, $\mathsf{lens}(v, w) \cap V(T_i) = \emptyset$ if and only if v and w are adjacent in T_i, as desired.

Case 4. v and w are in distinct components of $T - r$: Thus r is in T_i, v is in $T_{i,j}$ and $w \in T_{i,j'}$ for some $j \neq j'$, and v and w are not adjacent. By construction, v is drawn in $\mathsf{disc}(s_j, \varepsilon)$ and w is drawn in $\mathsf{disc}(s_{j'}, \varepsilon)$. Thus (b) implies that $q \in \mathsf{lens}(v, w)$. Thus r, which is drawn at q, is in $\mathsf{lens}(v, w)$, as desired. □

4. Drawings with Large Angular Resolution

Theorem 3 is implied by the following stronger result:

Theorem 7. *Let $\{T_1, \ldots, T_k\}$ be a partition of a rooted tree T into outdegree-3 subtrees. Then there is a non-crossing drawing of T with angular resolution at least*

$$\frac{\pi}{\max\{\Delta^+(T) - 1, 4\}}$$

such that each subtree T_i is drawn as the relative neighbourhood graph of its vertex set.

Theorem 7 is proved by induction with the following hypothesis.

Lemma 3. *Let $\{T_1, \ldots, T_k\}$ be a partition of a rooted tree T into outdegree-3 subtrees. Let r be the root of T. Let p and q be distinct points in the plane. Let δ be a real number with $0 < \delta < |pq|$. Then there is a non-crossing drawing of T contained in $\mathsf{lune}(p, q, \delta)$ such that:*

- *r, which is drawn at q, is in $\mathsf{lens}(x, p)$ for every vertex x of $T - r$, and*
- *for all $i \in \{1, \ldots, k\}$, the subtree T_i is drawn as the relative neighbourhood graph of its vertex set, and*
- *the drawing of T has angular resolution greater than*

$$\frac{\pi}{\max\{4, \Delta^+(T) - 1\}}.$$

Proof. We proceed by induction on $|V(T)|$. The result is trivial if $|V(T)| = 1$. Now assume that $|V(T)| \geq 2$. Let δ' be a real number with $0 < \delta' < \delta$.

Let $d := \text{outdeg}(r)$. For $i \in \{1, \ldots, k\}$, let d_i be the outdegree of r in T_i. So $d_i \in \{0, 1, 2, 3\}$ and $d = \sum_{i=1}^{k} d_i$. Let $v_{i,1}, \ldots, v_{i,d_i}$ be the neighbours of r in T_i. Let

$$X := \{i : d_i = 3\}, \quad Y := \{i : d_i = 2\}, \quad Z := \{i : d_i = 1\} \ .$$

Thus $d = 3|X| + 2|Y| + |Z|$. Partition $Z = Z' \cup Z''$ such that $|Z''| \leq |Z'| \leq |Z''| + 1$.

The circular arc $A := \mathsf{circle}(q, \delta') - \mathsf{disc}(p, |pq|)$ has an angle (measured from q) greater than π. Thus there are points $s_1, \ldots, s_d$ in this order on A such that the angle (measured from q) between distinct points s_a and s_b is greater than $\frac{\pi|b-a|}{d-1}$.

Let $\preceq$ be the total ordering of the neighbours of r such that $\{v_{i,1} : i \in X\} \preceq \{v_{i,1} : i \in Y\} \preceq \{v_{i,1} : i \in Z'\} \preceq \{v_{i,2} : i \in X\} \preceq \{v_{i,2} : i \in Y\} \preceq \{v_{i,1} : i \in Z''\} \preceq \{v_{i,3} : i \in X\}$, where within each set, the vertices are ordered by their i-value. Draw the neighbours of r in the order of $\preceq$ at $s_1, \ldots, s_d$. That is, the first vertex in $\preceq$ is drawn at s_1, the second vertex in $\preceq$ is drawn at s_2, and so on. Let $t_{i,j}$ be the point where $v_{i,j}$ is drawn.

