European Journal of Combinatorics 32 (2011) 1244–1252

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

On the maximum number of cliques in a graph embedded in a surface

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ARTICLE INFO

Article history:
Received 6 July 2009
Accepted 30 March 2011

ABSTRACT

This paper studies the following question: given a surface Σ and an integer n, what is the maximum number of cliques in an n-vertex graph embeddable in Σ? We characterise the extremal graphs for this question, and prove that the answer is between $8(n-\omega)+2^{\omega}$ and $8n+\frac{5}{2}2^{\omega}+o(2^{\omega})$, where ω is the maximum integer such that the complete graph K_ω embeds in Σ. For the surfaces $\mathbb{S}_0$, $\mathbb{S}_1$, $\mathbb{S}_2$, $\mathbb{N}_1$, $\mathbb{N}_2$, $\mathbb{N}_3$ and $\mathbb{N}_4$ we establish an exact answer.

1. Introduction

A *clique* in a graph[5] is a set of pairwise adjacent vertices. Let $c(G)$ be the number of cliques in a graph G. For example, every set of vertices in the complete graph K_n is a clique, and $c(K_n)=2^n$. This paper studies the following question at the intersection of topological and extremal graph theory:

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1 Supported by the Natural Sciences and Engineering Research Council of Canada.

2 Supported in part by the Slovenian Research Agency, Research Program P1-0297.

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4 Supported by a QEII Research Fellowship from the Australian Research Council.

5 We consider simple, finite, undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. A K_3 subgraph of G is called a *triangle* of G. For background graph theory, see [4].

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2011.04.001

given a surface Σ and an integer n, what is the maximum number of cliques in an n-vertex graph embeddable in Σ?

For previous bounds on the maximum number of cliques in certain graph families, see [5,6,13,14,22,23] for example. For background on graphs embedded in surfaces, see [11,21]. Every surface is homeomorphic to $\mathbb{S}_g$, the orientable surface with g handles, or to $\mathbb{N}_h$, the non-orientable surface with h crosscaps. The *Euler characteristic* of $\mathbb{S}_g$ is $2-2g$. The *Euler characteristic* of $\mathbb{N}_h$ is $2-h$. The *orientable genus* of a graph G is the minimum integer g such that G embeds in $\mathbb{S}_g$. The *non-orientable genus* of a graph G is the minimum integer h such that G embeds in $\mathbb{N}_h$. The orientable genus of K_n $(n \geq 3)$ is $\lceil \frac{1}{12}(n-3)(n-4) \rceil$, and its non-orientable genus is $\lceil \frac{1}{6}(n-3)(n-4) \rceil$, except that the non-orientable genus of K_7 is 3.

Throughout the paper, fix a surface Σ with Euler characteristic χ. If $\Sigma = \mathbb{S}_0$ then let $\omega = 3$, otherwise let ω be the maximum integer such that K_ω embeds in Σ. Thus $\omega = \lfloor \frac{1}{2}(7+\sqrt{49-24\chi}) \rfloor$ except for $\Sigma = \mathbb{S}_0$ and $\Sigma = \mathbb{N}_2$, in which case $\omega = 3$ and $\omega = 6$, respectively.

To avoid trivial exceptions, we implicitly assume that $|V(G)| \geq 3$ whenever $\Sigma = \mathbb{S}_0$.

Our first main result is to characterise the n-vertex graphs embeddable in Σ with the maximum number of cliques; see Theorem 1 in Section 2. Using this result we determine an exact formula for the maximum number of cliques in an n-vertex graph embeddable in each of the sphere $\mathbb{S}_0$, the torus $\mathbb{S}_1$, the double torus $\mathbb{S}_2$, the projective plane $\mathbb{N}_1$, the Klein bottle $\mathbb{N}_2$, as well as $\mathbb{N}_3$ and $\mathbb{N}_4$; see Section 3. Our third main result estimates the maximum number of cliques in terms of ω. We prove that the maximum number of cliques in an n-vertex graph embeddable in Σ is between $8(n-\omega)+2^\omega$ and $8n+\frac{5}{2}2^\omega+o(2^\omega)$; see Theorem 2 in Section 4.

