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ON THE GENERAL POSITION SUBSET SELECTION PROBLEM*

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Abstract. Let $f(n,\ell)$ be the maximum integer such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n,\ell)$ points with no three collinear. First we prove that if $\ell \leqslant O(\sqrt{n})$, then $f(n,\ell) \geqslant \Omega(\sqrt{n/\ln \ell})$. Second we prove that if $\ell \leqslant O(n^{(1-\epsilon)/2})$, then $f(n,\ell) \geqslant \Omega(\sqrt{n \log_\ell n})$, which implies all previously known lower bounds on $f(n,\ell)$ and improves them when ℓ is not fixed. A more general problem is to consider subsets with at most k collinear points in a point set with at most ℓ collinear. We also prove analogous results in this setting.

Key words. general position, Erdős problems, discrete geometry

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1. Introduction.

A set of points in the plane is in *general position* if it contains no three collinear points. The general position subset selection problem asks, given a finite set of points in the plane with at most ℓ collinear, how big is the largest subset in general position? That is, determine the maximum integer $f(n,\ell)$ such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n,\ell)$ points in general position. Throughout this paper we assume $\ell \geqslant 3$. Furthermore, as the results in this paper are all asymptotic in n, the expression "fixed ℓ" is shorthand for "ℓ a constant not dependent on n." Otherwise ℓ is allowed to grow as a function of n.

The problem was originally posed by Erdős, first for the case $\ell = 3$ [8], and later in a more general form [9]. Füredi [10] showed that the density version of the Hales–Jewett theorem [11] implies that $f(n,\ell) \leqslant o(n)$, and that a result of Phelps and Rödl [20] on independent sets in partial Steiner triple systems implies that

$$f(n,3) \geqslant \Omega(\sqrt{n \ln n}).$$

Until recently, the best known lower bound for $\ell \geqslant 4$ was $f(n,\ell) \geqslant \sqrt{2n/(\ell-2)}$, proved by a greedy selection algorithm. Lefmann [16] showed that for fixed ℓ,

$$f(n,\ell) \geqslant \Omega(\sqrt{n \ln n}).$$

(In fact, his results are more general; see section 3.)

In relation to the general position subset selection problem (and its relatives), Brass, Moser, and Pach [2, p. 318] write, "To make any further progress, one needs to explore the geometric structure of the problem." We do this by using the Szemerédi–Trotter theorem [25].

We give improved lower bounds on $f(n,\ell)$ when ℓ is not fixed, with the improvement being most significant for values of ℓ around $\sqrt{n}$. Our first result (Theorem 2.3)

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says that if $\ell \leqslant O(\sqrt{n})$, then $f(n,\ell) \geqslant \Omega(\sqrt{\frac{n}{\ln \ell}})$. Our second result (Theorem 2.5) says that if $\ell \leqslant O(n^{(1-\epsilon)/2})$, then $f(n,\ell) \geqslant \Omega(\sqrt{n \log_\ell n})$. For fixed ℓ, this implies Lefmann's lower bound on $f(n,\ell)$ mentioned above.

In section 3 we consider a natural generalization of the general position subset selection problem. Given $k < \ell$, Erdős [9] asked for the maximum integer $f(n,\ell,k)$ such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n,\ell,k)$ points with at most k collinear. Thus $f(n,\ell) = f(n,\ell,2)$. We prove results similar to Theorems 2.3 and 2.5 in this setting too.