Consider distinct vertices $v_{i,j}$ and $v_{i,\ell}$ in some subtree T_i such that $\ell > j$. Say $t_{i,j} = s_a$ and $t_{i,\ell} = s_b$. Observe that $b - a \geq |X| + |Y| + |Z''| \geq |X| + |Y| + \frac{1}{2}(|Z| - 1) \geq \frac{1}{3}(3|X| + 2|Y| + |Z| - 1) = \frac{d-1}{3}$. Hence the angle (measured from q) between $v_{i,j}$ and $v_{i,\ell}$ is greater than $\frac{\pi(d-1)/3}{d-1} = \frac{\pi}{3}$. This implies that $|t_{i,j}q| = |t_{i,\ell}q| < |t_{i,j}t_{i,\ell}|$. Thus $q \in \mathsf{lens}(t_{i,j}, t_{i,\ell})$ and $t_{i,\ell} \notin \mathsf{lens}(q, t_{i,j})$ and $t_{i,j} \notin \mathsf{lens}(q, t_{i,\ell})$.

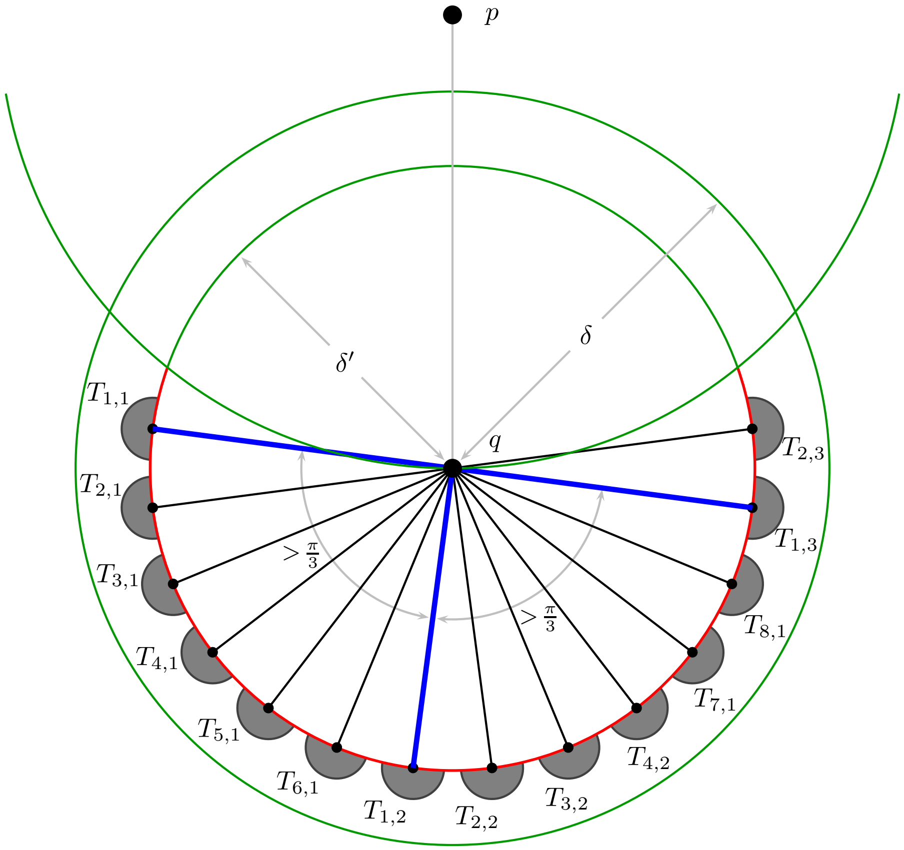

Fig. 5. Construction in the proof of Lemma 3. Here $X = \{1, 2\}$, $Y = \{3, 4\}$, $Z' = \{5, 6\}$ and $Z'' = \{7, 8\}$. The tree T_1 is highlighted.

For small enough discs around $s_1, \ldots, s_d$, these properties are extended to every point in the disc. More precisely, there is a real number $\varepsilon \in (0, \delta')$ such that:

(a) $\mathsf{disc}(s_a, \varepsilon) \subset \mathsf{lune}(p, q, \delta)$ for all $a \in \{1, \ldots, d\}$;
(b) $q \in \mathsf{lens}(x, y)$ for all points $x \in \mathsf{disc}(t_{i,j}, \varepsilon)$ and $y \in \mathsf{disc}(t_{i,\ell}, \varepsilon)$ for all distinct vertices $v_{i,j}$ and $v_{i,\ell}$ in the same subtree T_i;
(c) $q \notin \mathsf{lens}(x, y)$ for all points $x, y \in \mathsf{disc}(s_a, \varepsilon)$ for all $a \in \{1, \ldots, d\}$; and
(d) $\mathsf{lens}(x, y) \cap \mathsf{disc}(s_b, \varepsilon) = \emptyset$ for all distinct $a, b \in \{1, \ldots, d\}$ and for all points $x, y \in \mathsf{disc}(s_a, \varepsilon)$.