2. Characterisation of extremal graphs

The upper bounds proved in this paper are of the form: every graph G embeddable in Σ satisfies $c(G) \leq 8|V(G)|+f(\Sigma)$ for some function f. Define the *excess* of G to be $c(G)-8|V(G)|$. Thus the excess of G is at most Q if and only if $c(G) \leq 8|V(G)|+Q$. Theorem 2 proves that the maximum excess of a graph embeddable in Σ is finite.

In this section, we characterise the graphs embeddable in Σ with maximum excess. A *triangulation* of Σ is an embedding of a graph in Σ in which each facial walk has three vertices and three edges with no repetitions. (We assume that every face of a graph embedding is homeomorphic to a disc.)

Lemma 1. *Every graph G embeddable in Σ with maximum excess is a triangulation of Σ.*

Proof. Since adding edges within a face increases the number of cliques, the vertices on the boundary of each face of G form a clique.

Suppose that some face f of G has at least four distinct vertices in its boundary. Let G' be the graph obtained from G by adding one new vertex adjacent to four distinct vertices of f. Thus G' is embeddable in Σ, has $|V(G)|+1$ vertices, and has $c(G)+16$ cliques, which contradicts the choice of G. Now assume that every face of G has at most three distinct vertices.

Suppose that some face f of G has repeated vertices. Thus the facial walk of f contains vertices u, v, w, v in this order (where v is repeated in f). Let G' be the graph obtained from G by adding two new vertices p and q, where p is adjacent to $\{u, v, w, q\}$, and q is adjacent to $\{u, v, w, p\}$. So G' is embeddable in Σ and has $|V(G)|+2$ vertices. If $S \subseteq \{p, q\}$ and $S \neq \emptyset$ and $T \subseteq \{u, v, w\}$, then $S \cup T$ is a clique of G' but not of G. It follows that G' has $c(G)+24$ cliques, which contradicts the choice of G. Hence no face of G has repeated vertices, and G is a triangulation of Σ. □

Let G be a triangulation of Σ. An edge vw of G is *reducible* if vw is in exactly two triangles in G. We say G is *irreducible* if no edge of G is reducible [2,3,7,9,10,12,17,19,20]. Note that K_3 is a triangulation of $\mathbb{S}_0$, and by the above definition, K_3 is irreducible. In fact, it is the only irreducible triangulation of $\mathbb{S}_0$. We take this somewhat non-standard approach so that Theorem 1 holds for all surfaces.

Let vw be a reducible edge of a triangulation G of Σ. Let vwx and vwy be the two faces incident to vw in G. As illustrated in Fig. 1, let G/vw be the graph obtained from G by *contracting* vw; that is, delete the edges vw, wy, wx, and identify v and w into v. G/vw is a simple graph since x and y are the

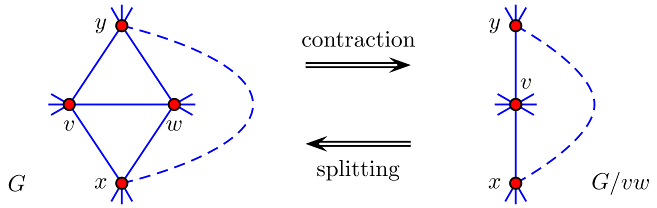

Fig. 1. Contracting a reducible edge.

only common neighbours of v and w. Indeed, G/vw is a triangulation of Σ. Conversely, we say that G is obtained from G/vw by *splitting* the path xvy at v. If, in addition, $xy \in E(G)$, then we say that G is obtained from G/vw by *splitting* the triangle xvy at v. Note that xvy need not be a face of G/vw. In the case that xvy is a face, splitting xvy is equivalent to adding a new vertex adjacent to each of x, v, y.

Graphs embeddable in Σ with maximum excess are characterised in terms of irreducible triangulations as follows.