2. Results. Our main tool is the following lemma.

LEMMA 2.1. *Let P be a set of n points in the plane with at most ℓ collinear. Then the number of collinear triples in P is at most $c(n^2 \ln \ell + \ell^2 n)$ for some constant c.*

Proof. For $2 \leqslant i \leqslant \ell$, let s_i be the number of lines containing exactly i points in P. A well-known corollary of the Szemerédi–Trotter theorem [25] states that for some constant $c \geqslant 1$, for all $i \geqslant 2$,

$$\sum_{j \geqslant i} s_j \leqslant c\left(\frac{n^2}{i^3} + \frac{n}{i}\right).$$

Thus the number of collinear triples is

$$\sum_{i=2}^{\ell} \binom{i}{3} s_i \leqslant \sum_{i=2}^{\ell} i^2 \sum_{j=i}^{\ell} s_j \leqslant \sum_{i=2}^{\ell} c i^2 \left(\frac{n^2}{i^3} + \frac{n}{i}\right)$$
$$\leqslant c \sum_{i=2}^{\ell} \left(\frac{n^2}{i} + in\right) \leqslant c(n^2 \ln \ell + \ell^2 n). \qquad \square$$

Note that Lefmann [15] proved Lemma 2.1 for the case of the $\sqrt{n} \times \sqrt{n}$ grid via a direct counting argument. A statement similar to Lemma 2.1 with $\ell = \sqrt{n}$ also appears in the book by Tao and Vu [26, Corollary 8.8].

To apply Lemma 2.1 it is useful to consider the 3-uniform hypergraph $H(P)$ determined by a set of points P, with vertex set P, and an edge for each collinear triple in P. A subset of P is in general position if and only if it is an independent set in $H(P)$. The size of the largest independent set in a hypergraph H is denoted $\alpha(H)$. Spencer [23] proved the following lower bound on $\alpha(H)$.

LEMMA 2.2 (Spencer [23]). *Let H be an r-uniform hypergraph with n vertices and m edges. If $m < n/r$, then $\alpha(H) > n/2$. If $m \geqslant n/r$, then*

$$\alpha(H) > \frac{r-1}{r^{r/(r-1)}} \frac{n}{(m/n)^{1/(r-1)}}.$$

Lemmas 2.1 and 2.2 imply our first result.

THEOREM 2.3. *Let P be a set of n points with at most ℓ collinear. Then P contains a set of $\Omega(n/\sqrt{n \ln \ell + \ell^2})$ points in general position. In particular, if $\ell \leqslant O(\sqrt{n})$, then P contains a set of $\Omega(\sqrt{\frac{n}{\ln \ell}})$ points in general position.*

Proof. Let m be the number of edges in $H(P)$. By Lemma 2.1, $m/n \leqslant c(n \ln \ell + \ell^2)$ for some constant c. Now apply Lemma 2.2 with $r = 3$. If $m < n/3$, then $\alpha(H(P)) > n/2$, as required. Otherwise,

$$\alpha(H(P)) > \frac{2n}{3^{3/2}(m/n)^{1/2}} \geqslant \frac{2n}{3^{3/2}\sqrt{c(n \ln \ell + \ell^2)}} = \frac{2}{3\sqrt{3c}} \frac{n}{\sqrt{n \ln \ell + \ell^2}}. \qquad \square$$

Note that Theorem 2.3 also shows that if $\ell^2/\ln \ell \geqslant n$, then $f(n,\ell) \geqslant \Omega(n/\ell)$. This improves upon the greedy bound mentioned in the introduction, and is within a constant factor of optimal, since there are point sets with at most ℓ collinear that can be covered by n/ℓ lines.

Theorem 2.3 answers, up to a logarithmic factor, a symmetric Ramsey-style version of the general position subset selection problem posed by Gowers [13]. He asked for the minimum integer $\mathrm{GP}(q)$ such that every set of at least $\mathrm{GP}(q)$ points in the plane contains q collinear points or q points in general position. Gowers noted that $\Omega(q^2) \leqslant \mathrm{GP}(q) \leqslant O(q^3)$. Theorem 2.3 with $\ell = q-1$ and $n = \mathrm{GP}(q)$ implies that $\Omega(\sqrt{\mathrm{GP}(q)/\ln(q-1)}) \leqslant q$ and so $\mathrm{GP}(q) \leqslant O(q^2 \ln q)$.