For $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, d_i\}$, let $T_{i,j}$ be the component of $T - r$ that contains $v_{i,j}$. Each subtree $T_{i,j}$ is rooted at $v_{i,j}$, and $\{T_1 \cap T_{i,j}, \ldots, T_k \cap T_{i,j}\}$ is a partition of $T_{i,j}$ into outdegree-3 subtrees. By induction, there is a non-crossing drawing of $T_{i,j}$ contained in $\mathsf{lune}(q, t_{i,j}, \varepsilon)$ such that:

(e) $v_{i,j}$, which is drawn at $t_{i,j}$, is in $\mathsf{lens}(x, q)$ for every vertex x of $T_{i,j} - v_{i,j}$; and
(f) for all $\ell \in \{1, \dots, k\}$, the subtree $T_\ell \cap T_{i,j}$ is drawn as the relative neighbourhood graph of its vertex set; and
(g) the drawing of $T_{i,j}$ has angular resolution greater than

$$\frac{\pi}{\max\{\Delta^+(T_{i,j}) - 1, 4\}},$$

which is at least

$$\frac{\pi}{\max\{\Delta^+(T) - 1, 4\}}.$$

Draw r at q, and draw a straight-line edge from r to each neighbour $v_{i,j}$ of r. The angle between two edges incident to r is at least

$$\frac{\pi}{d-1} \geq \frac{\pi}{\Delta^+(T) - 1}.$$

The angle between an edge $rv_{i,j}$ and each edge $v_{i,j}x$ in $T_{i,j}$ is at least $\frac{\pi}{4}$. With (g), this proves the third claim of the lemma.

Each subtree $T_{i,j}$ is drawn outside of $\mathsf{disc}(q, \delta')$, while the edges incident to r are contained within $\mathsf{disc}(q, \delta')$, and therefore do not cross any other edge. Hence the drawing of T is non-crossing. By (a), $T_{i,j}$ is drawn within $\mathsf{lune}(q, t_{i,j}, \varepsilon) \subset \mathsf{disc}(t_{i,j}, \varepsilon) \subset \mathsf{lune}(p, q, \delta)$. The edges incident to r are drawn within $\mathsf{lune}(p, q, \delta)$. Hence all of T is drawn within $\mathsf{lune}(p, q, \delta)$.

Now consider a vertex x of $T - r$. Then x is in $T_{i,j}$ for some $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, d_i\}$. Thus x is drawn in $\mathsf{disc}(q, \delta) - \mathsf{disc}(p, |pq|)$, implying $|xq| < \delta < |xp|$ and $|pq| < |px|$. Hence $q \in \mathsf{lens}(x, p)$, implying $r \in \mathsf{lens}(x, p)$. This proves the first claim of the lemma.

It remains to prove that each subtree T_i is drawn as the relative neighbourhood graph of its vertex set. Consider distinct vertices v and w in T_i. We must show that $\mathsf{lens}(v, w) \cap V(T_i) = \emptyset$ if and only if $vw \in E(T_i)$. Without loss of generality, $w \neq r$.

Case 1. $v = r$ and $vw \in E(T_i)$: So $w = v_{i,j}$ for some $j \in \{1, 2, 3\}$. Then v is drawn at q, and w is drawn at $t_{i,j}$. Now $\mathsf{lens}(q, t_{i,j}) \subset \mathsf{disc}(q, \delta')$, which contains no vertex except r (at q). Thus $\mathsf{lens}(v, w) \cap V(T) = \emptyset$, as desired.

Case 2. $v = r$ and $vw \notin E(T_i)$: Then w is in $T_{i,j}$ for some $j \in \{1, 2, 3\}$, but $w \neq v_{i,j}$. Since v is drawn at q, by (e), the vertex $v_{i,j}$, which is in T_i, is in $\mathsf{lens}(v, w)$, as desired.

Now assume that $v \neq r$ and $w \neq r$.

