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Polynomial treewidth forces a large grid-like-minor

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ABSTRACT

Robertson and Seymour proved that every graph with sufficiently large treewidth contains a large grid minor. However, the best known bound on the treewidth that forces an $\ell \times \ell$ grid minor is exponential in ℓ. It is unknown whether polynomial treewidth suffices. We prove a result in this direction. A *grid-like-minor of order* ℓ in a graph G is a set of paths in G whose intersection graph is bipartite and contains a K_ℓ-minor. For example, the rows and columns of the $\ell \times \ell$ grid are a grid-like-minor of order $\ell + 1$. We prove that polynomial treewidth forces a large grid-like-minor. In particular, every graph with treewidth at least $c\ell^4\sqrt{\log \ell}$ has a grid-like-minor of order ℓ. As an application of this result, we prove that the Cartesian product $G \square K_2$ contains a K_ℓ-minor whenever G has treewidth at least $c\ell^4\sqrt{\log \ell}$.

1. Introduction

A central theorem in Robertson and Seymour's theory of graph minors states that the grid[1] is a canonical witness for a graph to have large treewidth, in the sense that the $\ell \times \ell$ grid has treewidth ℓ, and every graph with sufficiently large treewidth contains an $\ell \times \ell$ grid minor [17]. See [18,9,16] for alternative proofs. The following theorem is the best-known explicit bound. See [7,5] for better bounds under additional assumptions.

Theorem 1.1 ([18]). *Every graph with treewidth at least* $20^{2\ell^5}$ *contains an* $\ell \times \ell$ *grid minor.*

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1 The $\ell \times \ell$ grid is the planar graph with vertex set $[\ell] \times [\ell]$, where vertices (x, y) and (p, q) are adjacent whenever $|x - p| + |y - q| = 1$.

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Robertson et al. [18] also proved that certain random graphs have treewidth proportional to $\ell^2 \log \ell$, yet do not contain an $\ell \times \ell$ grid minor. This is the best known lower bound on the function in Theorem 1.1. Thus it is open whether polynomial treewidth forces a large grid minor. This question is not only of theoretic interest — for example, it has a direct bearing on certain algorithmic questions [6]. In this paper we prove that polynomial treewidth forces a large 'grid-like-minor'.

A *grid-like-minor of order* ℓ in a graph G is a set $\mathcal{P}$ of paths in G, such that the intersection graph[2] of $\mathcal{P}$ is bipartite and contains a K_ℓ-minor. Observe that the intersection graph of the rows and columns of the $\ell \times \ell$ grid is the complete bipartite graph $K_{\ell,\ell}$, which contains a $K_{\ell+1}$-minor (formed by contracting a matching of $\ell - 1$ edges). Hence, the $\ell \times \ell$ grid contains a grid-like-minor of order $\ell + 1$. The following is our main result.

Theorem 1.2. *Every graph with treewidth at least $c\ell^4\sqrt{\log \ell}$ contains a grid-like-minor of order ℓ, for some constant c. Conversely, every graph that contains a grid-like-minor of order ℓ has treewidth at least $\lceil \frac{\ell}{2} \rceil - 1$.*

Theorem 1.2 proves that grid-like-minors serve as a canonical witness for a graph to have large treewidth, just like grid minors. The advantage of grid-like-minors is that a polynomial bound on treewidth suffices. The disadvantage of grid-like-minors is that they are a broader structure than grid minors (but not as broad as brambles; see Section 2).

Theorem 1.2 has an interesting corollary concerning the Cartesian product $G \square K_2$. This graph consists of two copies of G with an edge between corresponding vertices in the two copies. Motivated by Hadwiger's Conjecture for Cartesian products, the second author [23] showed that the maximum order of a complete minor in $G \square K_2$ is tied to the treewidth of G. In particular, if G has treewidth at most ℓ, then $G \square K_2$ has treewidth at most $2\ell + 1$ and thus contains no $K_{2\ell+3}$-minor. Conversely, if G has treewidth at least $2^{4\ell^4}$, then $G \square K_2$ contains a K_ℓ-minor. The proof of the latter result is based on the version of Theorem 1.1 due to [9]. The following theorem is a significant improvement.