The bound $\mathrm{GP}(q) \geqslant \Omega(q^2)$ comes from the $q \times q$ grid, which contains no $q+1$ collinear points, and no more than $2q+1$ in general position, since each row can have at most 2 points. Determining the maximum number of points in general position in the $q \times q$ grid is known as the *no-three-in-line problem*, first posed by Dudeney in 1917 [4]. See [14] for the best known bound and for more on its history.

As an aside, note that Pach and Sharir [18] proved a result somewhat similar to Lemma 2.1 for the number of triples in P determining a fixed angle $\alpha \in (0,\pi)$. Their proof is similar to that of Lemma 2.1 in its use of the Szemerédi–Trotter theorem. Also, Elekes [6] employed Lemma 2.2 to prove a similar result to Theorem 2.3 for the problem of finding large subsets with no triple determining a given angle $\alpha \in (0,\pi)$. Pach and Sharir and Elekes did not allow the case $\alpha = 0$, that is, collinear triples. This may be because their work did not consider the parameter ℓ, without which the case $\alpha = 0$ is exceptional since P could be entirely collinear, and all triples could determine the same angle.

The following lemma of Sudakov [24, Proposition 2.3] is a corollary of a result by Duke, Lefmann, and Rödl [5].

LEMMA 2.4 (Sudakov [24]). *Let H be a 3-uniform hypergraph on n vertices with m edges. Let $t \geqslant \sqrt{m/n}$ and suppose there exists a constant $\epsilon > 0$ such that the number of edges containing any fixed pair of vertices of H is at most $t^{1-\epsilon}$. Then $\alpha(H) \geqslant \Omega(\frac{n}{t}\sqrt{\ln t})$.*

Lemmas 2.1 and 2.4 can be used to prove our second result.

THEOREM 2.5. *Fix constants $\epsilon > 0$ and $d > 0$. Let P be a set of n points in the plane with at most ℓ collinear points, where $\ell \leqslant (dn)^{(1-\epsilon)/2}$. Then P contains a set of $\Omega(\sqrt{n \log_\ell n})$ points in general position.*

Proof. Let m be the number of edges in $H(P)$. By Lemma 2.1, for some constant $c \geqslant 1$,

$$m \leqslant c\ell^2 n + cn^2 \ln \ell < cdn^2 + cn^2 \ln \ell \leqslant (d+1)cn^2 \ln \ell.$$

Define $t := \sqrt{(d+1)cn \ln \ell}$. Thus $t \geqslant \sqrt{m/n}$. Each pair of vertices in H is in less than ℓ edges of H, and

$$\ell \leqslant (dn)^{(1-\epsilon)/2} < ((d+1)cn \ln \ell)^{(1-\epsilon)/2} = t^{1-\epsilon}.$$

Thus the assumptions in Lemma 2.4 are satisfied. So H contains an independent set of size $\Omega(\frac{n}{t}\sqrt{\ln t})$. Moreover,

$$\begin{aligned}\frac{n}{t}\sqrt{\ln t} &= \sqrt{\frac{n}{(d+1)c\ln \ell}}\sqrt{\ln \sqrt{(d+1)cn\ln \ell}} \\ &\geqslant \sqrt{\frac{n}{(d+1)c\ln \ell}}\sqrt{\frac{1}{2}\ln n}\end{aligned}$$

$$= \sqrt{\frac{1}{2(d+1)c}}\sqrt{\frac{n\ln n}{\ln \ell}}$$
$$= \Omega(\sqrt{n\log_\ell n}).$$

Thus P contains a subset of $\Omega(\sqrt{n\log_\ell n})$ points in general position. ∎

3. Generalizations. In this section we consider the function $f(n,\ell,k)$ defined to be the maximum integer such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n,\ell,k)$ points with at most k collinear, where $k<\ell$.

Brass [1] considered this question for fixed $\ell=k+1$ and showed that

$$o(n) \geqslant f(n,k+1,k) \geqslant \Omega(n^{(k-1)/k}(\ln n)^{1/k}).$$

This can be seen as a generalization of the results of Füredi [10] for $f(n,3,2)$. As in Füredi's work, the lower bound comes from a result on partial Steiner systems [22], and the upper bound comes from the density Hales–Jewett theorem [12]. Lefmann [16] further generalized these results for fixed ℓ and k by showing that

$$f(n,\ell,k) \geqslant \Omega(n^{(k-1)/k}(\ln n)^{1/k}).$$

The density Hales–Jewett theorem also implies the general bound $f(n,\ell,k)\leqslant o(n)$.

