COMBINATORICA
Bolyai Society – Springer-Verlag

Combinatorica 30pp.
DOI: 10.1007/s00493-020-3941-3

SEYMOUR'S CONJECTURE ON 2-CONNECTED GRAPHS OF LARGE PATHWIDTH

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Received February 9, 2018
Revised September 25, 2019

We prove a conjecture of Seymour (1993) stating that for every apex-forest H_1 and outerplanar graph H_2 there is an integer p such that every 2-connected graph of pathwidth at least p contains H_1 or H_2 as a minor. An independent proof was recently obtained by Dang and Thomas [3].

1. Introduction

Pathwidth is a graph parameter of fundamental importance, especially in graph structure theory. The *pathwidth* of a graph G is the minimum integer k for which there is a sequence of sets $B_1,\dots,B_n \subseteq V(G)$ such that $|B_i| \leqslant k+1$ for each $i \in [n]$, for every vertex v of G, the set $\{i \in [n] : v \in B_i\}$ is a non-empty interval, and for each edge vw of G, some B_i contains both v and w.

In the first paper of their graph minors series, Robertson and Seymour [7] proved the following theorem.

1.1. *For every forest F, there exists a constant p such that every graph with pathwidth at least p contains F as a minor.*

Mathematics Subject Classification (2010): 05C83

G. Joret acknowledges support from the Australian Research Council and from an *Action de Recherche Concertée* grant from the Wallonia-Brussels Federation in Belgium. P. Micek is partially supported by a Polish National Science Center grant (SONATA BIS 5; UMO-2015/18/E/ST6/00299). T. Huynh is supported by ERC Consolidator Grant 615640-ForEFront. T. Huynh, G. Joret, and P. Micek also acknowledge support from a joint grant funded by the Belgian National Fund for Scientific Research (F.R.S.–FNRS) and the Polish Academy of Sciences (PAN). Research of D. R. Wood is supported by the Australian Research Council.

The constant p was later improved to $|V(F)|-1$ (which is best possible) by Bienstock, Robertson, Seymour, and Thomas [1]. A simpler proof of this result was later found by Diestel [5].

Since forests have unbounded pathwidth, 1.1 implies that a minor-closed class of graphs has unbounded pathwidth if and only if it includes all forests. However, these certificates of large pathwidth are not 2-connected, so it is natural to ask for which minor-closed classes $\mathcal{C}$, does every *2-connected* graph in $\mathcal{C}$ have bounded pathwidth?

In 1993, Paul Seymour proposed the following answer (see [4]). A graph H is an *apex-forest* if $H - v$ is a forest for some $v \in V(H)$. A graph H is *outerplanar* if it has an embedding in the plane with all the vertices on the outerface. These classes are relevant since they both contain 2-connected graphs with arbitrarily large pathwidth. Seymour conjectured the following converse holds.

1.2. *For every apex-forest H_1 and outerplanar graph H_2 there is an integer p such that every 2-connected graph of pathwidth at least p contains H_1 or H_2 as a minor.*

Equivalently, 1.2 says that for a minor-closed class $\mathcal{C}$, every 2-connected graph in $\mathcal{C}$ has bounded pathwidth if and only if some apex-forest and some outerplanar graph are not in $\mathcal{C}$.

The original motivation for conjecturing 1.2 was to seek a version of 1.1 for matroids (see [3]). Observe that apex-forests and outerplanar graphs are planar duals (see 2.1). Since a matroid and its dual have the same pathwidth (see [6] for the definition of matroid pathwidth), 1.2 provides some evidence for a matroid version of 1.1.

In this paper we prove 1.2. An independent proof was recently obtained by Dang and Thomas [3].

We actually prove a slightly different, but equivalent version of 1.2. Namely, we prove that there are two unavoidable families of minors for 2-connected graphs of large pathwidth. We now describe our two unavoidable families.

A *binary tree* is a rooted tree such that every vertex has at most two children. For $\ell \geqslant 0$, the *complete binary tree of height* ℓ, denoted Γ_ℓ, is the binary tree with 2^ℓ leaves such that each root to leaf path has ℓ edges. It is well known that Γ_ℓ has pathwidth $\lceil \ell/2 \rceil$. Let Γ_ℓ^+ be the graph obtained from Γ_ℓ by adding a new vertex adjacent to all the leaves of Γ_ℓ. See Figure 1. Note that Γ_ℓ^+ is a 2-connected apex-forest, and its pathwidth grows as ℓ grows (since it contains Γ_ℓ).

Our second set of unavoidable minors is defined recursively as follows. Let ∇_1 be a triangle with a *root edge* e. Let H_1 and H_2 be copies of ∇_ℓ with

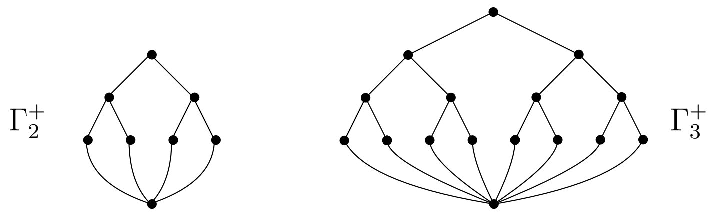

Figure 1. Complete binary trees with an extra vertex adjacent to all the leaves

root edges e_1 and e_2. Let ∇ be a triangle with edges e_1, e_2 and e_3. Define $\nabla_{\ell+1}$ by gluing each H_i to ∇ along e_i and then declaring e_3 as the new root edge. See Figure 2. Note that ∇_ℓ is a 2-connected outerplanar graph, and its pathwidth grows as ℓ grows (since it contains $\Gamma_{\ell-1}$).

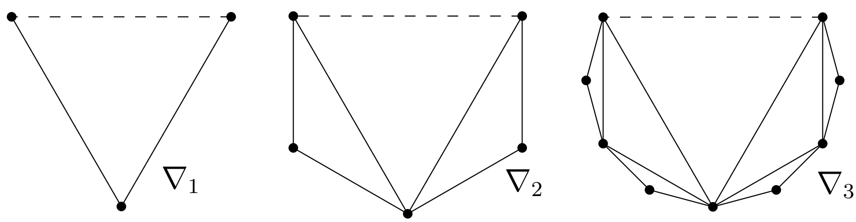

Figure 2. Universal outerplanar graphs. The root edges are dashed

The following is our main theorem.

1.3. *For every integer $\ell \geqslant 1$ there is an integer p such that every 2-connected graph of pathwidth at least p contains Γ_ℓ^+ or ∇_ℓ as a minor.*

In Section 2, we prove that every apex-forest is a minor of a sufficiently large Γ_ℓ^+ and every outerplanar graph is a minor of a sufficiently large ∇_ℓ. Thus, Theorem 1.3 implies Seymour's conjecture.

We actually prove the following theorem, which by 1.1, implies 1.3.

1.4. *For all integers $\ell \geqslant 1$, there exists an integer k such that every 2-connected graph G with a Γ_k minor contains Γ_ℓ^+ or ∇_ℓ as a minor.*

Our approach is different from that of Dang and Thomas [3], who instead observe that by the Grid Minor Theorem [8], one may assume that G has bounded treewidth but large pathwidth. Dang and Thomas then apply their machinery of 'non-branching tree decompositions' to prove 1.2.

The rest of the paper is organized as follows. Section 2 proves the universality of our two families. In Sections 3 and 4, we define 'special' ear decompositions and prove that special ear decompositions always yield $\mathit{\Gamma}_\ell^+$ or ∇_ℓ minors. In Section 5, we prove that a minimal counterexample to 1.4 always contains a special ear decomposition. Section 6 concludes with short derivations of our main results.

2. Universality

This section proves some elementary (and possibly well-known) results. We include the proofs for completeness.

2.1. *Outerplanar graphs and apex-forests are planar duals.*

Proof. Let G be an apex-forest, where G–v is a forest. Consider an arbitrary planar embedding of G. Note that every face of G includes v (otherwise G–v would contain a cycle). Let G^* be the planar dual of G. Let f be the face of G^* corresponding to v. Since every face of G includes v, every vertex of G^* is on f. So G^* is outerplanar.

Conversely, let G be an outerplanar graph. Consider a planar embedding of G, in which every vertex is on the outerface f. Let G^* be the planar dual of G. Let v be the vertex of G^* corresponding to f. If G^*-v contained a cycle C, then a face of G^*-v 'inside' C would correspond to a vertex of G that is not on f. Thus G^*-v is a forest, and G^* is an apex-forest. ∎

We now show that Theorem 1.3 implies Seymour's conjecture, by proving two universality results.

2.2. *Every apex-forest on $n \geqslant 2$ vertices is a minor of $\mathit{\Gamma}_{n-1}^+$.*

If H is a minor of G and $v \in V(H)$, the *branch set* of v is the set of vertices of G that are contracted to v. 2.2 is a corollary of the following.

2.3. *Every tree with $n \geqslant 1$ vertices is a minor of $\mathit{\Gamma}_{n-1}$, such that each branch set includes a leaf of $\mathit{\Gamma}_{n-1}$.*

Proof. We proceed by induction on n. The base case $n=1$ is trivial. Let T be a tree with $n \geqslant 2$ vertices. Let v be a leaf of T. Let w be the neighbour of v in T. By induction, $T-v$ is a minor of $\mathit{\Gamma}_{n-2}$, such that each branch set includes a leaf of $\mathit{\Gamma}_{n-2}$. In particular, the branch set for w includes some leaf x of $\mathit{\Gamma}_{n-2}$. Note that $\mathit{\Gamma}_{n-1}$ is obtained from $\mathit{\Gamma}_n$ by adding two new leaf vertices adjacent to each leaf of $\mathit{\Gamma}_{n-2}$. Let y and z be the leaf vertices of $\mathit{\Gamma}_{n-1}$ adjacent to x. Extend the branch set for w to include y and let $\{z\}$ be the branch set of v. For each leaf $u \neq x$ of $\mathit{\Gamma}_{n-2}$, if u is in the branch set of some vertex of $T-v$, then extend this branch set to include one of the new leaves in Γ_{n-1} adjacent to u. Now T is a minor of Γ_{n-1}, such that each branch set includes a leaf of Γ_{n-1}. ∎

Our second universality result is for outerplanar graphs.

2.4. *Every outerplanar graph on $n\geqslant 2$ vertices is a minor of ∇_{n-1}.*

2.4 is a corollary of the following.

2.5. *Every outerplanar triangulation G on $n\geqslant 3$ vertices is a minor of ∇_{n-1}, such that for every edge vw on the outerface of G, there is a non-root edge on the outerface of ∇_{n-1} joining the branch sets of v and w.*

Proof. We proceed by induction on n. The base case, $G=K_3$, is easily handled as illustrated in Figure 3. Let G be an outerplanar triangulation

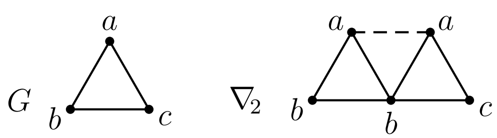

Figure 3. Proof of 2.5 in the base case

with $n\geqslant 4$ vertices. Every such graph has a vertex u of degree 2, such that if α and β are the neighbours of u, then $G-u$ is an outerplanar triangulation and $\alpha\beta$ is an edge on the outerface of $G-u$. By induction, $G-u$ is a minor of ∇_{n-2}, such that for every edge vw on the outerface of $G-u$, there is a non-root edge $v'w'$ on the outerface of ∇_{n-2} joining the branch sets of v and w. In particular, there is a non-root edge $\alpha'\beta'$ of ∇_{n-2} joining the branch sets of α and β. Note that ∇_{n-1} is obtained from ∇_{n-2} by adding, for each non-root edge pq on the outerface of ∇_{n-2}, a new vertex adjacent to p and q. Let the branch set of u be the vertex u' of $\nabla_{n-1}-V(\nabla_{n-2})$ adjacent to α' and β'. Thus ∇_{n-1} contains G as a minor. Every edge on the outerface of G is one of $u\alpha$ or $u\beta$, or is on the outerface of $G-u$. By construction, $u'\alpha'$ is a non-root edge on the outerface of ∇_{n-1} joining the branch sets of u and α. Similarly, $u'\beta'$ is a non-root edge on the outerface of ∇_{n-1} joining the branch sets of u and β. For every edge vw on the outerface of G, where $vw\not\in\{u\alpha,u\beta\}$, if z is the vertex in $\nabla_{n-1}-V(\nabla_{n-2})$ adjacent to v' and w', extend the branch set of v to include z. Now zw' is an edge on the outerface of ∇_{n-1} joining the branch sets for v and w. Thus for every edge vw on the outerface of G, there is a non-root edge of ∇_{n-1} joining the branch sets of v and w. ∎

3. Binary ear trees

Henceforth, all graphs in this paper are finite and simple. In particular, after contracting an edge, we suppress parallel edges and loops. Let H and G be graphs. We write $H \simeq G$ if H and G are isomorphic. Let $H \cup G$ be the graph with $V(H \cup G) = V(H) \cup V(G)$ and $E(H \cup G) = E(H) \cup E(G)$. If H is a subgraph of G, then an *H-ear* is a path in G with its two ends in $V(H)$ but with no internal vertex in $V(H)$. The *length* of a path is its number of edges.