Theorem 1.3. *If a graph G has treewidth at least $c\ell^4\sqrt{\log \ell}$, then $G \square K_2$ contains a K_ℓ-minor, for some constant c.*

2. Background

All graphs considered in this paper are undirected, simple, and finite. For undefined terminology, see [8]. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d. Mader [15] proved that every graph with no K_ℓ-minor is $2^{\ell-2}$-degenerate. Let $d(\ell)$ be the minimum integer such that every graph with no K_ℓ-minor is $d(\ell)$-degenerate. Kostochka [13] and Thomason [21,22] independently proved that $d(\ell) \in \Theta(\ell\sqrt{\log \ell})$.

Theorem 2.1 (*[13,21,22]*). *Every graph with no K_ℓ-minor is $d(\ell)$-degenerate, where $d(\ell) \leq c\ell\sqrt{\log \ell}$ for some constant c.*

Let G be a graph. Two subgraphs X and Y of G *touch* if $X \cap Y \neq \emptyset$ or there is an edge of G between X and Y. A *bramble* in G is a set of pairwise touching connected subgraphs. The subgraphs are called *bramble elements*. A set S of vertices in G is a *hitting set* of a bramble $\mathcal{B}$ if S intersects every element of $\mathcal{B}$. The *order* of $\mathcal{B}$ is the minimum size of a hitting set. The canonical example of a bramble of order ℓ is the set of crosses (union of a row and column) in the $\ell \times \ell$ grid. The following 'Treewidth Duality Theorem' shows the intimate relationship between treewidth and brambles.

Theorem 2.2 (*[19]*). *A graph G has treewidth at least ℓ if and only if G contains a bramble of order at least $\ell + 1$.*

[2] The intersection graph of a set X, whose elements are sets, has vertex set X where distinct vertices are adjacent whenever the corresponding sets have a non-empty intersection.

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See [2] for an alternative proof of Theorem 2.2. In light of Theorems 2.2 and 1.1 says that every bramble of large order contains a large grid minor, and Theorem 1.2 says that every bramble of polynomial order contains a large grid-like-minor.

3. Main proofs

In this section we prove Theorems 1.2 and 1.3. Let $e := 2.718\cdots$ and $[n] := \{1, 2, \dots, n\}$. The following lemma is by Birmelé et al. [3]; we include the proof for completeness.

Lemma 3.1 (*[3]*). *Let $\mathcal{B}$ be a bramble in a graph G. Then G contains a path that intersects every element of $\mathcal{B}$.*

Proof. Let P be a path in G that (1) intersects as many elements of $\mathcal{B}$ as possible, and (2) is as short as possible. Let v be an endpoint of P. There is a bramble element X that only intersects P at v, as otherwise we could delete v from P. Suppose on the contrary that P does not intersect some bramble element Z. Since X and Z touch, there is a path Q starting at v through X to some vertex in Z, and $Q \cap P = \{v\}$. Thus $P \cup Q$ is a path that also hits Z. This contradiction proves that P intersects every element of $\mathcal{B}$. □

Lemma 3.2. *Let G be a graph containing a bramble $\mathcal{B}$ of order at least $k\ell$ for some integers $k, \ell \geq 1$. Then G contains ℓ disjoint paths $P_1, \dots, P_\ell$, and for distinct $i, j \in [\ell]$, G contains k disjoint paths between P_i and P_j.*

Proof. By Lemma 3.1, there is a path $P = (v_1, \dots, v_n)$ in G that intersects every element of $\mathcal{B}$. For $1 \leq a \leq b \leq n$, let $P\langle a, b\rangle$ be the sub-path of P induced by $\{v_a, \dots, v_b\}$, and let $\mathcal{B}\langle a, b\rangle$ be the sub-bramble

$$\mathcal{B}\langle a, b\rangle := \{X \in \mathcal{B} : X \cap P\langle a, b\rangle \neq \emptyset, X \cap P\langle 1, a-1\rangle = \emptyset\}.$$