Case 3. v and w are in the same component $T_{\ell,j}$ of $T - r$, for some $\ell \in \{1, \dots, k\}$: Then v and w are drawn within $\mathsf{disc}(t_{\ell,j}, \varepsilon)$. Each vertex in T_i is r, is in $T_{\ell,j}$, or is in $T_{i,j'}$ for some $(i, j') \neq (\ell, j)$. Since r is drawn at q, (c) implies that $r \notin \mathsf{lens}(v, w)$. Since $T_{i,j'}$ is drawn within $\mathsf{disc}(t_{i,j'}, \varepsilon)$, by (d), $\mathsf{lens}(v, w) \cap V(T_{i,j'}) = \emptyset$. Hence $\mathsf{lens}(v, w) \cap V(T_i) = \emptyset$ if and only if $\mathsf{lens}(v, w) \cap V(T_{\ell,j}) \cap T_i = \emptyset$. By (f), $\mathsf{lens}(v, w) \cap V(T_i) = \emptyset$ if and only if v and w are adjacent in T_i, as desired.

Case 4. v and w are in distinct components of $T - r$: Thus r is in T_i, v is in $T_{i,j}$ and $w \in T_{i,j'}$ for some $j \neq j'$, and v and w are not adjacent. By construction, v is drawn in $\mathsf{disc}(t_{i,j}, \varepsilon)$ and w is drawn in $\mathsf{disc}(t_{i,j'}, \varepsilon)$. Thus (b) implies that $q \in \mathsf{lens}(v, w)$. Thus r, which is drawn at q, is in $\mathsf{lens}(v, w)$, as desired.

Therefore the subtree T_i is drawn as the relative neighbourhood graph of its vertex set. This completes the proof. □

5. Drawings Based on a Covering

Theorem 8 below establishes a result for relative neighbourhood graphs that implies Theorem 4 for minimum spanning trees. Before proving Theorem 8 we give a simpler proof of a weaker result, in which the obtained drawing might have crossings.

Proposition 1. *Let $\{T_1, T_2\}$ be a covering of a tree T by degree-5 subtrees. Then there is a drawing of T in which each T_i is drawn as the relative neighbourhood graph of its vertex set.*

Proof. We proceed by induction on $|V(T)|$. If $\Delta(T) \leq 5$ then $T \cong \mathsf{RNG}(P)$ for some point set P by Lemma 1. This drawing is crossing-free since it also a minimum spanning tree. Furthermore, each T_i is drawn as the relative neighbourhood graph of the subset of P representing T_i. Now assume that $\Delta(T) \geq 6$. Thus $\deg_T(v) \geq 6$ for some vertex v. Hence there are edges $vx \in E(T_1) - E(T_2)$ and $vy \in E(T_2) - E(T_1)$. Let T' be the tree obtained from T by identifying x and y into a new vertex w. (This operation is called an *elementary homomorphism* or *folding*; see [5, 10, 21, 23] and Fig. 6.) Let T'_i be the subtrees of T' determined by T_i for $i \in \{1, 2\}$. Note that the edge vw is in $T'_1 \cap T'_2$. Observe that $\{T'_1, T'_2\}$ is a covering of T' by degree-5 subtrees. By induction, there is a drawing of T' such that each T'_i is the relative neighbourhood graph of its vertex set. Moreover, for some $\varepsilon > 0$, if w is moved to any point in $\mathsf{disc}(w, \varepsilon)$ then in the resulting drawing of T', each T'_i is drawn as the relative neighbourhood graph of its vertex set. Consider a drawing of T in which every vertex in $V(T) - \{x, y\}$ inherits is position in the drawing of T', and x and y are assigned distinct points in $\mathsf{disc}(w, \varepsilon)$. Since $x \in V(T_1) - V(T_2)$ and $y \in V(T_2) - V(T_1)$, each T_i is drawn as the relative neighbourhood graph of its vertex set in the drawing of T. □

We now strengthen Proposition 1 by showing that the drawing of T can be made crossing-free. Theorem 4 is implied by the following stronger result:

Theorem 8. *Let $\{T_1, T_2\}$ be a covering of a tree T by degree-5 subtrees. Then there is a non-crossing drawing of T such that each T_i is drawn as the relative neighbourhood graph of its vertex set.*

The proof of Theorem 8 depends on the following definition. A *combinatorial embedding* of a graph is a cyclic ordering of the edges incident to each vertex. We define a combinatorial embedding of a graph G, with respect to a covering $\{G_1, G_2\}$