For a vertex v in a rooted tree T, let T_v be the subtree of T rooted at v. A vertex v of T is said to be *branching* if v has at least two children.

A *binary ear tree* in a graph G is a pair $(T, \mathcal{P})$, where T is a binary tree, and $\mathcal{P} = \{P_x \colon x \in V(T)\}$ is a collection of paths in G of length at least 2 such that, for every non-root vertex x of T the following holds:

(i) P_x is a P_y-ear, where y is the parent of x in T, and
(ii) no internal vertex of P_x is in $\bigcup_{z \in V(T) \setminus V(T_x)} V(P_z)$.

A binary ear tree $(T, \mathcal{P})$ is *clean* if for every non-leaf vertex y of T, there is an end of P_y that is not contained in any P_x where x is a child of y.

The main result of this section is the following.

3.1. *For every integer $\ell \geqslant 1$, if G has a clean binary ear tree $(T, \mathcal{P})$ such that $T \simeq \Gamma_{3\ell-2}$, then G contains Γ_ℓ^+ or ∇_ℓ as a minor.*

Before starting the proof, we first set up notation for a Ramsey-type result that we will need.

If p and q are vertices of a tree T, then let pTq denote the unique pq-path in T. If T' is a subdivision of a tree T, the vertices of T' coming from T are called *original vertices* and the other vertices of T' are called *subdivision vertices*. Given a colouring of the vertices of $T = \Gamma_n$ with colours $\{\mathsf{red}, \mathsf{blue}\}$, we say that T contains a *red subdivision of Γ_k*, if it contains a subdivision T' of Γ_k such that all the original vertices of T' are red, and for all $a, b \in V(T')$ with b a descendant of a, the path aTb is descending. (Here a path is *descending* if it is contained in a path that starts at the root.) Define $R(k, \ell)$ to be the minimum integer n such that every colouring of Γ_n with colours $\{\mathsf{red}, \mathsf{blue}\}$ contains a red subdivision of Γ_k or a blue subdivision of Γ_ℓ. We will use the following easy result.

3.2. *$R(k, \ell) \leqslant k + \ell$ for all integers $k, \ell \geqslant 0$.*

Proof. We proceed by induction on $k + \ell$. As base cases, it is clear that $R(k, 0) = k$ and $R(0, \ell) = \ell$ for all k, ℓ. For the inductive step, assume $k, \ell \geqslant 1$ and let T be a $\{\mathsf{red}, \mathsf{blue}\}$-coloured copy of $\Gamma_{k+\ell}$. By symmetry, we may assume that the root r of T is coloured red. Let T_1 and T_2 be the components of $T-r$, both of which are copies of $\Gamma_{k+\ell-1}$. If T_1 or T_2 contains a blue subdivision of Γ_ℓ, then so does T and we are done. By induction, $R(k-1,\ell)\leqslant k-1+\ell$, so both T_1 and T_2 contain a red subdivision of Γ_{k-1}. Add the paths from r to the roots of these red subdivisions. We obtain a red subdivision of Γ_k, as desired. ∎

The following observation will be helpful when considering subdivision vertices.

3.3. *Let G be a graph having a clean binary ear tree $(T,\mathcal{P})$ with $\mathcal{P}=\{P_v\colon v\in V(T)\}$. Suppose that y is a degree-2 vertex in T with parent x and child z. Then there is a clean binary ear tree $(T/yz,\mathcal{P}')$ of G, with $\mathcal{P}'=\{P'_v\colon v\in V(T/yz)\}$ where $P'_v=P_v$ for all $v\in V(T)\setminus\{y,z\}$, and P'_{yz} is the unique P_x-ear contained in $P_y\cup P_z$ that contains P_z, where the vertex resulting from the contraction of edge yz is denoted yz as well.*

Proof. Property (i) of the definition of binary ear trees holds for vertex yz of T/yz by our choice of P'_{yz}. Property (ii) holds for yz because it held for y and for z in $(T,\mathcal{P})$. Also, these two properties hold for children of yz in T/yz (if any) because they held for z before. Thus, $(T/yz,\mathcal{P}')$ is a binary ear tree. Finally, note that cleanliness of the binary ear tree $(T/yz,\mathcal{P}')$ follows from that of $(T,\mathcal{P})$, and the fact that the ends of P'_{yz} are the same as the ones of P_y. ∎

We now prove 3.1.

Proof of 3.1. Let t be a non-leaf vertex of T. Let u and v be the children of t. Let u_1 and u_2 be the ends of P_u. Let v_1 and v_2 be the ends of P_v. We say that t is *nested* if $u_1P_tu_2\subseteq v_1P_tv_2$ or $v_1P_tv_2\subseteq u_1P_tu_2$. If t is not nested, then t is *split*. See Figures 4 and 5. Regarding split and nested as colours, we apply 3.2 to the tree T with the leaves removed, and obtain a tree T^* which is a split subdivision of $\Gamma_{\ell-1}$ or a nested subdivision of $\Gamma_{2\ell-2}$. For each leaf of T^*, add back its two children in T. This way, we deduce that T contains either a subdivision of Γ_ℓ with all branching vertices split, or a subdivision of $\Gamma_{2\ell-1}$ with all branching vertices nested. In the first case, we will find a ∇_ℓ minor, while in the second we will find a Γ_ℓ^+ minor. The two cases are covered by 3.4 and 3.5.

3.4. *If T contains a subdivision T^1 of Γ_ℓ such that every branching vertex is split, then $\bigcup_{t\in V(T^1)}P_t$ contains ∇_ℓ as a minor.*

Subproof. Consider the clean binary ear tree 'induced by' the subtree T^1, that is, the pair $(T^1,\mathcal{P}^1)$ where $\mathcal{P}^1=\{P_t\colon t\in V(T^1)\}$. First, for every subdivision vertex y of T^1 with child z, we apply 3.3 to $(T^1,\mathcal{P}^1)$ in order to

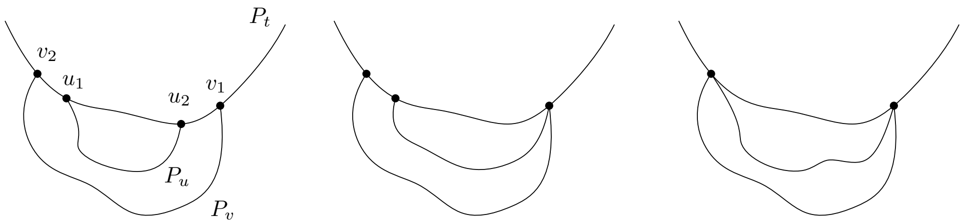

Figure 4. Examples of a nested vertex t with a path P_t in a clean binary ear tree

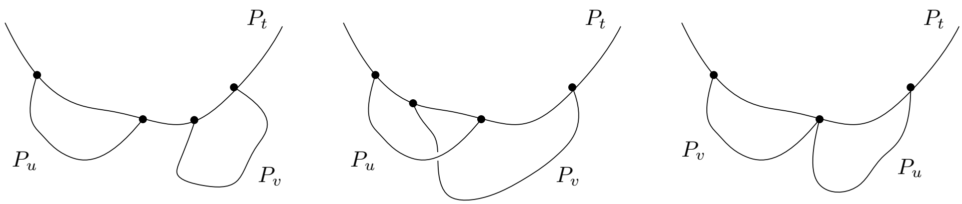

Figure 5. Examples of a split vertex t with a path P_t in a clean binary ear tree

suppress vertex y. Note that every branching vertex of T^1 stays split. In particular, this is true if z is branching. Hence, we may assume from now on that T^1 has no subdivision vertices.

Let P be a path in a graph G. Let ∇_ℓ^- be the graph obtained from ∇_ℓ by deleting its root edge xy. We say that a ∇_ℓ^- minor in G is *rooted on P* if the two roots of the ∇_ℓ^- minor are the ends of P. (By 'roots' we mean the ends of the root edge.)

We prove the following technical statement. Let $m \geqslant 0$ be an integer, and let T' be a subtree of T^1 isomorphic to Γ_m such that all branching vertices of T' are split, then $\bigcup_{t\in V(T')} P_t$ contains a ∇_{m+1}^- minor rooted on P_r, where r is the root of T'.

This proves 3.4 for $\ell \geqslant 2$, since $\nabla_{\ell+1}^-$ contains a ∇_ℓ minor. For $\ell = 1$, 3.4 is straightforward.

We prove the above technical statement by induction on m. The case $m=0$ is clear since then T' is a single vertex v and ∇_1^- is just a path with three vertices. (Here we use that $|V(P_v)| \geqslant 3$.)

For the inductive step, let a and b be the children of r. By induction, $G_a := \bigcup_{t\in V(T'_a)} P_t$ contains a ∇_m^- minor H_a rooted on P_a, and $G_b := \bigcup_{t\in V(T'_b)} P_t$ contains a ∇_m^- minor H_b rooted on P_b.

We prove that G_a and G_b are vertex-disjoint, except possibly at a vertex of $V(P_a) \cap V(P_b)$ (there is at most one such vertex since r is split). Suppose v is a vertex appearing in both G_a and G_b. Let x be the vertex in T'_a closest to the root such that $v \in V(P_x)$ and let y be the vertex in T'_b closest to the root such that $v \in V(P_y)$. By property (ii) of binary ear trees we know that no internal vertex of P_x lies in $\bigcup_{z\in V(T^1)\setminus V(T'_x)} V(P_z)$. Since $y\in V(T^1)\setminus V(T'_x)$ and $v\in V(P_y)$, we conclude that v is an end of P_x. This means that v lies in T'_p where p is the parent of x in T'. By the choice of x this is only possible when $x=a$. Thus, v is an end of P_a and lies in P_r. By a symmetric argument we conclude that v is an end of P_b as well, as desired.