If S is a hitting set of $\mathcal{B}\langle a, b\rangle$, then $S \cup \{v_{b+1}\}$ is a hitting set of $\mathcal{B}\langle a, b+1\rangle$. Thus the order of $\mathcal{B}\langle a, b+1\rangle$ is at most the order of $\mathcal{B}\langle a, b\rangle$ plus 1. Hence for each $a \in [n]$, either the order of $\mathcal{B}\langle a, n\rangle$ is less than k, or for some $b \geq a$ the order of $\mathcal{B}\langle a, b\rangle$ equals k. Thus there are positive integers $a_1 < a_2 < \cdots < a_s \leq n$ such that for each $i \in [s]$ the order of $\mathcal{B}_i := \mathcal{B}\langle a_{i-1} + 1, a_i\rangle$ equals k (where $a_0 = 0$), and the order of $\mathcal{B}_{s+1} := \mathcal{B}\langle a_s + 1, n\rangle$ is less than k. Since $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_{s+1}$, the order of $\mathcal{B}$ is at most the sum of the orders of $\mathcal{B}_1, \dots, B_{s+1}$, which is strictly less than $(s+1)k$. Since the order of $\mathcal{B}$ is at least $k\ell$, we have $s \geq \ell$. Let $P_i := P\langle a_{i-1} + 1, a_i\rangle$ for $i \in [\ell]$. Thus $P_1, \dots, P_\ell$ are disjoint paths in G.

Suppose that there is a set $S \subseteq V(G)$ separating some pair of distinct paths P_i and P_j, where $|S| \leq k - 1$. Thus S is not a hitting set of $\mathcal{B}_i$, since $\mathcal{B}_i$ has order k. Hence some element $X \in \mathcal{B}_i$ does not intersect S. Similarly, some element $Y \in \mathcal{B}_j$ does not intersect S. Thus S separates X from Y, and hence X and Y do not touch. This contradiction proves that every set of vertices separating P_i and P_j has at least k vertices. By Menger's Theorem, there are k disjoint paths between P_i and P_j, as desired. □

We now prove the main result.

Proof of the first part of Theorem 1.2. Let $k := \lceil 4e \binom{\ell}{2} d(\ell)\rceil$. Let G be a graph with treewidth at least $c\ell^4\sqrt{\log \ell}$, which is at least $k\ell - 1$ for an appropriate value of c. By Theorem 2.2, G has a bramble of order at least $k\ell$. By Lemma 3.2, G contains ℓ disjoint paths $P_1, \dots, P_\ell$, and for distinct $i, j \in [\ell]$, G contains a set $\mathcal{Q}_{i,j}$ of k disjoint paths between P_i and P_j.

For distinct $i, j \in [\ell]$ and distinct $a, b \in [\ell]$ with $\{i, j\} \neq \{a, b\}$, let $H_{i,j,a,b}$ be the intersection graph of $\mathcal{Q}_{i,j} \cup \mathcal{Q}_{a,b}$. Since $H_{i,j,a,b}$ is bipartite, if K_ℓ is a minor of $H_{i,j,a,b}$, then $\mathcal{Q}_{i,j} \cup \mathcal{Q}_{a,b}$ is a grid-like-minor of order ℓ. Now assume that K_ℓ is not a minor of $H_{i,j,a,b}$. By Theorem 2.1, $H_{i,j,a,b}$ is $d(\ell)$-degenerate.

Let H be the intersection graph of $\cup\{\mathcal{Q}_{i,j} : 1 \leq i < j \leq \ell\}$; that is, H is the union of the $H_{i,j,a,b}$. Then H is $\binom{\ell}{2}$-colourable, where each colour class is some $\mathcal{Q}_{i,j}$. Each colour class of H has k vertices, and each pair of colour classes in H induce a $d(\ell)$-degenerate subgraph. By Lemma 4.3 (in the following

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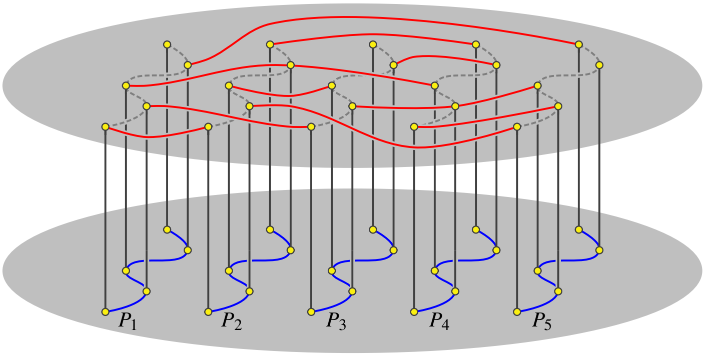