Theorem 1. *Let Q be the maximum excess of an irreducible triangulation of Σ. Let X be the set of irreducible triangulations of Σ with excess Q. Then the excess of every graph G embeddable in Σ is at most Q, with equality if and only if G is obtained from some graph in X by repeatedly splitting triangles.*

Proof. We proceed by induction on $|V(G)|$. By Lemma 1, we may assume that G is a triangulation of Σ. If G is irreducible, then the claim follows from the definition of X and Q. Otherwise, some edge vw of G is in exactly two triangles vwx and vwy. By induction, the excess of G/vw is at most Q, with equality if and only if G/vw is obtained from some $H \in X$ by repeatedly splitting triangles. Hence $c(G/vw) \leq 8|V(G/vw)| + Q$.

Observe that every clique of G that is not in G/vw is in $\{A \cup \{w\} : A \subseteq \{x, v, y\}\}$. Thus $c(G) \leq c(G/vw) + 8$, with equality if and only if xvy is a triangle. Hence $c(G) \leq 8|V(G)| + Q$; that is, the excess of G is at most Q.

Now suppose that the excess of G equals Q. Then the excess of G/vw equals Q, and $c(G) = c(G/vw)+8$ (implying xvy is a triangle). By induction, G/vw is obtained from H by repeatedly splitting triangles. Therefore G is obtained from H by repeatedly splitting triangles.

Conversely, suppose that G is obtained from some $H \in X$ by repeatedly splitting triangles. Then xvy is a triangle and G/vw is obtained from H by repeatedly splitting triangles. By induction, the excess of G/vw equals Q, implying the excess of G equals Q. □

3. Low-genus surfaces

To prove an upper bound on the number of cliques in a graph embedded in Σ, by Theorem 1, it suffices to consider irreducible triangulations of Σ with maximum excess. The complete list of irreducible triangulations is known for $\mathbb{S}_0$, $\mathbb{S}_1$, $\mathbb{S}_2$, $\mathbb{N}_1$, $\mathbb{N}_2$, $\mathbb{N}_3$ and $\mathbb{N}_4$. In particular, Steinitz and Rademacher [16] proved that K_3 is the only irreducible triangulation of $\mathbb{S}_0$ (under our definition of irreducible). Lavrenchenko [9] proved that there are 21 irreducible triangulations of $\mathbb{S}_1$, each with between 7 and 10 vertices. Sulanke [17] proved that there are 396,784 irreducible triangulations of $\mathbb{S}_2$, each with between 10 and 17 vertices. Barnette [1] proved that the embeddings of K_6 and $K_7 - K_3$ in $\mathbb{N}_1$ are the only irreducible triangulations of $\mathbb{N}_1$. Sulanke [20] proved that there are 29 irreducible triangulations of $\mathbb{N}_2$, each with between 8 and 11 vertices (correcting an earlier result by Lawrencenko and Negami [10]). Sulanke [17] proved that there are 9708 irreducible triangulations of $\mathbb{N}_3$, each with between 9 and 16 vertices. Sulanke [17] proved that there are 6,297,982 irreducible triangulations of $\mathbb{N}_4$, each with between 9 and 22 vertices. Using the lists of all irreducible triangulations due to Sulanke [18] and a naive algorithm for counting cliques,[6] we have computed the set X in Theorem 1 for each of the above surfaces; see Table 1. This data with Theorem 1 implies the following results.

[6] The code is available from the authors upon request.

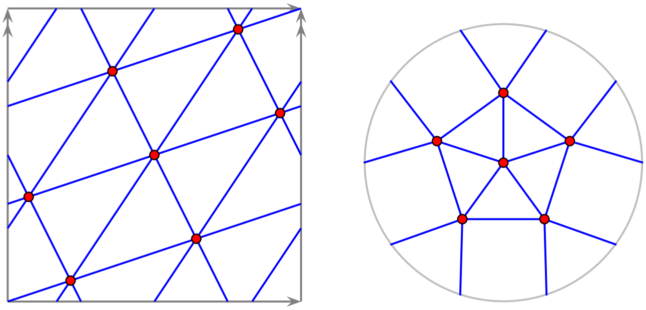

Fig. 2. K_7 embedded in the torus, and K_6 embedded in the projective plane.

Table 1
The maximum excess of an n-vertex irreducible triangulation of Σ.