The result of Lefmann may be generalized to include the dependence of $f(n,\ell,k)$ on ℓ for fixed $k\geqslant 3$, analogously to Theorems 2.3 and 2.5 for $k=2$. The first result we need is a generalization of Lemma 2.1. It is proved in the same way.

LEMMA 3.1. *Let P be a set of n points in the plane with at most ℓ collinear. Then, for $k\geqslant 4$, the number of collinear k-tuples in P is at most $c(\ell^{k-3}n^2+l^{k-1}n)$ for some absolute constant c.*

Lemmas 2.2 and 3.1 imply the following theorem, which is proved in the same way as Theorem 2.3.

THEOREM 3.2. *If $k\geqslant 3$ is fixed and $\ell\leqslant O(\sqrt{n})$, then $f(n,\ell,k)\geqslant\Omega\big(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}\big)$.*

For $\ell=\sqrt{n}$ and fixed $k\geqslant 3$, Theorem 3.2 implies $f(n,\sqrt{n},k)\geqslant\Omega\big(\frac{n^{(k-1)/k}}{n^{(k-2)/2k}}\big)=\Omega\left(n^{(2k-2-k+2)/2k}\right)=\Omega(\sqrt{n})$. This answers completely a generalized version of Gowers' question [13], namely, to determine the minimum integer $\mathrm{GP}_k(q)$ such that every set of at least $\mathrm{GP}_k(q)$ points in the plane contains q collinear points or q points with at most k collinear, for $k\geqslant 3$. Thus $\mathrm{GP}_k(q)\leqslant O(q^2)$. The bound $\mathrm{GP}_k(q)\geqslant\Omega(q^2)$ comes from the following construction. Let $m:=\lfloor(q-1)/k\rfloor$ and let P be the $m\times m$ grid. Then P has at most m points collinear, and $m<q$. If S is a subset of P with at most k collinear, then S has at most k points in each row. So $|S|\leqslant km\leqslant q-1$.

Theorem 2.5 can be generalized using Lemma 3.1 and a theorem of Duke, Lefmann, and Rödl [5] (the one that implies Lemma 2.4).

THEOREM 3.3 (Duke, Lefmann, and Rödl [5]). *Let H be a k-uniform hypergraph with maximum degree $\Delta(H)\leqslant t^{k-1}$ where $t\gg k$. Let $p_j(H)$ be the number of pairs of edges of H sharing exactly j vertices. If $p_j(H)\leqslant nt^{2k-j-1-\gamma}$ for $j=2,\dots,k-1$ and some $\gamma>0$, then $\alpha(H)\geqslant C(k,\gamma)\frac{n}{t}(\ln n)^{1/(k-1)}$ for some constant $C(k,\gamma)>0$.*

THEOREM 3.4. *Fix constants $d>0$ and $\epsilon\in(0,1)$. If $k\geqslant 3$ is fixed and $4\leqslant\ell\leqslant dn^{(1-\epsilon)/2}$, then*

$$f(n,\ell,k)\geqslant\Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}(\ln n)^{1/k}\right).$$

Proof. Given a set P of n points with at most ℓ collinear, a subset with at most k collinear points corresponds to an independent set in the $(k+1)$-uniform hypergraph

$H_{k+1}(P)$ of collinear $(k+1)$-tuples in P. By Lemma 3.1, the number of edges in $H_{k+1}(P)$ is $m \leqslant c(n^2\ell^{k-2} + nl^k)$ for some constant c. The first term dominates since $\ell \leqslant o(\sqrt{n})$. For n large enough, $m/n \leqslant 2cn\ell^{k-2}$.