Fig. 1. Construction of a K_ℓ-minor in $G \square K_2$.

section) with $n = k$ and $r = \binom{\ell}{2}$ and $d = d(\ell)$, H has an independent set with one vertex from each colour class. That is, in each set $\mathcal{Q}_{i,j}$ there is one path $Q_{i,j}$ such that $Q_{i,j} \cap Q_{a,b} = \emptyset$ for distinct pairs i, j and a, b. Consider the set of paths

$$\mathcal{P} := \{P_i : i \in [\ell]\} \cup \{Q_{i,j} : 1 \le i < j \le \ell\}.$$

The intersection graph of $\mathcal{P}$ is bipartite and contains the 1-subdivision of K_ℓ, which contains a K_ℓ-minor. Therefore $\mathcal{P}$ is a grid-like-minor of order ℓ in G. □

The next lemma with $r = 2$ implies that if a graph G contains a grid-like-minor of order ℓ, then the treewidth of G is at least $\lceil \frac{\ell}{2} \rceil - 1$, which is the second part of Theorem 1.2.

Lemma 3.3. *Let H be the intersection graph of a set $\mathcal{X}$ of connected subgraphs in a graph G. If H contains a K_ℓ-minor, and H contains no K_{r+1}-subgraph, then the treewidth of G is at least $\lceil \frac{\ell}{r} \rceil - 1$.*

Proof. Let $H_1, \dots, H_\ell$ be the branch sets of a K_ℓ-minor in H. Each H_i corresponds to a subset $\mathcal{X}_i \subseteq \mathcal{X}$, such that $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset$ for distinct $i, j \in [\ell]$. Let G_i be the subgraph of G formed by the union of the subgraphs in $\mathcal{X}_i$. Since H_i is connected and each subgraph in $\mathcal{X}_i$ is connected, G_i is connected. For distinct $i, j \in [\ell]$, some vertex in H_i is adjacent to some vertex in H_j. That is, some subgraph in $\mathcal{X}_i$ intersects some subgraph in $\mathcal{X}_j$. Hence G_i and G_j share a vertex in common, and $\mathcal{B} := \{G_1, \dots, G_\ell\}$ is a bramble in G. Since H has no K_{r+1}-subgraph, every vertex of G is in at most r bramble elements of $\mathcal{B}$. Thus every hitting set of $\mathcal{B}$ has at least $\lceil \frac{\ell}{r} \rceil$ vertices. Hence $\mathcal{B}$ has order at least $\lceil \frac{\ell}{r} \rceil$. By Theorem 2.2, G has treewidth at least $\lceil \frac{\ell}{r} \rceil - 1$. □

Theorem 1.3 follows from Theorem 1.2 and the next lemma.

Lemma 3.4. *Let $\mathcal{P}$ be a grid-like-minor in a graph G. Then the intersection graph H of $\mathcal{P}$ is a minor of $G \square K_2$.*

Proof. Let $\mathcal{A} \cup \mathcal{B}$ be a bipartition of $V(H)$. If $XY \in E(H)$ for some $X, Y \in \mathcal{P}$, then $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, and some vertex v of G is in $X \cap Y$. Thus in $G \square K_2$, the copy of v in the first copy of G is adjacent to the copy of v in the second copy of G. Thus H is obtained by contracting each path in $\mathcal{A}$ in the first copy of G, and by contracting each path in $\mathcal{B}$ in the second copy of G, as illustrated in Fig. 1. □

Note that Lemma 3.4 generalises as follows: If H is the intersection graph of a set of connected subgraphs of a graph G, then H is a minor of $G \square K_{\chi(H)}$.