Σ	χ	ω	$n=3$	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	Max
$\mathbb{S}_0$	2	3	**−16**																		−16
$\mathbb{S}_1$	0	7			**72**	48	40	32													72
$\mathbb{S}_2$	−2	8						**208**	160	136	128	120	96	88	80						208
$\mathbb{N}_1$	1	6		**16**	8																16
$\mathbb{N}_2$	0	6				**48**	**48**	40	32												48
$\mathbb{N}_3$	−1	7					**104**	**104**	96	80	80	72	64	56							104
$\mathbb{N}_4$	−2	8					**216**	208	152	136	136	136	128	120	112	107	99	91	83	75	216

Proposition 1. *Every planar graph G with $|V(G)| \geq 3$ has at most $8|V(G)| - 16$ cliques, as proved by Wood [22]. Moreover, a planar graph G has $8|V(G)| - 16$ cliques if and only if G is obtained from the embedding of K_3 in $\mathbb{S}_0$ by repeatedly splitting triangles.*

Proposition 2. *Every toroidal graph G has at most $8|V(G)|+72$ cliques. Moreover, a toroidal graph G has $8|V(G)| + 72$ cliques if and only if G is obtained from the embedding of K_7 in $\mathbb{S}_1$ by repeatedly splitting triangles (see Fig. 2).*

Proposition 3. *Every graph G embeddable in $\mathbb{S}_2$ has at most $8|V(G)| + 208$ cliques. Moreover, a graph G embeddable in $\mathbb{S}_2$ has $8|V(G)| + 208$ cliques if and only if G is obtained from one of the following two graph embeddings in $\mathbb{S}_2$ by repeatedly splitting triangles[7]:*

graph #1: bcde,aefdghic,abiehfgd,acgbfihe,adhcigfb,begchjid,bdcfeijh,bgjfcedi,bhdfjgec,fhgi

graph #6: bcde,aefdghijc,abjehfgd,acgbfjihe,adhcjgfb,begchjd,bdcfejh,bgjfcedi,bhdj,bidfhgec.

Proposition 4. *Every projective planar graph G has at most $8|V(G)| + 16$ cliques. Moreover, a projective planar graph G has $8|V(G)| + 16$ cliques if and only if G is obtained from the embedding of K_6 in $\mathbb{N}_1$ by repeatedly splitting triangles (see Fig. 2).*

Proposition 5. *Every graph G embeddable in the Klein bottle $\mathbb{N}_2$ has at most $8|V(G)| + 48$ cliques. Moreover, a graph G embeddable in $\mathbb{N}_2$ has $8|V(G)| + 48$ cliques if and only G is obtained from one of the following three graph embeddings in $\mathbb{N}_2$ by repeatedly splitting triangles (see Fig. 3):*

graph #3: bcdef,afgdhec,abefd,acfhbge,adghbcf,aecdhgb,bfhed,bdfge

graph #6: bcde,aefdghc,abhegd,acgbfhe,adhcgfb,beghd,bdcefh,bgfdec

graph #26: bcdef,afghidec,abefd,acfhgibe,adbcf,aecdhigb,bfidh,bgdfi,bhfgd.

[7] This representation describes a graph with vertex set $\{a, b, c, \ldots\}$ by adjacency lists of the vertices in order $a, b, c, \ldots$. The graph # refers to the position in Sulanke's file [18].

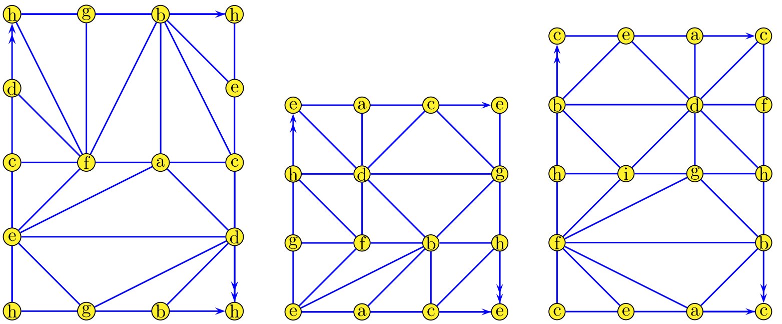

Fig. 3. Irreducible triangulations of $\mathbb{N}_2$ with maximum excess: left-to-right #3, #6, #26.