Let a_1 and a_2 be the ends of P_a, b_1 and b_2 be the ends of P_b, and r_1 and r_2 be the ends of P_r. By symmetry, we may assume that the ordering of these points along P_r is either r_1,a_1,b_1,a_2,b_2,r_2 or r_1,a_1,a_2,b_1,b_2,r_2. (Note that some vertices may coincide.) Using the observation from the previous paragraph, we obtain a ∇^-_{m+1} minor rooted on P_r by considering the union of the ∇^-_m minor rooted on P_a and the ∇^-_m minor rooted on P_b that we were given, and contracting the following three subpaths of P_r: $r_1P_ra_1$, $a_2P_rb_1$, and $b_2P_rr_2$. Notice that if G_a and G_b have a vertex v in common, then $v=a_2=b_1$. See Figure 6 for an illustration of the construction. ∎

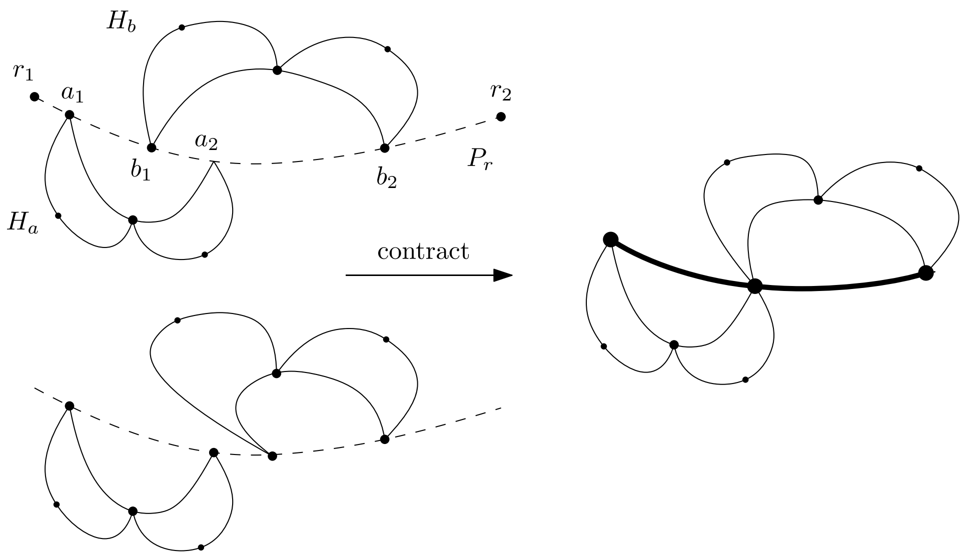

Figure 6. Inductively constructing a ∇^-_3 minor

3.5. *If T contains a subdivision T^2 of $\Gamma_{2\ell-1}$ such that every branching vertex is nested, then $\bigcup_{t\in V(T^2)} P_t$ contains Γ^+_ℓ as a minor.*

Subproof. Consider the clean binary ear tree $(T^2,\mathcal{P}^2)$ where $\mathcal{P}^2=\{P_t\colon t\in V(T^2)\}$. First, for every subdivision vertex y of T^2 with child z, we apply 3.3

to $(T^2, \mathcal{P}^2)$ in order to suppress vertex y. Note that every branching vertex of T^2 stays nested. In particular, this is true if z is branching. Hence, we may assume from now on that T^2 has no subdivision vertices.

Orient each path in $\mathcal{P}^2$ inductively as follows. Let r be the root of T^2 and orient P_r arbitrarily. If P_s has already been oriented and t is a child of s in T^2, then orient P_t so that $P_s \cup P_t$ does not contain a directed cycle. Consider each path in $\mathcal{P}^2$ to be oriented from left to right, and thus with left and right ends.

Let t be a non-leaf vertex of T^2 and let u and v be the children of t. Define t to be *left-good* if the left end of P_t is not in P_u nor P_v. Define t to be *right-good* if the right end of P_t is not in P_u nor P_v. Since $(T^2, \mathcal{P}^2)$ is clean we know that every non-leaf vertex t of T^2 is left-good or right-good. We colour the non-leaf vertices of T^2 with left and right in such a way that when a vertex is coloured left (right), then it is left-good (right-good). Applying 3.2 on the tree T^2 with branching vertices coloured this way in which we remove all the leaves, we obtain a subdivision T^* of $\Gamma_{\ell-1}$ such that all original vertices are coloured left, or all are coloured right, say without loss of generality left. For every leaf of T^*, add back to T^* its two children in T^2, and denote by T^3 the resulting tree. Note that T^3 is a subdivision of Γ_ℓ and all branching vertices of T^3 are left-good.

We focus on the clean binary ear tree $(T^3, \mathcal{P}^3)$ induced by T^3, where $\mathcal{P}^3 = \{P_t \colon t \in V(T^3)\}$. Then, for every subdivision vertex y of T^3 with child z, we apply 3.3 to $(T^3, \mathcal{P}^3)$ in order to suppress vertex y, as before. Note that every branching vertex of T^3 stays nested and left-good. Hence, we may assume from now on that T^3 has no subdivision vertices.

Let t be a non-leaf vertex of T^3 and u and v be the children of t in T^3. Let $f(t)$ be the first vertex of P_t that is a left end of either P_u or of P_v. Note that $f(t)$ is not the left end of P_t, since t is left-good. Let $e(t)$ be the last edge of P_t incident to a left end of either P_u or P_v. If t is a leaf of T^3, we define $f(t)$ to be any internal vertex of P_t and $e(t)$ to be the last edge of P_t incident to $f(t)$.

Let $H := \bigcup_{t \in V(T^3)} P_t$ and $M := \{e(t) \colon t \in V(T^3)\}$. Since every branching vertex of T^3 is nested, $H \setminus M$ contains two components H_{left} and H_{right} such that H_{left} contains all left ends of $\{P_t \colon t \in V(T^3)\}$ and H_{right} contains all right ends of $\{P_t \colon t \in V(T^3)\}$. Using that every branching vertex of T^3 is left-good, it is easy to see that H_{left} contains a subdivision T^4 of Γ_ℓ whose set of original vertices is $\{f(t) \colon t \in V(T^3)\}$; see Figure 7. By construction, each leaf of T^4 is incident to an edge in M. Also, H_{right} is clearly connected. Therefore, after contracting all edges of H_{right}, $T^4 \cup M \cup H_{\text{right}}$ contains a Γ_ℓ^+ minor. ∎

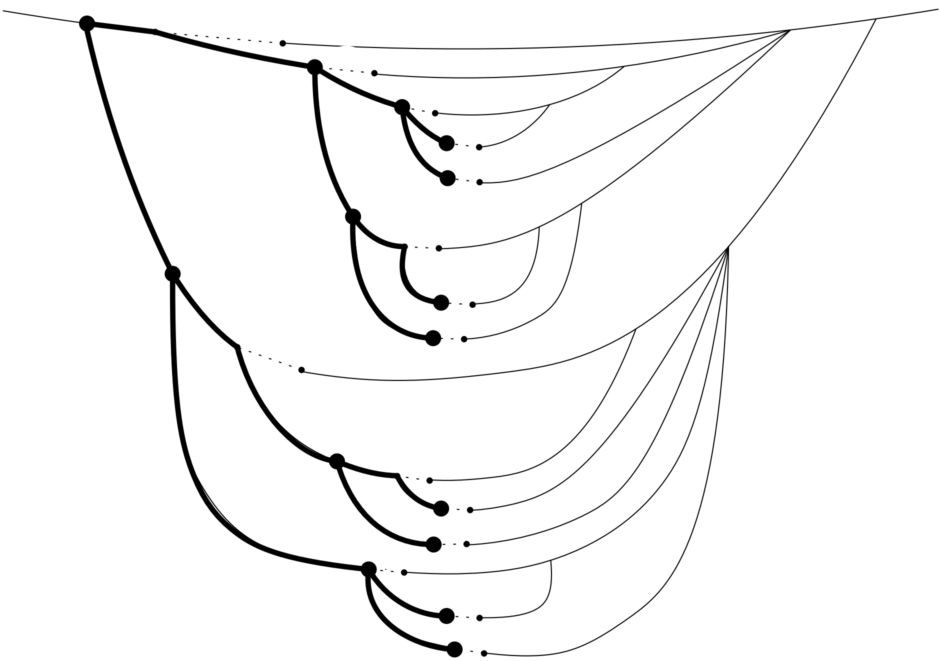

Figure 7. A Γ_3 minor in H_{left}

This ends the proof of 3.1. ∎

4. Binary pear trees

In order to prove our main theorem, we need something slightly more general than binary ear trees, which we now define. A *binary pear tree* in a graph G is a pair $(T, \mathcal{B})$, where T is a binary tree, and $\mathcal{B} = \{(P_x, Q_x)\colon x \in V(T)\}$ is a collection of pairs of paths of G of length at least 2 such that $P_x \subseteq Q_x$ for all $x \in V(T)$, and the following properties are satisfied for each non-root vertex $x \in V(T)$.

(i) Q_x is a P_y-ear, where y is the parent of x in T;

(ii) if x has no sibling, then no internal vertex of Q_x is in

$$\bigcup_{z \in V(T) \setminus V(T_x)} V(Q_z);$$

(iii) if x has a sibling x', then

- no internal vertex of Q_x is in $\bigcup_{z \in V(T)\setminus(V(T_x)\cup V(T_{x'}))} V(Q_z)$, and
- no internal vertex of P_x is in $Q_{x'}$.

Furthermore, the binary pear tree is *clean* if for every non-leaf vertex y of T, there is an end of P_y that is not contained in any Q_x where x is a child of y.

Note that if $(T, \{P_x : x \in V(T)\})$ is a clean binary ear tree, then $(T, \{(P_x, P_x) : x \in V(T)\})$ is a clean binary pear tree. We now prove the following converse.

4.1. *If G has a clean binary pear tree $(T, \mathcal{B})$, then G has a minor H such that H has a clean binary ear tree $(T, \mathcal{P})$.*

Proof. Say $\mathcal{B} = \{(P_v, Q_v) : v \in V(T)\}$. We prove the stronger result that there exist H and $(T, \{P'_v : v \in V(T)\})$ such that H is a minor of G, $(T, \{P'_v : v \in V(T)\})$ is a clean binary ear tree in H, and $P_v \subseteq P'_v$ for all leaves v of T. This last property will be referred to as the *leaf property*; note that this is a property of $(T, \{P'_v : v \in V(T)\})$ w.r.t. the pair $(T, \mathcal{B})$ (which is fixed). Arguing by contradiction, suppose that this result is not true. Among all counterexamples, choose $(G, (T, \mathcal{B}))$ such that $|E(G)|$ is minimum. This clearly implies that $|V(T)| > 1$.

Let y be a deepest leaf in T. If y has a sibling, let z denote this sibling, which is also a leaf of T. Let x be the parent of y in T. Delete from G the internal vertices of Q_y and Q_z (if z exists), and denote by G^- the resulting graph. Note that $|E(G^-)| < |E(G)|$ since Q_y has length at least 2. Let T^- be the tree obtained from T by removing y and z (if z exists). Notice that no internal vertex of Q_y or Q_z appears in a path Q_v with $v \in V(T^-)$, by properties (ii) and (iii) of the definition of binary pear trees. Thus $(T^-, \{(P_v, Q_v) : v \in V(T^-)\})$ is a clean binary pear tree. By minimality, G^- has a minor H^- such that H^- has a clean binary ear tree $(T^-, \{P^-_v : v \in V(T^-)\})$ such that $P_v \subseteq P^-_v$ for all leaves v of T^-. Since x is a leaf of T^-, we have $P_x \subseteq P^-_x$.

Notice that Q_y and Q_z (if z exists) are P^-_x-ears. If z does not exist, then let $P^-_y := Q_y$ and observe that $(T, \{P^-_v : v \in V(T)\})$ is a clean binary ear tree satisfying the leaf property, contradicting the fact that $(G, (T, \mathcal{B}))$ is a counterexample. Thus, z must exist.

Consider an internal vertex v of Q_y. If v is included in Q_z, then v cannot be an end of Q_z, because ends of Q_z are in P_x, which would imply that v is an end of Q_y as well. Thus, if Q_y and Q_z have a vertex in common, either this vertex is a common end of both paths, or it is internal to both paths.

If Q_y and Q_z have no internal vertex in common, let $P^-_y := Q_y$ and $P^-_z := Q_z$. Note that $(T, \{P^-_v : v \in V(T)\})$ is a clean binary ear tree satisfying the leaf property, a contradiction. Hence, Q_y and Q_z must have at least one internal vertex in common.

Next, given an edge $e\in E(G)$ and a path P in G, define $P/\!\!/e$ to be P if $e\notin E(P)$ and P/e if $e\in E(P)$, and let $\mathcal{B}/e:=\{(P_v/\!\!/e,Q_v/\!\!/e)\colon v\in V(T)\}$. Suppose that there is an edge $e\in E(Q_y)\cap E(Q_z)$. Since $|E(P_y)|\geqslant 2$ and $|E(P_z)|\geqslant 2$, property (iii) of the definition of binary pear trees implies that $e\notin E(P_y)\cup E(P_z)$. Thus $P_y/\!\!/e=P_y$ and $P_z/\!\!/e=P_z$. It follows that $(T,\mathcal{B}/e)$ is a clean binary pear tree of G/e, which contradicts the minimality of the counterexample. Hence, no such edge e exists.

So far we established that the two paths Q_y and Q_z have at least one internal vertex in common and are edge-disjoint. The rest of the proof is split into a number of cases. In each case, we show that either there is an edge e of G such that $G\backslash e$ still has a clean binary pear tree which is indexed by the same tree T, or that there is a way to modify $(T,\mathcal{B})$ so that it remains a clean binary pear tree of G, and after the modification the two paths Q_y and Q_z have at least one edge in common. Note that each outcome contradicts the minimality of our counterexample; in the latter case, this is because we can then apply the argument of the previous paragraph and obtain a smaller counterexample.