To limit the maximum degree of $H_{k+1}(P)$, discard vertices of degree greater than $2(k+1)m/n$. Let $\tilde{n}$ be the number of such vertices. Considering the sum of degrees, $(k+1)m \geqslant \tilde{n}2(k+1)m/n$, and so $\tilde{n} \leqslant n/2$. Thus discarding these vertices yields a new point set P' such that $|P'| \geqslant n/2$ and $\Delta(H_{k+1}(P')) \leqslant 4(k+1)cn\ell^{k-2}$. Note that an independent set in $H_{k+1}(P')$ is also independent in $H_{k+1}(P)$.

Set $t := (4(k+1)cn\ell^{k-2})^{1/k}$, so $m \leqslant \frac{1}{2(k+1)}nt^k$ and $\Delta(H_{k+1}(P')) \leqslant t^k$, as required for Theorem 3.3. By assumption, $\ell \leqslant dn^{(1-\epsilon)/2}$. Thus

$$\ell \leqslant d\left(\frac{t^k\ell^{2-k}}{4(k+1)c}\right)^{\frac{1-\epsilon}{2}}.$$

Hence $\ell^{\frac{2}{1-\epsilon}+k-2} \leqslant \frac{d^{2/(1-\epsilon)}t^k}{4(k+1)c}$, implying $\ell \leqslant C_1(k)t^{\frac{k}{\frac{2}{1-\epsilon}+k-2}} = C_1(k)t^{\frac{1-\epsilon}{1-\epsilon+\frac{2\epsilon}{k}}}$ for some constant $C_1(k)$. Define $\epsilon' := 1 - \frac{1-\epsilon}{1-\epsilon+\frac{2\epsilon}{k}}$, so $\epsilon' > 0$ (since $\epsilon < 1$) and $\ell \leqslant C_1(k)t^{1-\epsilon'}$. To bound $p_j(H_{k+1}(P'))$ for $j = 2, \ldots, k$, first choose one edge (which determines a line), then choose the subset to be shared, then choose points from the line to complete the second edge of the pair. Thus for $\gamma := \epsilon'/2$ and sufficiently large n,

$$\begin{aligned} p_j(H_{k+1}(P')) &\leqslant m\binom{k+1}{j}\binom{\ell-k-1}{k+1-j} \\ &\leqslant C_2(k)nt^k\ell^{k+1-j} \\ &\leqslant C_2(k)(C_1(k))^{k+1-j}nt^kt^{(1-\epsilon')(k+1-j)} \\ &\leqslant nt^{2(k+1)-j-1-\gamma}. \end{aligned}$$

Hence the second requirement of Theorem 3.3 is satisfied. Thus

$$\begin{aligned} \alpha(H_{k+1}(P')) &\geqslant \Omega\left(\frac{n}{t}(\ln t)^{1/k}\right) \\ &\geqslant \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}\left(\ln((n\ell^{k-2})^{1/k})\right)^{1/k}\right) \\ &\geqslant \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}(\ln n)^{1/k}\right). \qquad \square \end{aligned}$$

4. Conjectures. Theorem 3.2 suggests the following conjecture, which would completely answer Gowers's question [13], showing that $\mathrm{GP}(q) = \Theta(q^2)$. It is true for the $\sqrt{n} \times \sqrt{n}$ grid [14], [7, Appendix].

CONJECTURE 4.1. $f(n, \sqrt{n}) \geqslant \Omega(\sqrt{n})$.

A natural variation of the general position subset selection problem is to color the points of P with as few colors as possible, such that each color class is in general position. An easy application of the Lovász local lemma shows that under this requirement, n points with at most ℓ collinear are colorable with $O(\sqrt{\ell n})$ colors. The following conjecture would imply Conjecture 4.1. It is also true for the $\sqrt{n} \times \sqrt{n}$ grid [27].

CONJECTURE 4.2. *Every set P of n points in the plane with at most $\sqrt{n}$ collinear can be colored with $O(\sqrt{n})$ colors such that each color class is in general position.*

The following proposition is somewhat weaker than Conjecture 4.2.