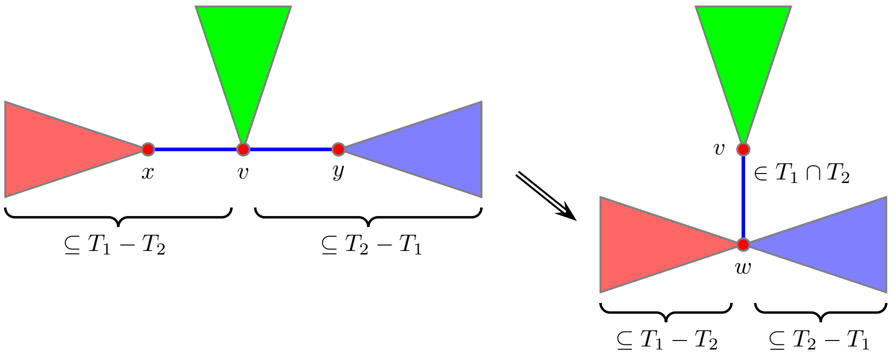

Fig. 6. Folding the tree T in the proof of Proposition 1.

of G, to be *good* if for each vertex v of G, in the clockwise ordering of the edges incident to v, the edges in $E(G_1) - E(G_2)$ are grouped together, followed by the edges in $E(G_1) \cap E(G_2)$, followed by the edges in $E(G_2) - E(G_1)$. Since every tree covered by two subtrees obviously has a good embedding, Theorem 8 now follows from the next lemma:

Lemma 4. *Let $\{T_1, T_2\}$ be a covering of a tree T by degree-5 subtrees. For every good combinatorial embedding of T, with respect to $\{T_1, T_2\}$, there is a non-crossing drawing of T such that each T_i is drawn as the relative neighbourhood graph of its vertex set, and the given combinatorial embedding of T is preserved in the drawing.*

Proof. We proceed by induction on $|V(T)|$. If $\Delta(T) \leq 5$ then $T \cong \mathsf{RNG}(P)$ for some point set P by Lemma 1. This drawing is crossing-free since it also a minimum spanning tree. Moreover, by examining the proof of Lemma 1, it is easily seen that any given combinatorial embedding of T can be preserved in the drawing. Each T_i is drawn as the relative neighbourhood graph of the subset of P representing T_i. Now assume that $\deg_T(v) \geq 6$ for some vertex v. Hence there are edges $vx \in E(T_1) - E(T_2)$ and $vy \in E(T_2) - E(T_1)$ such that vx and vy are consecutive in the cyclic ordering of the edges incident to v.

Let T' be the tree obtained from T by identifying x and y into a new vertex w. Let T'_i be the subtrees of T' determined by T_i for $i \in \{1, 2\}$. Note that the edge vw is in $T'_1 \cap T'_2$. The cyclic ordering of the edges in T' incident to v is obtained from the cyclic ordering of the edges in T incident to v by replacing vx and vy (which are consecutive) by vw. And $N_{T'}(w)$ is ordered $(N_{T_1 - E(T_2)}(x), wv, N_{T_2 - E(T_1)}(y))$. Other vertices keep their ordering in T.

Observe that $\{T'_1, T'_2\}$ is a covering of T' by degree-5 subtrees. By induction, there is a non-crossing drawing of T' such that each T'_i is the relative neighbourhood graph of its vertex set, and the given combinatorial embedding of T is preserved in the drawing. For some $\varepsilon > 0$, if w is moved to any point in $\mathsf{disc}(w, \varepsilon)$ then in the resulting drawing of T', each T'_i is drawn as the relative neighbourhood graph of its vertex set, and the given combinatorial embedding of T is preserved. Consider a drawing of T in which every vertex in $V(T)-\{x,y\}$ inherits is position in the drawing of T', and x and y are assigned distinct points in $\mathsf{disc}(w,\varepsilon)$. Since $x \in V(T_1)-V(T_2)$ and $y \in V(T_2)-V(T_1)$, each T_i is drawn as the relative neighbourhood graph of its vertex set in the drawing of T. It remains to assign points for x and y in $\mathsf{disc}(w,\varepsilon)$ so that the drawing of T is crossing-free. In the drawing of T', the edges incident to w are ordered $(N_{T_1-E(T_2)}(x), wv, N_{T_2-E(T_1)}(y))$. Let R be a ray centred at w that separates the edges in $T_1 - E(T_2)$ incident to w and those in $T_2 - E(T_1)$ incident to w, such that v is not on the extension of R. At most one of x and y, say x, has neighbours on both sides of the extension of R. As illustrated in Fig. 7, position x at w, and position y on R and inside $\mathsf{disc}(w,\varepsilon)$. It follows that there are no crossings and the correct ordering of edges is preserved at v, x and y. □