Let us now proceed with the case analysis, see Figure 8 for an illustration of the different cases. Choose an orientation of P_x from left to right, let x_1 denote its left end and x_2 denote its right end, and let y_1,y_2 and z_1,z_2 be the two ends of respectively Q_y and Q_z on P_x, ordered from left to right. Given two vertices u,v of P_x, let us simply write $u\leqslant v$ if $u=v$ or u is to the left of v on P_x. Without loss of generality, we may assume that $y_1\leqslant z_1$.

Recalling that Q_y and Q_z have an internal vertex in common, let v_1 be the first such vertex on the path Q_y starting from y_1. Note that either $P_y\subseteq y_1Q_yv_1$ or $P_y\subseteq v_1Q_yy_2$, and similarly either $P_z\subseteq z_1Q_zv_1$ or $P_z\subseteq v_1Q_zz_2$, by property (iii) of the definition of binary pear trees.

First suppose that $P_y\subseteq y_1Q_yv_1$ and $P_z\subseteq z_1Q_zv_1$. Let $Q_y^1:=y_1Q_yv_1Q_zz_2$. (The superscript denotes the case number.) It is easily checked that replacing Q_y with Q_y^1 in $(T,\mathcal{B})$ gives another clean binary pear tree of G. Moreover, Q_y^1 and Q_z have the path $v_1Q_zz_2$ in common, which contains at least one edge, as desired.

Next suppose that $P_y\subseteq y_1Q_yv_1$ and $P_z\subseteq v_1Q_zz_2$. We consider whether some internal vertex of the path $v_1Q_zz_1$ is in Q_y. If there is one, let v_2 be the last such vertex that is met when going along Q_y from y_1 to y_2. Let $Q_y^2:=y_1Q_yv_1Q_zv_2Q_yy_2$, and replace Q_y with Q_y^2 in $(T,\mathcal{B})$ as in the previous paragraph. Note that Q_y^2 and Q_z have the path $v_1Q_zv_2$ in common, and thus at least one edge in common, as desired.

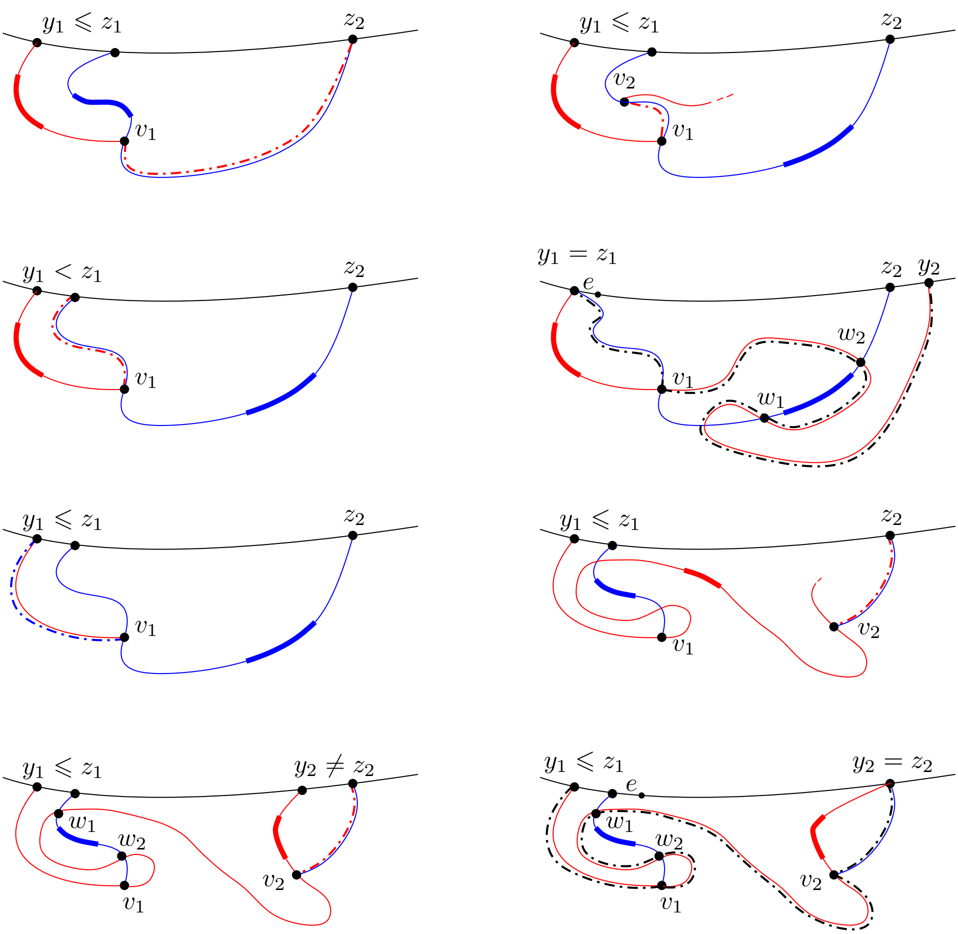

Figure 8. Cases in the proof of 4.1. P_x is drawn in black, Q_y in red, and Q_z in blue. The bold subpaths of Q_y and Q_z denote respectively P_y and P_z. The dotted lines illustrate the modifications of the paths P_x, Q_y, Q_z.

If no internal vertex of $v_1 Q_z z_1$ is in Q_y, we consider whether $y_1 < z_1$ or $y_1 = z_1$. If $y_1 < z_1$, let $Q_y^3 := y_1 Q_y v_1 Q_z z_1$, and replace Q_y with Q_y^3 in $(T, \mathcal{B})$. In particular, Q_y^3 and Q_z now have the path $v_1 Q_z z_1$ in common, and thus at least one edge in common, as desired.

If $y_1 = z_1$, we adopt a different strategy. Let $P_x^4 := x_1 P_x y_1 Q_z v_1 Q_y y_2 P_x x_2$ and let Q_x^4 be the path obtained from Q_x by replacing the P_x section with P_x^4. Let $Q_y^4 := y_1 Q_y v_1$. Let w_1 be the first vertex of Q_y that is met when starting in P_z and walking along Q_z toward z_1. (Note that possibly $w_1 = v_1$.)

Let w_2 be the first vertex of Q_y that is met when starting in P_z and walking along Q_z toward z_2, if there is one. Let $Q_z^4 := w_1 Q_z w_2$ if w_2 exists, otherwise let $Q_z^4 := w_1 Q_z z_2 P_x y_2$. Finally, let e be the edge of P_x incident to z_1 that is to the right of z_1. Observe that e is not included in any of the three paths Q_x^4, Q_y^4, Q_z^4. Now, it can be checked that replacing P_x, Q_x, Q_y, Q_z in $(T, \mathcal{B})$ with their newly defined counterparts produces a clean binary pear tree of $G \backslash e$, giving the desired contradiction. This concludes the case that $P_y \subseteq y_1 Q_y v_1$ and $P_z \subseteq v_1 Q_z z_2$.

Next suppose that $P_y \subseteq v_1 Q_y y_2$ and $P_z \subseteq v_1 Q_z z_2$. Let $Q_z^5 := y_1 Q_y v_1 Q_z z_2$. Replacing Q_z with Q_z^5 in $(T, \mathcal{B})$ gives another clean binary pear tree of G. Moreover, Q_y and Q_z^5 have the path $y_1 Q_y v_1$ in common, which contains at least one edge, as desired.

Finally, suppose that $P_y \subseteq v_1 Q_y y_2$ and $P_z \subseteq z_1 Q_z v_1$. Let v_2 be the first common internal vertex of Q_y and Q_z that is met when starting in z_2 and walking along Q_z toward v_1. (Note that possibly $v_2 = v_1$.) If $P_y \subseteq v_1 Q_y v_2$, then let $Q_y^6 := y_1 Q_y v_2 Q_z z_2$. Replacing Q_y with Q_y^6 in $(T, \mathcal{B})$ gives another clean binary pear tree of G. Moreover, Q_y^6 and Q_z have the path $v_2 Q_z z_2$ in common, which contains at least one edge, as desired.

If $P_y \subseteq v_2 Q_y y_2$, then consider whether $y_2 = z_2$. If $y_2 \neq z_2$ then let $Q_y^7 := y_2 Q_y v_2 Q_z z_2$. Replacing Q_y with Q_y^7 in $(T, \mathcal{B})$ gives another clean binary pear tree of G. Moreover, Q_y^7 and Q_z have the path $v_2 Q_z z_2$ in common, which contains at least one edge, as desired.

If $y_2 = z_2$, then let $P_x^8 := x_1 P_x y_1 Q_y v_2 Q_z z_2 P_x x_2$ and let Q_x^8 be the path obtained from Q_x by replacing the P_x section with P_x^8. Let $Q_y^8 := v_2 Q_y y_2$. Let w_1 be the first vertex of Q_y that is met when starting in P_z and walking along Q_z toward z_1, if there is one. Let w_2 be the first vertex of Q_y that is met when starting in P_z and walking along Q_z toward z_2. (Note that possibly $w_2 = v_1$.) Let $Q_z^8 := w_1 Q_z w_2$ if w_1 exists, otherwise let $Q_z^8 := y_1 P_x z_1 Q_z w_2$. Let e be the edge of P_x incident to z_1 that is to the right of z_1. Observe that e is not included in any of the three paths Q_x^8, Q_y^8, Q_z^8. Now, it can be checked that replacing P_x, Q_x, Q_y, Q_z in $(T, \mathcal{B})$ with their newly defined counterparts produces a clean binary pear tree of $G \backslash e$, giving the desired contradiction. This concludes the proof. ∎

5. Finding binary pear trees

A binary tree is *full* if every internal vertex has exactly two children. The main result of this section is the following.

5.1. *For all integers* $\ell \geqslant 1$ *and* $k \geqslant 9\ell^2 - 3\ell + 1$, *if* G *is a minor-minimal* 2*-connected graph containing a subdivision of* Γ_k *and* T^1 *is a full binary tree of height at most* $3\ell - 2$, *then either* G *contains* Γ_ℓ^+ *as a minor, or* G *contains a clean binary pear tree* $(T^1, \mathcal{B})$.

We proceed via a sequence of lemmas.

5.2. *If* G *is a minor-minimal* 2*-connected graph containing a subdivision of* Γ_k, *then every subdivision of* Γ_k *in* G *is a spanning tree.*

Proof. Let T be a subdivision of Γ_k in G. We use the well-known fact that for all $e \in E(G)$, at least one of $G \backslash e$ or G/e is 2-connected. Therefore, if some edge e of G has an end not in $V(T)$, then $G \backslash e$ or G/e is a 2-connected graph containing a subdivision of Γ_k, which contradicts the minor-minimality of G. ∎

5.3. *Let* $1 \leqslant \ell \leqslant k$ *and let* T *be a tree isomorphic to* Γ_k *with root* r. *Suppose that a non-empty subset of vertices of* T *are marked. Then*

(i) T *contains a subdivision of* Γ_ℓ, *all of whose leaves are marked, or*

(ii) *there exist a vertex* $v \in V(T)$ *and a child* w *of* v *such that* T_v *has at least one marked vertex but* T_w *has none, and* w *is at distance at most* ℓ *from* r.

Proof. A vertex v in T is *good* if there is a marked vertex in T_v, and is *bad* otherwise. Let T' be the subtree of T induced by vertices at distance at most ℓ from r in T. If each leaf of T' is good, then for each such leaf u we can find a marked vertex m_u in T_u, and $T' \cup \bigcup\{uTm_u : u \text{ leaf of } T'\}$ is a Γ_ℓ subdivision with all leaves marked, as required by (i). Now assume that some leaf u of T' is bad. Let w be the bad vertex closest to r on the rTu path. Since some vertex in T is marked, r is good. Thus $w \neq r$. Moreover, the parent v of w is good, by our choice of w. Also, w is at distance at most ℓ from r. Therefore, v and w satisfy (ii). ∎

Our main technical tools are 5.4 and 5.5 below, which are lemmas about 2-connected graphs G containing a subdivision T of Γ_k as a spanning tree. In order to state them, we need to introduce some definitions and notation.