PROPOSITION 4.3. *Every set P of n points in the plane with at most $\sqrt{n}$ collinear can be colored with $O(\sqrt{n}\ln^{3/2} n)$ colors such that each color class is in general position.*

Proof. Color P by iteratively selecting a largest subset in general position and giving it a new color. Let $P_0 := P$. Let C_i be a largest subset of P_i in general position and let $P_{i+1} := P_i \setminus C_i$. Define $n_i := |P_i|$. Applying Lemma 2.1 to P_i shows that $H(P_i)$ has $O(n_i^2 \ln \ell + \ell^2 n_i)$ edges. Thus the average degree of $H(P_i)$ is at most $O(n_i \ln \ell + \ell^2)$, which is $O(n \ln n)$ since $n_i \leqslant n$ and $\ell \leqslant \sqrt{n}$.

Applying Lemma 2.2 gives $|C_i| = \alpha(H(P_i)) > cn_i/\sqrt{n \ln n}$ for some constant $c > 0$. Thus $n_i \leqslant n(1 - c/\sqrt{n \ln n})^i$. It is well known (and not difficult to show) that if a sequence of numbers m_i satisfies $m_i \leqslant m(1 - 1/x)^i$ for some $x > 1$ and if $j > x \ln m$, then $m_j \leqslant 1$. Hence if $k \geqslant \sqrt{n \ln n} \ln n/c$, then $n_k \leqslant 1$, so the number of colors used is $O(\sqrt{n}\ln^{3/2} n)$. ◻

The problem of determining the correct asymptotics of $f(n, \ell)$ (and $f(n, \ell, k)$) for fixed ℓ remains wide open. The Szemerédi–Trotter theorem is essentially tight for the $\sqrt{n} \times \sqrt{n}$ grid [19], but says nothing for point sets with bounded collinearities. For this reason, the lower bounds on $f(n, \ell)$ for fixed ℓ remain essentially combinatorial. Finding a way to bring geometric information to bear in this situation is an interesting challenge.

CONJECTURE 4.4. *If ℓ is fixed, then $f(n, \ell) \geqslant \Omega(n/\operatorname{polylog}(n))$.*

The point set that gives the upper bound $f(n, \ell) \leqslant o(n)$ (from the density Hales–Jewett theorem) is the generic projection to the plane of the $\lfloor \log_\ell n \rfloor$-dimensional $\ell \times \ell \times \cdots \times \ell$ integer lattice (henceforth $[\ell]^d$, where $d := \lfloor \log_\ell(n) \rfloor$). The problem of finding large general position subsets in this point set for $\ell = 3$ is known as Moser's cube problem [17, 21], and the best known asymptotic lower bound is $\Omega(n/\sqrt{\ln n})$ [3, 21].

In the coloring setting, the following conjecture is equivalent to Conjecture 4.4 by an argument similar to that of Proposition 4.3.

CONJECTURE 4.5. *For all fixed $\ell \geqslant 3$, every set of n points in the plane with at most ℓ collinear can be colored with $O(\operatorname{polylog}(n))$ colors such that each color class is in general position.*

Conjecture 4.5 is true for $[\ell]^d$, which can be colored with $O(d^{\ell-1})$ colors as follows. For each $x \in [\ell]^d$, define a signature vector in $\mathbb{Z}^\ell$ whose entries are the number of entries in x equal to $1, 2, \ldots, \ell$. The number of such signatures is the number of partitions of d into at most ℓ parts, which is $O(d^{\ell-1})$. Give each set of points with the same signature its own color. To see that this is a proper coloring, suppose that $\{a, b, c\} \subset [\ell]^d$ is a monochromatic collinear triple, with b between a and c. Permute the coordinates so that the entries of b are nondecreasing. Consider the first coordinate i in which a_i, b_i, and c_i are not all equal. Then without loss of generality $a_i < b_i$. But this implies that a has more entries equal to a_i than b does, contradicting the assumption that the signatures are equal.

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