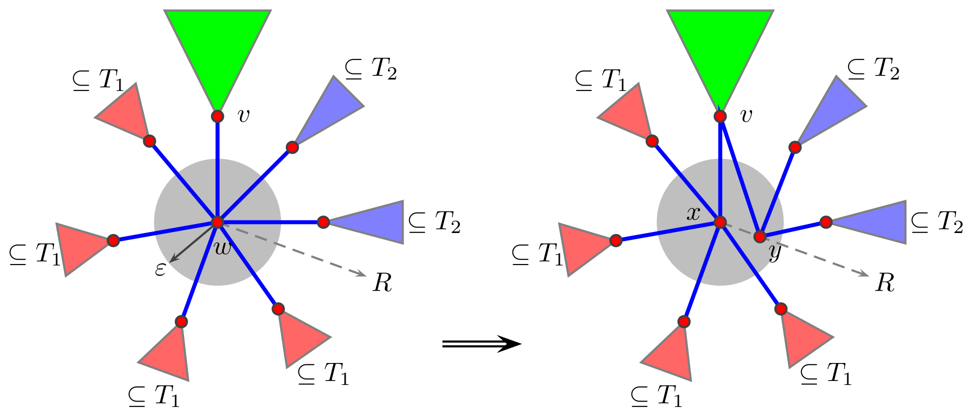

Fig. 7. Producing a drawing of T given a drawing of T' in the proof of Lemma 4.

We now show that Theorem 4 cannot be generalised for coverings by three or more subtrees. (Thus neither Proposition 1 nor Theorem 8 can be similarly generalised.) Let T be the 6-star with root r and leaves $v_1,\dots,v_6$. Let $\{T_1,T_2,T_3\}$ be the following covering of T. Let T_1 be the subtree of T induced by $\{r,v_1,v_2,v_3,v_4\}$. Let T_2 be the subtree of T induced by $\{r,v_1,v_2,v_5,v_6\}$. Let T_3 be the subtree of T induced by $\{r,v_3,v_4,v_5,v_6\}$. Thus each T_i is a 4-star. Suppose on the contrary that T has a drawing such that each T_i is drawn as a minimum spanning tree of its vertex set. The angle $\angle v_i r v_j$ between some pair of consecutive edges rv_i and rv_j (in the cyclic order around r) is less than $\frac{\pi}{3}$ since no three vertices are collinear. Since v_i and v_j are each in two subtrees, and r is in every subtree, the vertices r, v_i, v_j are in a common subtree T_ℓ. Every minimum spanning tree has angular resolution at least $\frac{\pi}{3}$. Thus T_ℓ is not drawn as a minimum spanning tree. This contradiction proves there is no drawing of T such that each T_i is drawn as a minimum spanning tree of its vertex set. Note that this argument generalises to show that if $P_1, \ldots, P_{15}$ are the $\binom{6}{2}$ paths through the root of the 6-star T, then in every drawing of T, some P_i is not a minimum spanning tree of its vertex set.

6. Further Research

This paper has not analysed the area of the drawings produced by our algorithms. It would be interesting to consider whether there are drawings whose area is polynomial in the number of vertices of the given tree, for example when the tree is partitioned into outdegree-3 subtrees. While the problem of drawing a tree as a minimum spanning tree in polynomial area is open in the general case [31], Frati and Kaufmann [22] proved that every degree-4 tree has a drawing as a minimum spanning tree in polynomial area; also see [32].

A second direction for further research is to extend the approach used in this paper to other types of proximity drawings of trees; see [30] for appropriate definitions. For example, every degree-4 tree admits a w-β-drawing for all values of β in $(\cos(\frac{2\pi}{5})^{-1}, \infty)$; see [12, Theorem 7]. Given a partition of a rooted tree T into outdegree-3 subtrees and a value of β in the above interval, is there a drawing of T in which each subtree is drawn as a w-β-drawing?