For the next two paragraphs, let G be a 2-connected graph containing a subdivision T of Γ_k as a spanning tree. For each vertex $v \in V(G)$, let $\mathsf{h}(v)$ be the number of original non-leaf vertices on the path vTw, where w is any leaf of T_v. We stress the fact that *subdivision vertices are not counted* when computing $\mathsf{h}(v)$. Since the length of a path in Γ_k from a fixed vertex to any leaf is the same, $\mathsf{h}(v)$ is independent of the choice of w. We also use the shorthand notation $\mathsf{Out}(v) := V(G) \backslash V(T_v)$ when G and T are clear from the context. For $X,Y \subseteq V(G)$, we say that X *sees* Y if $xy \in E(G)$ for some $x \in X$ and $y \in Y$. If P is a path with ends x and y, and Q is a path with ends y and z, then let PQ be the walk that follows P from x to y and then follows Q from y to z.

A path P of G is (x,a,y)-*special* if $|V(P)| \geqslant 3$, and x,y are the ends of P, and a is a child of x such that $V(P) \setminus \{x,y\} \subseteq V(T_a)$ and $y \notin V(T_a)$. A vertex w is *safe* for an (x,a,y)-special path P if w satisfies the following properties:

- the parent v of w is in $V(P) \setminus \{x,y\}$;
- $\mathsf{h}(v) \geqslant \mathsf{h}(x) - 2\ell$;
- $V(P) \cap V(T_w) = \emptyset$;
- $V(T_w)$ does not see $\mathsf{Out}(a) \setminus \{x\}$, and
- if v is an original vertex and u is its child distinct from w, then either $V(P) \cap V(T_u) \neq \emptyset$ or $V(T_u)$ does not see $\mathsf{Out}(a) \setminus \{x\}$.

5.4. *Let $1 \leqslant \ell \leqslant k$. Let G be a minor-minimal 2-connected graph containing a subdivision of Γ_k. Let T be a subdivision of Γ_k in G, $v \in V(T)$ with $\mathsf{h}(v) \geqslant 3\ell + 1$, and w be a child of v. Then, either G contains a Γ_ℓ^+ minor, or there is a (v_0, w_0, v_0')-special path P and two distinct safe vertices for P such that:*

- *$V(P) \subseteq V(T_w)$,*
- *$\mathsf{h}(v_0) \geqslant \mathsf{h}(v) - \ell$,*
- *$V(T_{v_0})$ sees $\mathsf{Out}(w) \setminus \{v\}$,*
- *$V(T_{w_0})$ does not see $\mathsf{Out}(w) \setminus \{v\}$, and*
- *$V(T_{u_0})$ sees $\mathsf{Out}(v_0)$ if v_0 is an original vertex and u_0 is its child distinct from w_0.*

Proof. By 5.2, T is a spanning tree of G. Colour red each vertex of T_w that sees a vertex in $\mathsf{Out}(w) \setminus \{v\}$. Observe that there is at least one red vertex. Indeed, $V(T_w)$ must see $\mathsf{Out}(w) \setminus \{v\}$, for otherwise v would be a cut vertex separating $V(T_w)$ from $\mathsf{Out}(w) \setminus \{v\}$ in G.

Let $\tilde{T}_w$ be the complete binary tree obtained from T_w by iteratively contracting each edge of the form pq with p a subdivision vertex and q the child of p into vertex q. Declare q to be coloured red after the edge contraction if at least one of p,q was coloured red beforehand. Since $\mathsf{h}(w) \geqslant \mathsf{h}(v) - 1 \geqslant 3\ell$, the tree $\tilde{T}_w$ has height at least 3ℓ.

If $\tilde{T}_w$ contains a subdivision of Γ_ℓ with all leaves coloured red, then so does T_w. Therefore, G contains Γ_ℓ^+ as a minor, because $\mathsf{Out}(w)$ induces a connected subgraph of G which is vertex-disjoint from $V(T_w)$ and which sees all the leaves of T_w. Thus, by 5.3, we may assume there is a vertex $\tilde{v}_0$ of $\tilde{T}_w$ and a child $\tilde{w}_0$ of $\tilde{v}_0$ with $\mathsf{h}(\tilde{w}_0) \geqslant \mathsf{h}(w) - \ell$ such that $T_{\tilde{v}_0}$ has at least one red vertex but $T_{\tilde{w}_0}$ has none. Going back to T_w, we deduce that there is a vertex v_0 of T_w and a child w_0 of v_0 with $\mathsf{h}(w_0) \geqslant \mathsf{h}(w) - \ell$ such that T_{v_0} has at least one red vertex but T_{w_0} has none. To see this, choose v_0 as the deepest red vertex in the preimage of $\tilde{v}_0$. Note that v_0 or w_0 could be subdivision vertices.

If v_0 is an original vertex, let u_0 denote the child of v_0 distinct from w_0. Since v_0 is not a cut vertex of G, one of the two subtrees T_{u_0} and T_{w_0} sees $\mathsf{Out}(v_0)$. If T_{u_0} does not see $\mathsf{Out}(v_0)$, then T_{u_0} has no red vertex and T_{w_0} sees $\mathsf{Out}(v_0)$. Therefore, by exchanging u_0 and w_0 if necessary, we guarantee that the following two properties hold when u_0 exists.

(1) $\quad T_{u_0}$ sees $\mathsf{Out}(v_0)$ $\quad$ and $\quad$ T_{w_0} has no red vertex.

We iterate this process in T_{w_0}. Colour blue each vertex of T_{w_0} that sees a vertex in $\mathsf{Out}(w_0)\backslash\{v_0\}$. There is at least one blue vertex, since otherwise v_0 would be a cut vertex of G separating $V(T_{w_0})$ from $\mathsf{Out}(w_0)\backslash\{v_0\}$. Defining $\tilde{T}_{w_0}$ similarly as above, if $\tilde{T}_{w_0}$ contains a subdivision of Γ_ℓ with all leaves coloured blue, then G has a Γ_ℓ^+ minor. Applying 5.3 and going back to T_{w_0}, we may assume there is a vertex v_1 of T_{w_0} and a child w_1 of v_1 with $\mathsf{h}(w_1) \geqslant \mathsf{h}(w_0) - \ell$ such that T_{v_1} has at least one blue vertex but T_{w_1} has none.

We now define the (v_0, w_0, v_0')-special path P, and identify two distinct safe vertices for P. To do so, we will need to consider different cases. In all cases, the end v_0' will be a vertex of $\mathsf{Out}(w_0) \setminus \{v_0\}$ seen by a (carefully chosen) blue vertex in T_{v_1}, thus $v_0' \notin V(T_{w_0})$, and the path P will be such that $V(P) \setminus \{v_0, v_0'\} \subseteq V(T_{w_0})$. Note that the end v_0 of P satisfies $\mathsf{h}(v_0) \geqslant \mathsf{h}(v) - \ell$, as desired.

Before proceeding with the case analysis, we point out the following property of G. If st is an edge of G such that G/st contains a subdivision of Γ_k, then G/st is not 2-connected by the minor-minimality of G, and it follows that $\{s, t\}$ is a cutset of G. Note that this applies if st is an edge of T such that at least one of s, t is a subdivision vertex, or if st is an edge of $E(G)\backslash E(T)$ linking two subdivision vertices of T that are on the same subdivided path of T. This will be used below.

Case 1. v_1 is a subdivision vertex:

In this case, v_1 is the unique blue vertex in T_{v_1}. Let v_0' be a vertex of $\mathsf{Out}(w_0) \setminus \{v_0\}$ seen by the blue vertex v_1. Since v_1 is not a cut vertex of G, there is an edge st with $s \in V(T_{w_1})$ and $t \in \mathsf{Out}(v_1)$. Note that $t \in V(T_{w_0}) \cup \{v_0\}$, since T_{w_1} has no blue vertex.

Case 1.1. There is at least one original vertex on the path v_1Ts:

Let q be the first original vertex on the path v_1Ts. Let s_1 denote a child of q not on the qTs path. Let q' be the first original vertex distinct from q on the qTs path if any, and otherwise let $q' := s$ (note that possibly $q' = q = s$). Let s_2 be a child of q' not on the qTs path, and distinct from s_1 if $q' = q$. As illustrated in Figure 9, define

$$P := v_0 T t s T v_1 v_0'.$$

Observe that $V(P) \setminus \{v_0, v_0'\} \subseteq V(T_{w_0})$, by construction. Observe also that the parent q' of s_2 satisfies $\mathsf{h}(q') \geqslant \mathsf{h}(q) - 1 = \mathsf{h}(v_1) - 1 \geqslant \mathsf{h}(v_0) - \ell - 1 \geqslant \mathsf{h}(v_0) - 2\ell$. It can be checked that s_1, s_2 are two distinct safe vertices for P, as desired.

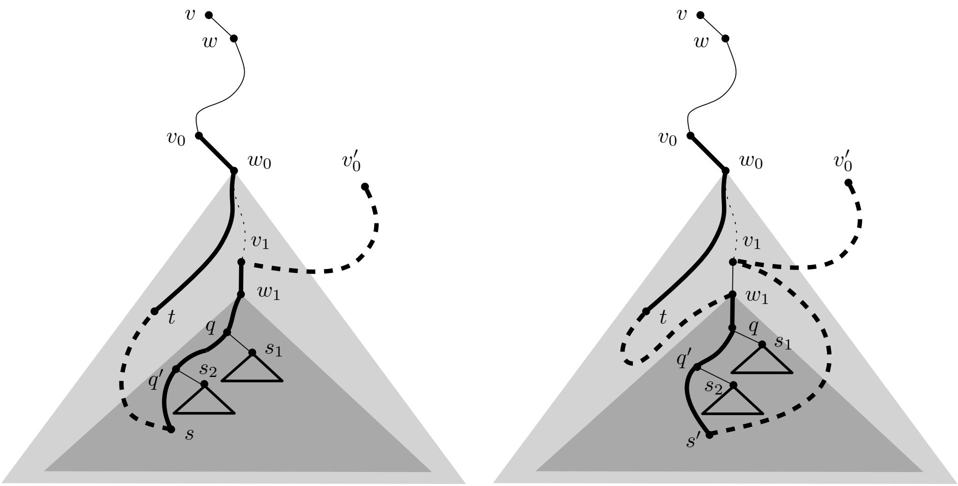

Figure 9. Path P and the safe vertices s_1, s_2. Cases 1.1 and 1.2.

Case 1.2. All vertices of the path $v_1 T s$ are subdivision vertices:

In particular, w_1 is a subdivision vertex. We show that the unique child q of w_1 is an original vertex, and therefore $s = w_1$. Indeed, assume not, and let q' denote the child of q. Since v_1 is not a cut vertex of G but $\{v_1, w_1\}$ is a cutset of G, we deduce that w_1 sees a vertex w_1' in $\mathsf{Out}(v_1)$ and that $V(T_q)$ does not see $\mathsf{Out}(v_1)$. Similarly, because w_1 is not a cut vertex of G but $\{w_1, q\}$ is a cutset of G, we deduce that $qv_1 \in E(G)$ and that $V(T_{q'})$ does not see $\mathsf{Out}(w_1)$. Since q is not a cut vertex, some vertex $q'' \in V(T_{q'})$ sees $\mathsf{Out}(q)$, and hence sees w_1 (since $V(T_{q'})$ does not see $\mathsf{Out}(v_1)$). But then, because of the edges $q''w_1$ and $w_1 w_1'$, we see that $\{v_1, q\}$ cannot be a cutset of G. It follows that $G/v_1 q$ is 2-connected and contains a Γ_k minor, contradicting our assumption on G.

Hence, q is an original vertex, and $s = w_1$. Since w_1 is not a cut vertex of G, there is an edge linking $V(T_q)$ to $\mathsf{Out}(w_1)$. Since $\{v_1, w_1\}$ is a cutset of G, this edge links some vertex $s' \in V(T_q)$ to v_1.

Let s_1 denote a child of q not on the qTs' path. Let q' be the first original vertex distinct from q on the qTs' path if any, and otherwise let $q' := s'$ (note that possibly $q' = s' = q$). Let s_2 be a child of q' not on the qTs' path, and distinct from s_1 if $q' = q$. As illustrated in Figure 9, define

$$P := v_0 T t w_1 T s' v_1 v_0'.$$

Again, note that $V(P) \setminus \{v_0, v_0'\} \subseteq V(T_{w_0})$ by construction. Observe also that the parent q' of s_2 satisfies $\mathsf{h}(q') \geqslant \mathsf{h}(q) - 1 = \mathsf{h}(v_1) - 1 \geqslant \mathsf{h}(v_0) - \ell - 1 \geqslant \mathsf{h}(v_0) - 2\ell$. It is easy to see that s_1, s_2 are two distinct safe vertices for P, as desired.