4. Independent transversals

An *independent transversal* in a coloured graph is an independent set with exactly one vertex in each colour class. Many results are known that say that if each colour class is large compared to the maximum degree and the number of colours, then an independent transversal exists [12,14,1,24,25,11,4,20]. Here we prove two similar results, in which the maximum degree assumption is relaxed. This result is used in the proof of Theorem 1.2. The proof is based on the Lovász Local Lemma.

Lemma 4.1 ([10]). *Let $\mathcal{X}$ be a set of events, such that each event in $\mathcal{X}$ has probability at most p and is mutually independent of all but D other events in $\mathcal{X}$. If $ep(D+1) \leq 1$ then with positive probability no event in $\mathcal{X}$ occurs.*

Lemma 4.2. *Let $V_1, \dots, V_r$ be the colour classes in an r-colouring of a graph H. For $i \in [r]$, let $n_i := |V_i|$, and let m_i be the number of edges with one endpoint in V_i. Suppose that $n_i \geq 2et$ and $m_i \leq tn_i$ for some $t > 0$ and for all $i \in [r]$. Then there exists an independent set $\{x_1, \dots, x_r\}$ of H such that each $x_i \in V_i$.*

Proof. Let $n := \lceil 2et \rceil$. Suppose that $n_i > n$ for some $i \in [r]$. Some vertex $v \in V_i$ has degree at least $\frac{m_i}{n_i}$. Thus $\frac{m_i - \deg(v)}{n_i - 1} \leq \frac{m_i}{n_i} \leq t$. Hence $H - v$ satisfies the assumptions. By induction, $H - v$ contains the desired independent set. Now assume that $n_i = n$ for all $i \in [r]$.

For each $i \in [r]$, independently and randomly choose one vertex $x_i \in V_i$. Each vertex in V_i is chosen with probability $\frac{1}{n}$. Consider an edge vw, where $v \in V_i$ and $w \in V_j$. Let X_{vw} be the event that both v and w are chosen. Thus X_{vw} has probability $p := \frac{1}{n^2}$. Observe that X_{vw} is mutually independent of every event X_{xy} where $x \notin V_i \cup V_j$ and $y \notin V_i \cup V_j$. Thus X_{vw} is mutually independent of all but at most $D := m_i + m_j - 1$ other events.

Now $2em_i \leq 2etn \leq n^2$ and $2em_j \leq 2etn \leq n^2$. Thus $e(m_i + m_j) \leq n^2$. That is, $ep(D+1) \leq 1$. By Lemma 4.1, with positive probability no event X_{vw} occurs. Hence there exists $x_1, \dots, x_r$ such that no event X_{vw} occurs. That is, $\{x_1, \dots, x_r\}$ is the desired independent set. □

Lemma 4.3. *Let $V_1, \dots, V_r$ be the colour classes in an r-colouring of a graph H. Suppose that $|V_i| \geq 4e(r-1)d$ for all $i \in [r]$, and $H[V_i \cup V_j]$ is d-degenerate for distinct $i, j \in [r]$. Then there exists an independent set $\{x_1, \dots, x_r\}$ of H such that each $x_i \in V_i$.*

Proof. Let $n := \lceil 4e(r-1)d \rceil$. For each $i \in [r]$, we may assume that $|V_i| = n$ (since deleting vertices from V_i does not change the degeneracy assumption). Let m_i be the number of edges with one endpoint in V_i. Every d-degenerate graph with N vertices has at most dN edges. Thus $m_i \leq 2(r-1)dn$. Let $t := 2(r-1)d$. The result follows from Lemma 4.2 since $n \geq 2et$ and each $m_i \leq tn$. □

We now give an example that shows that the lower bound on $|V_i|$ in Lemma 4.3 is best possible up to a constant factor. Say V_1 has $d(r-1)$ vertices. Partition V_1 into sets $W_2, \dots, W_r$ each of size d. Connect every vertex in W_i to every vertex in V_i by an edge. Each bichromatic subgraph (ignoring isolated vertices) is the complete bipartite graph $K_{d,n}$ (for some n), which is d-degenerate. However, since every vertex in V_1 dominates some colour class, no independent set has one vertex from each colour class. It is interesting to determine the best possible lower bound on the size of each colour class in Lemma 4.3. It is possible that $|V_i| \geq d(r-1) + c$ suffices.

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