The results of this paper motivate studying coverings and partitions of trees by subtrees of bounded degree. We consider these purely combinatorial problems in our companion paper [37]. For example, given a tree T and integer d, we present there a formula for the minimum number of degree-d subtrees that partition T, and describe a polynomial time algorithm that finds such a partition. Similarly, we present a polynomial time algorithm that finds a covering of T by the minimum number of degree-d subtrees.

References

1. B. Aronov, M. Dulieu and F. Hurtado, Witness (Delaunay) graphs, *Comput. Geom.* **44**(6-7) (2011) 329–344, doi: 10.1016/j.comgeo.2011.01.001.
2. B. Aronov, M. Dulieu and F. Hurtado, Witness Gabriel graphs, *Comput. Geom.* **46**(7) (2013) 894–908, doi: 10.1016/j.comgeo.2011.06.004.
3. J. Barát, J. Matoušek and D. R. Wood, Bounded-degree graphs have arbitrarily large geometric thickness, *Electron. J. Combin.* **13**(1) (2006) R3, http://www.combinatorics.org/Volume_13/Abstracts/v13i1r3.html.
4. P. Bose, W. Lenhart and G. Liotta, Characterizing proximity trees, *Algorithmica* **16**(1) (1996) 83–110, doi: 10.1007/s004539900038.
5. F. Buckley and L. Superville, Extremal results for the ajointed number, *Proc. 12th Southeastern Conf. Combinatorics, Graph Theory and Computing, Vol. I*, Vol. 32 of *Congr. Numer.* (1981), pp. 163–171.
6. P. Carmi, V. Dujmović, P. Morin and D. R. Wood, Distinct distances in graph drawings, *Electron. J. Combin.* **15** (2008) R107, http://www.combinatorics.org/Volume_15/Abstracts/v15i1r107.html.
7. T. M. Chan, A near-linear area bound for drawing binary trees, *Algorithmica* **34**(1) (2002) 1–13, doi: 10.1007/s00453-002-0937-x.

8. T. M. Chan, M. T. Goodrich, S. R. Kosaraju and R. Tamassia, Optimizing area and aspect ratio in straight-line orthogonal tree drawings, *Comput. Geom.* **23**(2) (2002) 153–162, doi: 10.1016/S0925-7721(01)00066-9.
9. R. F. Cohen, G. Di Battista, R. Tamassia and I. G. Tollis, Dynamic graph drawings: Trees, series-parallel digraphs, and planar *st*-digraphs, *SIAM J. Comput.* **24**(5) (1995) 970–1001, doi: 10.1137/S0097539792235724.
10. C. R. Cook and A. B. Evans, Graph folding, *Proc.* 10*th Southeastern Conf. Combinatorics, Graph Theory and Computing*, Vol. XXIII–XXIV of *Congr. Numer.* (1979), pp. 305–314.
11. P. Crescenzi, G. Di Battista and A. Piperno, A note on optimal area algorithms for upward drawings of binary trees, *Comput. Geom.* **2**(4) (1992) 187–200, doi: 10.1016/0925-7721(92)90021-J.
12. G. Di Battista, G. Liotta and S. H. Whitesides, The strength of weak proximity, *J. Discr. Algorithms* **4**(3) (2006) 384–400, doi: 10.1016/j.jda.2005.12.004.
13. M. B. Dillencourt, D. Eppstein and D. S. Hirschberg, Geometric thickness of complete graphs, *J. Graph Algorithms Appl.* **4**(3) (2000) 5–17, doi: 10.7155/jgaa.00023.
14. V. Dujmović, D. Eppstein, M. Suderman and D. R. Wood, Drawings of planar graphs with few slopes and segments, *Comput. Geom. Theor. Appl.* **38** (2007) 194–212, doi: 10.1016/j.comgeo.2006.09.002.
15. V. Dujmović and D. R. Wood, Graph treewidth and geometric thickness parameters, *Discr. Comput. Geom.* **37**(4) (2007) 641–670, doi: 10.1007/s00454-007-1318-7.
16. C. A. Duncan, On graph thickness, geometric thickness, and separator theorems, *Comput. Geom.* **44**(2) (2011) 95–99, doi: 10.1016/j.comgeo.2010.09.005.
17. C. A. Duncan, D. Eppstein and S. G. Kobourov, The geometric thickness of low degree graphs, *Proc.* 20*th ACM Symp. Computational Geometry* (*SoCG* ′04) (2004), pp. 340–346, doi: 10.1145/997817.997868.
18. P. Eades and S. Whitesides, The realization problem for Euclidean minimum spanning trees is NP-hard, *Algorithmica* **16**(1) (1996) 60–82, doi: 10.1007/s004539900037.
19. D. Eppstein, Separating geometric thickness from book thickness (2001), arXiv: math/0109195.
20. D. Eppstein, Separating thickness from geometric thickness, *Towards a Theory of Geometric Graphs*, ed. J. Pach, Vol. 342 of *Contemporary Mathematics* (Amer. Math. Soc., 2004), pp. 75–86, arXiv: math/0204252.
21. A. B. Evans, Absolutely 3-chromatic graphs, *J. Graph Theor.* **10**(4) (1986) 511–521, doi: 10.1002/jgt.3190100409.
22. F. Frati and M. Kaufmann, Polynomial area bounds for MST embeddings of trees, *Comput. Geom. Theor. Appl.* **44** (2011) 529–543, doi: 10.1016/j.comgeo.2011.05.005.
23. G.-G. Gao and G. Hahn, Minimal graphs that fold onto K_n, *Discr. Math.* **142**(1-3) (1995) 277–280, doi: 10.1016/0012-365X(93)E0227-U.
24. A. Garg and A. Rusu, Straight-line drawings of binary trees with linear area and arbitrary aspect ratio, *J. Graph Algorithms Appl.* **8**(2) (2004) 135–160, doi: 10.7155/jgaa.00086.
25. S.-H. Hong and P. Eades, Drawing trees symmetrically in three dimensions, *Algorithmica* **36**(2) (2003) 153–178, doi: 10.1007/s00453-002-1011-4.
26. J. P. Hutchinson, T. C. Shermer and A. Vince, On representations of some thickness-two graphs, *Comput. Geom. Theor. Appl.* **13**(3) (1999) 161–171, doi: 10.1016/S0925-7721(99)00018-8.
27. S. K. Kim, Logarithmic width, linear area upward drawing of AVL trees, *Inform. Process. Lett.* **63**(6) (1997) 303–307, doi: 10.1016/S0020-0190(97)00137-3.