Case 2. v_1 is an original vertex:

Let u_1 denote the child of v_1 distinct from w_1. If T_{u_1} has no blue vertex, then v_1 is the unique blue vertex in T_{v_1}. Let v_0' be a vertex of $\mathsf{Out}(w_0) \setminus \{v_0\}$ seen by the blue vertex v_1. Define

$$P := v_0 T v_1 v_0'.$$

Clearly, $V(P) \setminus \{v_0, v_0'\} \subseteq V(T_{w_0})$, and u_1, w_1 are two distinct safe vertices for P.

Next, assume that T_{u_1} has a blue vertex. In this case, we need to define an extra pair v_2, w_2 of vertices. Observe that $\mathsf{h}(u_1) \geqslant \mathsf{h}(w_0) - \ell \geqslant \mathsf{h}(w) - 2\ell = \mathsf{h}(v) - 2\ell - 1 \geqslant \ell$. Let $\tilde{T}_{u_1}$ be the tree obtained from T_{u_1}, as before. Again, if $\tilde{T}_{u_1}$ contains a subdivision of $\varGamma_\ell$ all of whose leaves are blue, then G contains an $\varGamma_\ell^+$ minor. Thus, by 5.3, we may assume there is a vertex v_2 of T_{u_1} and a child w_2 of v_2 with $\mathsf{h}(w_2) \geqslant \mathsf{h}(u_1) - \ell = \mathsf{h}(w_1) - \ell$ such that T_{v_2} has at least one blue vertex, but T_{w_2} has none.

Case 2.1. v_2 is a subdivision vertex:

Here, v_2 is the unique blue vertex in T_{v_2}. Let v_0' be a vertex of $\mathsf{Out}(w_0) \setminus \{v_0\}$ seen by v_2. As illustrated in Figure 10, define

$$P := v_0 T v_2 v_0'.$$

Observe that $V(P) \setminus \{v_0, v_0'\} \subseteq V(T_{w_0})$ by construction, and that w_1, w_2 are two distinct safe vertices for P.

Case 2.2. v_2 is an original vertex:

Let u_2 be the child of v_2 distinct from w_2. Let b_2 denote a blue vertex in $V(T_{u_2}) \cup \{v_2\}$, distinct from v_2 if possible. Let v_0' be a vertex of $\mathsf{Out}(w_0) \setminus \{v_0\}$ seen by the blue vertex b_2. Define

$$P := v_0 T b_2 v_0'.$$

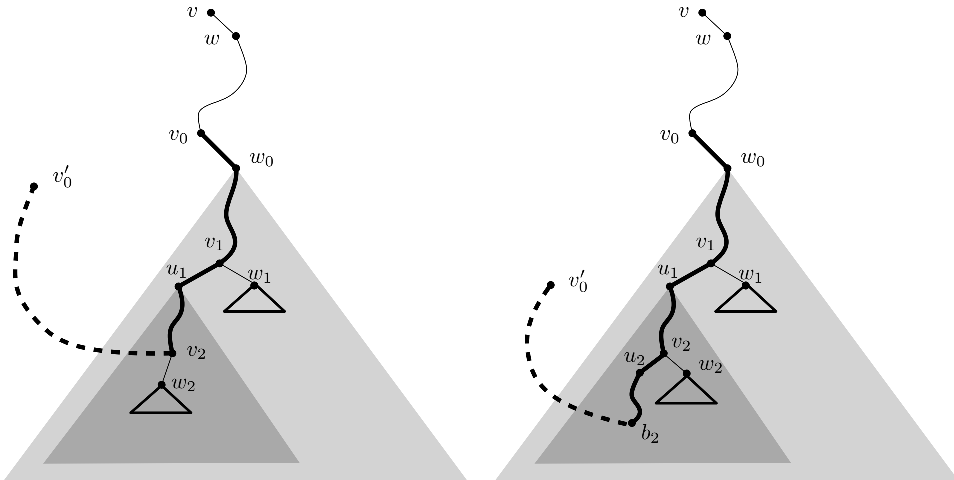

Figure 10. Path P and the safe vertices w_1, w_2. Cases 2.1 and 2.2.

Again, $V(P) \setminus \{v_0, v_0'\} \subseteq V(T_{w_0})$ by construction.

If $b_2 \neq v_2$, then P intersects $V(T_{u_2})$. If $b_2 = v_2$, then P avoids $V(T_{u_2})$, and $V(T_{u_2})$ has no blue vertex. That is, $V(T_{u_2})$ does not see $\mathsf{Out}(w_0) \setminus \{v_0\}$. Using these observations, one can check that w_1, w_2 are two distinct safe vertices for P in both cases; see Figure 10. ∎

5.5. *Let $1 \leqslant \ell \leqslant k$. Let G be a minor-minimal 2-connected graph containing a subdivision of Γ_k and let T be a subdivision of Γ_k in G. Let S be an (x, a, y)-special path with $\mathsf{h}(x) \geqslant 5\ell + 1$. Let w be a safe vertex for S and let $v \in V(S)$ denote the parent of w in T. Then, either G contains a Γ_ℓ^+ minor, or there is a (v_0, w_0, v_0')-special path P, two distinct safe vertices w_1, w_2 for P, and an S-ear Q such that:*

(a) $V(P) \subseteq V(T_w)$,
(b) $\mathsf{h}(v_0) \geqslant \mathsf{h}(x) - 3\ell$,
(c) *$V(T_{w_0})$ does not see $\mathsf{Out}(w) \setminus \{v\}$,*
(d) $P \subseteq Q$,
(e) $V(Q) \setminus V(P) \subseteq \mathsf{Out}(w_0) \setminus \{v_0\}$,
(f) $V(Q) \subseteq V(T_a) \cup \{x\}$,
(g) *$V(Q) \cap V(T_{w_i}) = \emptyset$ for $i = 1, 2$, and*
(h) *if $e \in E(Q) \setminus E(T)$ and no end of e is in $V(T_w)$, then v is an original vertex with children u, w, the path S is disjoint from $V(T_u)$, and e links $V(T_u)$ to $\mathsf{Out}(v)$.*

Proof. By 5.2, T is a spanning tree. Also, G does not contain Γ_ℓ^+ as a minor (otherwise, we are done). Applying 5.4 on vertex v and its child w, we obtain a (v_0, w_0, v_0')-special path P and two distinct safe vertices w_1, w_2 for P such that $V(P) \subseteq V(T_w)$; $\mathsf{h}(v_0) \geqslant \mathsf{h}(v) - \ell \geqslant \mathsf{h}(x) - 3\ell$; $V(T_{v_0})$ sees $\mathsf{Out}(w) \setminus \{v\}$; $V(T_{w_0})$ does not see $\mathsf{Out}(w) \setminus \{v\}$; and if v_0 is an original vertex and u_0 is the child of v_0 distinct from w_0, then $V(T_{u_0})$ sees $\mathsf{Out}(v_0)$. It remains to extend P into an S-ear Q satisfying properties (d)–(h). The proof is split into twelve cases, all of which are illustrated in Figure 11.

If v is an original vertex, let u denote the child of v distinct from w. In order to simplify the arguments below, we let $V(T_u)$ be the empty set if u is not defined (same for u_0).

First assume that $v_0' \notin V(T_{u_0})$. Then $v_0' \in \mathsf{Out}(v_0) \cap V(T_w)$. Recall that $V(T_{v_0}) \setminus V(T_{w_0}) = V(T_{u_0}) \cup \{v_0\}$ sees $\mathsf{Out}(w) \setminus \{v\} = V(T_u) \cup \mathsf{Out}(v)$. Suppose that there is an edge $st \in E(G)$ with $s \in V(T_{u_0}) \cup \{v_0\}$ and $t \in \mathsf{Out}(v)$. Note that $t \in V(T_a) \cup \{x\}$, since w is a safe vertex for S. Let v' be the closest ancestor of t in T that lies on S. Note that $v' \in V(T_a) \cup \{x\}$. Define

$$Q_1 := vTv_0'Pv_0TstTv'.$$

Next, suppose that there is no such edge st. Then, there must be an edge st with $s \in V(T_{u_0}) \cup \{v_0\}$ and $t \in V(T_u)$. In particular, u exists. If the path S intersects $V(T_u)$, then let v' be a vertex in $V(S) \cap V(T_u)$ that is closest to t in T. Define

$$Q_2 := vTv_0'Pv_0TstTv'.$$

Otherwise, we have $V(S) \cap V(T_u) = \emptyset$. Since w is a safe vertex for S, $V(T_u)$ does not see $\mathsf{Out}(a) \setminus \{x\}$ in this case. If $V(T_u)$ sees $\mathsf{Out}(v)$, then let $s't'$ be an edge with $s' \in V(T_u)$ and $t' \in \mathsf{Out}(v)$, and let v' be the closest ancestor of t' in T that lies on S. Note that both t' and v' lie in $V(T_a) \cup \{x\}$. Define

$$Q_3 := vTv_0'Pv_0TstTs't'Tv'.$$

Otherwise, $V(T_u)$ does not see $\mathsf{Out}(v)$. Since v is not a cut vertex in G, we deduce that $V(T_w)$ sees $\mathsf{Out}(v)$. As we already know that neither $V(T_{w_0})$ nor $V(T_{u_0}) \cup \{v_0\}$ sees $\mathsf{Out}(v)$, there is an edge $s''t'' \in E(G)$ with $s'' \in V(T_w) \setminus V(T_{v_0})$ and $t'' \in \mathsf{Out}(v)$. Again, since w is safe for S, we know that $t'' \in V(T_a) \cup \{x\}$. Let v' be the closest ancestor of t'' in T that lies on S. Note that $v' \in V(T_a) \cup \{x\}$. Define

$$Q_4 := vTtsTv_0Pv_0'Ts''t''Tv'.$$

Next, assume that $v_0' \in V(T_{u_0})$. In particular, u_0 exists. Recall that $V(T_{u_0})$ sees $\mathsf{Out}(v_0)$. If $V(T_{u_0})$ sees $\mathsf{Out}(v)$, then let st be an edge with $s \in V(T_{u_0})$