28. S. K. Kim, Order-preserving, upward drawing of binary trees using fewer bends, *Discr. Appl. Math.* **143**(1–3) (2004) 318–323, doi: 10.1016/j.dam.2004.02.011.
29. G. Liotta and H. Meijer, Voronoi drawings of trees, *Comput. Geom. Theor. Appl.* **24**(3) (2003) 147–178, doi: 10.1016/S0925-7721(02)00137-2.
30. G. Liotta, Proximity drawings, *Handbook of Graph Drawing and Visualization*, ed. R. Tamassia, chap. 4 (Chapman and Hall/CRC Press, 2013), pp. 115–154.
31. C. Monma and S. Suri, Transitions in geometric minimum spanning trees, *Discr. Comput. Geom.* **8**(3) (1992) 265–293, doi: 10.1007/BF02293049.
32. P. Penna and P. Vocca, Proximity drawings in polynomial area and volume, *Comput. Geom. Theor. Appl.* **29**(2) (2004) 91–116, doi: 10.1016/j.comgeo.2004.03.015.
33. C.-S. Shin, S. K. Kim and K.-Y. Chwa, Area-efficient algorithms for straight-line tree drawings, *Comput. Geom. Theor. Appl.* **15**(4) (2000) 175–202, doi: 10.1016/S0925-7721(99)00053-X.
34. M. Suderman, Pathwidth and layered drawings of trees, *Int. J. Comput. Geom. Appl.* **14**(3) (2004) 203–225, doi: 10.1142/S0218195904001433.
35. G. T. Toussaint, The relative neighbourhood graph of a finite planar set, *Patt. Recogn.* **12**(4) (1980) 261–268, doi: 10.1016/0031-3203(80)90066-7.
36. L. Trevisan, A note on minimum-area upward drawing of complete and Fibonacci trees, *Inform. Process. Lett.* **57**(5) (1996) 231–236, doi: 10.1016/0020-0190(96)81422-0.
37. D. R. Wood, Partitions and coverings of trees by bounded-degree subtrees (2010), arXiv: 1008.3190.