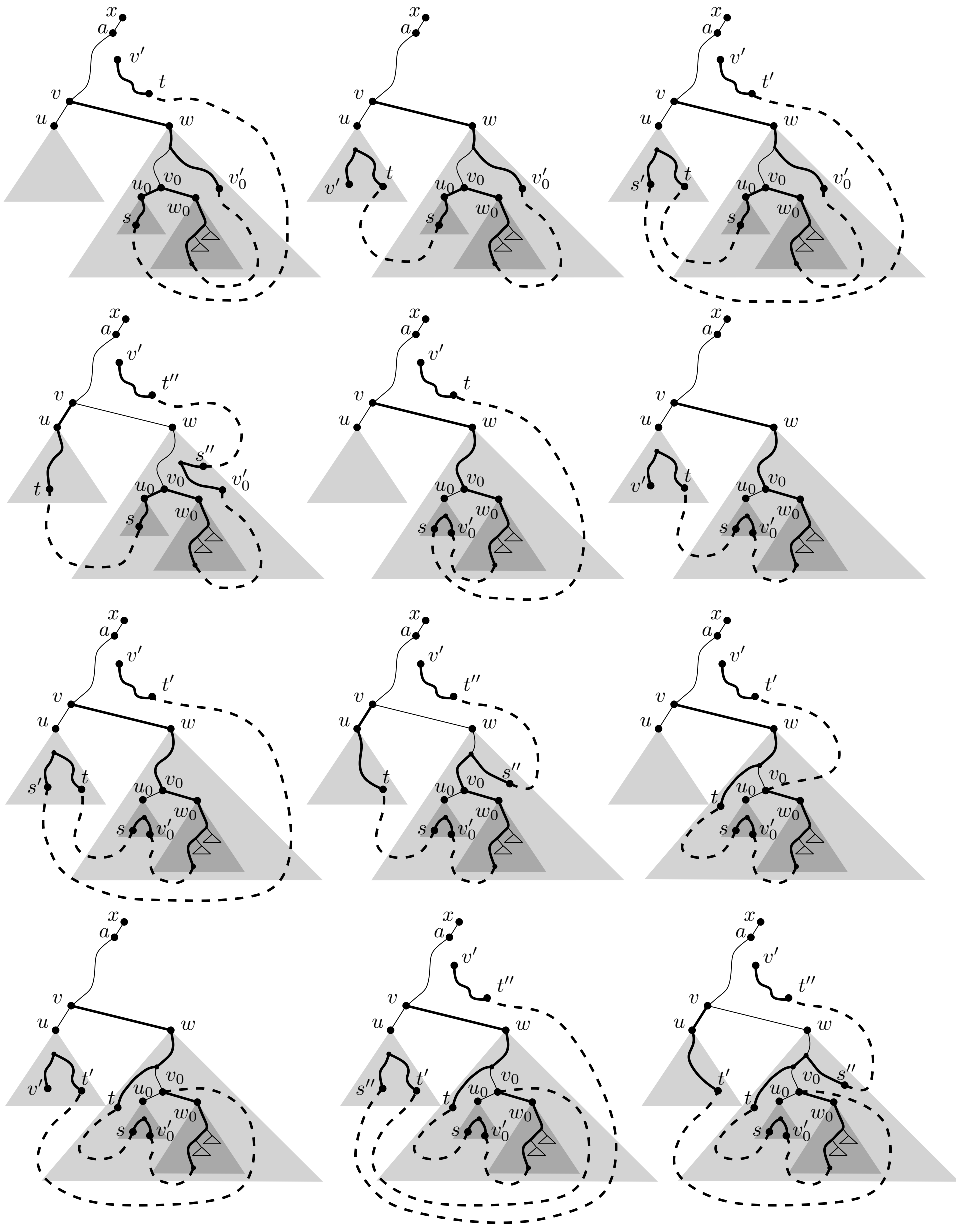

Figure 11. Definition of S-ears $Q_1,\ldots,Q_{12}$

and $t \in \mathsf{Out}(v)$. Observe that $t \in V(T_a) \cup \{x\}$ since w is safe for S. Let v' be the closest ancestor of t in T that lies on S. Note that $v' \in V(T_a) \cup \{x\}$ as well. Define

$$Q_5 := vTv_0Pv_0'TstTv'.$$

Next, suppose that $V(T_{u_0})$ does not see $\mathsf{Out}(v)$. If $V(T_{u_0})$ sees $V(T_u)$, then let st be an edge with $s \in V(T_{u_0})$ and $t \in V(T_u)$. In particular, u exists. If S intersects $V(T_u)$, then let v' be a vertex in $V(S) \cap V(T_u)$ that is closest to t in T. Define

$$Q_6 := vTv_0Pv_0'TstTv'.$$

Otherwise, we have $V(S) \cap V(T_u) = \emptyset$. Since w is a safe vertex for S, $V(T_u)$ does not see $\mathsf{Out}(a) \setminus \{x\}$ in this case. If $V(T_u)$ sees $\mathsf{Out}(v)$, then let $s't'$ be an edge with $s' \in V(T_u)$ and $t' \in \mathsf{Out}(v)$ and let v' be the closest ancestor of t' in T that lies on S. Note that both t' and v' lie in $V(T_a) \cup \{x\}$. Define

$$Q_7 := vTv_0Pv_0'TstTs't'Tv'.$$

Next, suppose that $V(T_u)$ does not see $\mathsf{Out}(v)$. Since v is not a cut vertex in G, we deduce that $V(T_w)$ sees $\mathsf{Out}(v)$. As we already know that neither $V(T_{w_0})$ nor $V(T_{u_0})$ sees $\mathsf{Out}(v)$, there is an edge $s''t'' \in E(G)$ with $s'' \in (V(T_w) \setminus V(T_{v_0})) \cup \{v_0\}$ and $t'' \in \mathsf{Out}(v)$. Again, since w is safe for S, $t'' \in V(T_a) \cup \{x\}$. Let v' be the closest ancestor of t'' in T that lies on S. Note that $v' \in V(T_a) \cup \{x\}$. Define

$$Q_8 := vTtsTv_0'Pv_0Ts''t''Tv'.$$

We are done with the cases where $V(T_{u_0})$ sees $\mathsf{Out}(v)$ or $V(T_u)$. Next, assume that $V(T_{u_0})$ sees neither of these two sets. Since $V(T_{u_0})$ sees $\mathsf{Out}(v_0)$, there is an edge st with $s \in V(T_{u_0})$ and $t \in V(T_w) \setminus V(T_{v_0})$. Recall that $V(T_{v_0})$ sees $\mathsf{Out}(w) \setminus \{v\}$. Since neither $V(T_{u_0})$ nor $V(T_{w_0})$ sees $\mathsf{Out}(w) \setminus \{v\}$, we conclude that v_0 sees $\mathsf{Out}(w) \setminus \{v\}$. If v_0 sees $\mathsf{Out}(v)$, then let v_0t' be an edge with $t' \in \mathsf{Out}(v)$. Let v' be the closest ancestor of t' in T. As before, $\{t', v'\} \subseteq V(T_a) \cup \{x\}$. Define

$$Q_9 := vTtsTv_0'Pv_0t'Tv'.$$

Otherwise, v_0 sees $V(T_u)$. Let v_0t' be an edge with $t' \in V(T_u)$. If S intersects $V(T_u)$, then let v' be a vertex in $V(S) \cap V(T_u)$ that is closest to t' in T. Define

$$Q_{10} := vTtsTv_0'Pv_0t'Tv'.$$

Otherwise, $V(S) \cap V(T_u) = \emptyset$. Since w is a safe vertex for S, we know that $V(T_u)$ does not see $\mathsf{Out}(a) \setminus \{x\}$ in this case. If $V(T_u)$ sees $\mathsf{Out}(v)$, then let $s''t''$ be an edge with $s'' \in V(T_u)$ and $t'' \in \mathsf{Out}(v)$ and let v' be the closest ancestor of t'' in T that lies on S. Note that both t'' and v' lie in $V(T_a)\cup\{x\}$. Define

$$Q_{11} := vTtsTv_0'Pv_0t'Ts''t''Tv'.$$

Otherwise, $V(T_u)$ does not see $\mathsf{Out}(v)$. Since v is not a cut vertex in G, we deduce that $V(T_w)$ sees $\mathsf{Out}(v)$. As we already know that neither $V(T_{w_0})$ nor $V(T_{u_0})\cup\{v_0\}$ sees $\mathsf{Out}(v)$, there is an edge $s''t''\in E(G)$ with $s''\in V(T_w)\setminus V(T_{v_0})$ and $t''\in\mathsf{Out}(v)$. Again, since w is safe for S, $t''\in V(T_a)\cup\{x\}$. Let v' be the closest ancestor of t'' in T that lies on S. Note that $v'\in V(T_a)\cup\{x\}$. Define

$$Q_{12} := vTt'v_0Pv_0'TstTs''t''Tv'.$$

One can check that for all $i\in[12]$, if we set $Q=Q_i$, then Q is an S-ear satisfying properties (d)–(h). ∎

We now prove 5.1 using 5.4 and 5.5.

Proof of 5.1. Let T be a subdivision of Γ_k in G, which is a spanning tree of G by 5.2. Also, G has no Γ_ℓ^+ minor (otherwise, we are done). As before, for $v\in V(G)$, we let $\mathsf{h}(v)$ be the number of original non-leaf vertices on the path vTw, where w is any leaf of T_v. The *depth* of $x\in V(T^1)$, denoted $\mathsf{d}(x)$, is the number of edges in xT^1r, where r is the root of T^1.

We prove the stronger statement that G contains a clean binary pear tree $(T^1,\{(P_x,Q_x)\colon x\in V(T^1)\})$ such that:

(1) for all $x\in V(T^1)$, the path P_x is a (v_x,w_x,v_x')-special path for some vertices v_x,w_x,v_x' of G such that $\mathsf{h}(v_x)\geqslant k-3\ell\mathsf{d}(x)-\ell$, and P_x has two distinguished safe vertices; moreover, if x is not a leaf we associate these safe vertices with the two children y,z of x and denote them s_{xy} and s_{xz};

(2) for all $x,y\in V(T^1)$, v_x is an ancestor of v_y in T if and only if x is an ancestor of y in T^1;

(3) for all $x,y\in V(T^1)$ such that y is a child of x, the paths P_y and Q_y are obtained by applying 5.5 on P_x with safe vertex s_{xy};

(4) for all $y,z\in V(T^1)$ such that y and z are siblings, no vertex of Q_z meets T_{w_y}, and no vertex of Q_y meets T_{w_z};

(5) for all leaves x of T^1, $V(T_{w_x})$ and $\bigcup_{p\in V(T^1)\setminus\{x\}}V(Q_p)$ are disjoint.

The proof is by induction on $|V(T^1)|$. For the base case $|V(T^1)|=1$, the tree T^1 is a single vertex x. Applying 5.4 with v the root of T and w a child of v in T, we obtain a (v_x,w_x,v_x')-special path P_x and two distinct safe vertices for P_x. Let $Q_x:=P_x$. Then $(T^1,\{(P_x,Q_x)\})$ is a binary pear tree in G. Observe that $\mathsf{d}(x)=0$ and $\mathsf{h}(v_x) \geqslant \mathsf{h}(v) - \ell = k - \ell$, thus (1) holds. Properties (2)–(5) hold vacuously since x is the only vertex of T^1.

Next, for the inductive case, assume $|V(T^1)| > 1$. Let x be a vertex of T^1 with two children y, z that are leaves of T^1. Applying induction on the binary tree $T^1 - \{y, z\}$, we obtain a binary pear tree $(T^1 - \{y, z\}, \{(P_p, Q_p) : p \in V(T^1 - \{y, z\})\})$ in G satisfying the claim.

Note that $\mathsf{d}(x) \leqslant 3\ell - 3$, and thus $\mathsf{h}(v_x) \geqslant k - 3\ell\mathsf{d}(x) - \ell \geqslant (9\ell^2 - 3\ell + 1) - 3\ell(3\ell - 3) - \ell \geqslant 5\ell + 1$. By (1), the path P_x comes with two distinguished safe vertices. Considering now the two children y, z of x in the tree T, we associate these safe vertices to y and z, as expected, and denote them s_{xy} and s_{xz}. Let v_{xy} and v_{xz} denote their respective parents in T. First, apply 5.5 with the path P_x and safe vertex s_{xy}, giving a (v_y, w_y, v'_y)-special path P_y with two distinct safe vertices, and a P_x-ear Q_y satisfying the properties of 5.5. Next, apply 5.5 with the path P_x and safe vertex s_{xz}, giving a (v_z, w_z, v'_z)-special path P_z with two distinct safe vertices, and a P_x-ear Q_z satisfying the properties of 5.5.

Observe that, by property (b) of 5.5, $\mathsf{h}(v_y) \geqslant \mathsf{h}(v_x) - 3\ell \geqslant k - 3\ell\mathsf{d}(x) - 4\ell = k - 3\ell\mathsf{d}(y) - \ell$, and similarly $\mathsf{h}(v_z) \geqslant k - 3\ell\mathsf{d}(z) - \ell$. Thus, property (1) is satisfied. Clearly, property (2) and property (3) are satisfied as well. To establish property (4), it only remains to show that no vertex of Q_z meets T_{w_y}, and that no vertex of Q_y meets T_{w_z}. By symmetry it is enough to show the former, which we do now.

Arguing by contradiction, assume that Q_z meets T_{w_y}. Since $V(T_{w_y}) \subseteq V(T_{s_{xy}})$ and $V(Q_x) \cap V(T_{s_{xy}}) = \emptyset$ (by property (g) of 5.5), and since the two ends of Q_z are on Q_x, we see that the two ends of Q_z are outside $V(T_{w_y})$. Thus, at least two edges of Q_z have exactly one end in $V(T_{w_y})$, and there is an edge st which is not an edge of T (i.e. $st \neq v_y w_y$). By symmetry, $s \in V(T_{w_y})$ and $t \notin V(T_{w_y})$.

Clearly, $s \notin V(T_{s_{xz}})$ since $V(T_{w_y}) \subseteq V(T_{s_{xy}})$, and $V(T_{s_{xy}}) \cap V(T_{s_{xz}}) = \emptyset$. Moreover, $t \notin V(T_{s_{xz}})$, since $V(T_{s_{xz}}) \subseteq \mathsf{Out}(s_{xy}) \setminus \{v_{xy}\}$ and since $V(T_{w_y})$ does not see $\mathsf{Out}(s_{xy}) \setminus \{v_{xy}\}$ by property (c) of 5.5. Since st is an edge of Q_z not in T with neither of its ends in $V(T_{s_{xz}})$, it follows from property (h) of 5.5 that v_{xz} is an original vertex with children u_{xz} and s_{xz}; the path P_x avoids $V(T_{u_{xz}})$; and the edge st has one end in $V(T_{u_{xz}})$ and the other in $\mathsf{Out}(v_{xz})$. (We remark that we do not know which end is in which set at this point.)

First, suppose $s_{xy} = u_{xz}$. Then $v_{xy} = v_{xz}$. Since $s \in V(T_{w_y}) \subseteq V(T_{s_{xy}})$ and $s_{xy} = u_{xz}$, we deduce that $s \in V(T_{u_{xz}})$ and $t \in \mathsf{Out}(v_{xz})$ in this case. However, $V(T_{w_y})$ does not see $\mathsf{Out}(s_{xy}) \setminus \{v_{xy}\}$ (by property (c) of 5.5), and $t \in \mathsf{Out}(v_{xz}) \subseteq \mathsf{Out}(u_{xz}) \setminus \{v_{xz}\} = \mathsf{Out}(s_{xy}) \setminus \{v_{xy}\}$, a contradiction.

Next, assume that $s_{xy} \neq u_{xz}$. Then $s_{xy} \notin V(T_{u_{xz}})$, because the parent v_{xy} of s_{xy} is on the path P_x, and P_x avoids $V(T_{u_{xz}})$. Since $s_{xy} \notin V(T_{s_{xz}})$ and $s_{xy} \neq v_{xz}$, it follows that $s_{xy} \in \mathsf{Out}(v_{xz})$. Since $s \in V(T_{w_y}) \subseteq V(T_{s_{xy}})$ and since s_{xy} is not an ancestor of v_{xz} (otherwise $V(T_{s_{xy}})$ would contain v_{xz}, which is on the path P_x), we deduce that $V(T_{s_{xy}}) \subseteq \mathsf{Out}(v_{xz})$, and thus $s \in \mathsf{Out}(v_{xz})$. It then follows that $t \in V(T_{u_{xz}})$. Observe that u_{xz} is neither an ancestor of v_{xy} (otherwise $V(T_{u_{xz}})$ would contain v_{xy}, which is on the path P_x) nor a descendant of s_{xy} (otherwise $V(T_{s_{xy}})$ would contain v_{xz} since $u_{xz} \neq s_{xy}$, which is a vertex of P_x). Hence, we deduce that $V(T_{u_{xz}}) \subseteq \mathsf{Out}(s_{xy}) \setminus \{v_{xy}\}$. However, the edge st then contradicts the fact that $V(T_{w_y})$ does not see $\mathsf{Out}(s_{xy}) \setminus \{v_{xy}\}$ (c.f. property (c) of 5.5). Therefore, $V(Q_z) \cap V(T_{w_y}) = \emptyset$, as claimed. Property (4) follows.

We now verify property (5). First, we show (5) holds for the leaf y of T^1. Note that $V(T_{w_y}) \subseteq V(T_{s_{xy}}) \subseteq V(T_{w_x})$. Thus, $V(T_{w_y})$ and $\bigcup_{p \in V(T^1) \setminus \{x,y,z\}} V(Q_p)$ are disjoint by induction and property (5) for the leaf x of $T^1 - \{y,z\}$. Since $V(T_{w_y}) \subseteq V(T_{s_{xy}})$ and $V(T_{s_{xy}}) \cap V(Q_x) = \emptyset$ (by property (g) of 5.5), we deduce that $V(T_{w_y}) \cap V(Q_x) = \emptyset$. Moreover, $V(T_{w_y}) \cap V(Q_z) = \emptyset$, by property (4) shown above. This proves property (5) for the leaf y of T^1, and also for the leaf z by symmetry.

Every other leaf q of T^1 is also a leaf in $T^1 - \{y,z\}$. By induction, $V(T_{w_q})$ and $\bigcup_{p \in V(T^1) \setminus \{q,y,z\}} V(Q_p)$ are disjoint. Moreover, $V(T_{v_q})$ and $V(T_{v_x})$ are disjoint, by property (2). Since $V(Q_y)$ and $V(Q_z)$ are contained in $V(T_{v_x})$ (by property (f) of 5.5) and $V(T_{w_q}) \subseteq V(T_{v_q})$, it follows that $V(T_{w_q})$ and $V(Q_y) \cup V(Q_z)$ are also disjoint. Property (5) follows.

To conclude the proof, it only remains to verify that $(T^1, \{(P_p, Q_p) \colon p \in V(T^1)\})$ is a binary pear tree in G, and that it is clean. Recall that $(T^1 - \{y,z\}, \{(P_p, Q_p) \colon p \in V(T^1 - \{y,z\})\})$ is a binary pear tree, by induction. By construction, $P_y \subseteq Q_y$ and $P_z \subseteq Q_z$, P_y and P_z each have length at least 2, and both are P_x-ears. Clearly, property (i) of the definition of binary pear trees holds. Property (ii) holds vacuously, since T^1 is a full binary tree, and thus every non-root vertex of T^1 has a sibling. Hence, it only remains to show that property (iii) holds.

Let p be a non-root vertex of T^1, and let p' denote its sibling. First we want to show that no internal vertex of Q_p is in $\bigcup_{q \in V(T^1) \setminus (V(T^1_p) \cup V(T^1_{p'}))} V(Q_q)$.

If p is an ancestor of x in T^1 (including x), then this holds thanks to property (iii) of the binary pear tree $(T^1 - \{y,z\}, \{(P_q, Q_q) \colon q \in V(T^1 - \{y,z\})\})$.

Next, suppose p is not an ancestor of x in T^1 and p is not y nor z. Then we already know that no internal vertex of Q_p is in

$\bigcup_{q \in V(T^1-\{y,z\}) \setminus (V(T^1_p) \cup V(T^1_{p'}))} V(Q_q)$, again by property (iii) of the binary pear tree $(T^1-\{y,z\}, \{(P_q,Q_q): q \in V(T^1-\{y,z\})\})$. Thus it only remains to show that if some internal vertex of Q_p is in Q_y, then y is a descendant of p or of p', and that the same holds for Q_z. By symmetry, it is enough to prove this for Q_y. So let us assume that some internal vertex of Q_p is in Q_y. Note that $V(Q_y) \subseteq V(T_{w_x}) \cup \{v_x\}$, by property (f) of 5.5. By property (5) of the inductive statement, $V(T_{w_x})$ is disjoint from $V(Q_p)$. Thus, the only vertex that the paths Q_p and Q_y can have in common is v_x. Since v_x is an internal vertex of Q_p (by our assumption) and since $v_x \in V(Q_x)$, from property (iii) of the binary pear tree $(T^1-\{y,z\}, \{(P_q,Q_q): q \in V(T^1-\{y,z\})\})$ we deduce that x is a descendant of p or p', and hence so is y, as desired.

Finally, consider the case where p is y or z, say y. Recall that $V(Q_y) \subseteq V(T_{w_x}) \cup \{v_x\}$. Note also that v_x cannot be an internal vertex of Q_y, since $v_x \in V(P_x)$ and Q_y is a P_x-ear. Hence, all internal vertices of Q_y are in $V(T_{w_x})$. Since $V(T_{w_x})$ and $V(Q_q)$ are disjoint for all $q \in V(T^1) \setminus \{x,y,z\}$ (by induction, using property (5) on the leaf x of $T^1 - \{y,z\}$). Thus, it only remains to show that no internal vertex of Q_y is in Q_x. This is the case, because Q_y is a P_x-ear, and $V(Q_x) \setminus V(P_x) \subseteq \mathsf{Out}(w_x) \setminus \{v_x\}$ (by property (e) of 5.5).

To establish property (iii), it remains to show that no internal vertex of P_p is in $Q_{p'}$, for every two siblings p,p' of T^1. If $\{p,p'\} \neq \{y,z\}$, this is true by property (iii) of the binary pear tree $(T^1 - \{y,z\}, \{(P_q,Q_q): q \in V(T^1 - \{y,z\})\})$. Thus by symmetry, it is enough to show that no internal vertex of P_y is in Q_z. This holds because all internal vertices of P_y are in $V(T_{w_y})$ (since P_y is a (v_y, w_y, v'_y)-special path) and $V(Q_z) \cap V(T_{w_y}) = \emptyset$ by (4).

This concludes the proof that $(T^1, \{(P_p,Q_p): p \in V(T^1)\})$ is a binary pear tree. Finally, note that it is clean because the binary pear tree $(T^1 - \{y,z\}, \{(P_q,Q_q): q \in V(T^1 - \{y,z\})\})$ is clean (by induction), and the end v'_x of P_x is not in Q_y, since $V(Q_y) \subseteq V(T_{w_x}) \cup \{v_x\}$ (by property (f) of 5.5), and since $v'_x \notin V(T_{w_x}) \cup \{v_x\}$, and similarly v'_x is not in Q_z either. ∎

6. Proof of main theorems

We have the following quantitative version of 1.4.

6.1. *For all integers $\ell \geqslant 1$ and $k \geqslant 9\ell^2 - 3\ell + 1$, every 2-connected graph G with a Γ_k minor contains Γ^+_ℓ or ∇_ℓ as a minor.*

Proof. Among all 2-connected graphs containing Γ_k as a minor, but containing neither Γ^+_ℓ nor ∇_ℓ as a minor, choose G with $|E(G)|$ minimum. Since

$\mathit{\Gamma}_k$ has maximum degree 3, G contains a subdivision of $\mathit{\Gamma}_k$. Therefore, G is a minor-minimal 2-connected graph containing a subdivision of $\mathit{\Gamma}_k$. By 5.1, G has a clean binary pear tree $(T^1,\mathcal{B})$, with $T^1 \simeq \mathit{\Gamma}_{3\ell-2}$. By 4.1, G has a minor H such that H has a clean binary ear tree $(T^1,\mathcal{P})$, with $T^1 \simeq \mathit{\Gamma}_{3\ell-2}$. By 3.1, H contains $\mathit{\Gamma}_\ell^+$ or ∇_ℓ as a minor, and hence so does G. ∎

We have the following quantitative version of 1.3.

6.2. *For every integer $\ell \geqslant 1$, every 2-connected graph G of pathwidth at least $2^{9\ell^2-3\ell+2}-2$ contains $\mathit{\Gamma}_\ell^+$ or ∇_ℓ as a minor.*

Proof. As mentioned in Section 1, Bienstock et al. [1] proved that for every forest F, every graph with pathwidth at least $|V(F)|-1$ contains F as a minor. Let $k:=9\ell^2-3\ell+1$. Note that $|V(\mathit{\Gamma}_k)|=2^{k+1}-1$. By assumption, G has pathwidth at least $2^{k+1}-2$. Thus G contains $\mathit{\Gamma}_k$ as a minor. The result follows from 6.1. ∎

Finally, we have the following quantitative version of 1.2.

6.3. *For every apex-forest H_1 and outerplanar graph H_2, if $\ell := \max\{|V(H_1)|,|V(H_2)|,2\}-1$, then every 2-connected graph G of pathwidth at least $2^{9\ell^2-3\ell+2}-2$ contains H_1 or H_2 as a minor.*

Proof. By 6.2, G contains $\mathit{\Gamma}_\ell^+$ or ∇_ℓ as a minor. In the first case, by 2.2, H_1 is a minor of $\mathit{\Gamma}^+_{|V(H_1)|-1}$ and thus of G (since $\ell \geqslant |V(H_1)|-1$). In the second case, by 2.4, H_2 is a minor of $\nabla_{|V(H_2)|-1}$ and thus of G (since $\ell \geqslant |V(H_2)|-1$). ∎

Acknowledgements. We would like to thank Robin Thomas for informing us of the PhD thesis of Thanh N. Dang [2]. We are also grateful to an anonymous referee for several helpful remarks and suggestions.

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