Proposition 6. *Every graph G embeddable in* $\mathbb{N}_3$ *has at most* $8|V(G)| + 104$ *cliques. Moreover, a graph G embeddable in* $\mathbb{N}_3$ *has* $8|V(G)| + 104$ *cliques if and only if G is obtained from one of the following 15 graph embeddings in* $\mathbb{N}_3$ *by repeatedly splitting triangles:*

graph #1: bcde,aefdghic,abiegfd,acfbgie,adicghfb,behigcd,bdifceh,bgefi,bhfgdec
graph #3: bcde,aefdghic,abiehd,achfbgie,adichgfb,begihd,bdifeh,bgecdfi,bhfgdec
graph #4: bcde,aefdghic,abiehd,achifbge,adgichfb,behgid,bdeifh,bgfecdi,bhdfgec
graph #6: bcde,aefdghic,abiehfd,acfbgihe,adhcifb,beighcd,bdifh,bgfcedi,bhdgfec
graph #8: bcde,aefdghic,abiehgfd,acfbgihe,adhcigfb,begcd,bdiefch,bgcedi,bhdgec
graph #10: bcde,aefdghic,abifegd,acgbfhie,adigcfb,becighd,bdceifh,bgfdi,bhdegfc
graph #12: bcde,aefdghic,abifehd,achfbgie,adihcfb,becighd,bdifh,bgfdcei,bhedgfc
graph #14: bcde,aefdghic,abigehd,achfbgie,adihcgfb,beghd,bdicefh,bgfdcei,bhedgc
graph #16: bcde,aefgdhic,abiegd,acgbhfie,adicghfb,behdig,bfihecd,bdfegi,bhgfdec
graph #19: bcde,aefghdic,abiehd,achbifge,adgichfb,behidg,bfdeih,bgifecd,bdfhgec
graph #20: bcde,aefghdic,abigehd,achbifge,adgchifb,beidg,bfdecih,bgiecd,bdfehgc
graph #21: bcde,aefghdic,abihegd,acgfhbie,adigchfb,behdg,bfdceih,bgicefd,bdeghc
graph #22: bcde,aefghdic,abihegd,acgfibhe,adhcgifb,beidg,bfdceih,bgiced,bdfeghc
graph #82: bcdef,afgdhiec,abefd,acfigbhe,adhgibcf,aecdihgb,bfheid,bdegfi,bhfdge
graph #2464: bcdef,afghijdec,abefd,acfhigjbe,adbcf,aecdhjigb,bfidjh,bgjfdi,bhdgfj,bifhgd.

Proposition 7. *Every graph G embeddable in* $\mathbb{N}_4$ *has at most* $8|V(G)| + 216$ *cliques. Moreover, a graph G embeddable in* $\mathbb{N}_4$ *has* $8|V(G)| + 216$ *cliques if and only if G is obtained from one of the following three graph embeddings in* $\mathbb{N}_4$ *by repeatedly splitting triangles:*

graph #1: bcdef,afdgehic,abiegfhd,achgbfie,adicgbhf,aehcgidb,bdhifce,befcdgi,bhgfdec
graph #2: bcdef,afdgehic,abifehgd,acgbfhie,adigbhcf,aecighdb,bdchfie,becgfdi,bhdegfc
graph #3: bcdef,afdgheic,abihfegd,acgbfihe,adhbigcf,aechgidb,bdceifh,bgfcide,begfdhc.

Note that the three embeddings in Proposition 7 are of the same graph.

4. A bound for all surfaces

Recall that Σ is a surface with Euler characteristic χ, and if $\Sigma = \mathbb{S}_0$ then $\omega = 3$, otherwise ω is the maximum integer such that K_ω embeds in Σ. We start with the following upper bound on the minimum degree of a graph.

Lemma 2. *Assume* $\Sigma \neq \mathbb{S}_0$. *Then every graph G embeddable in* Σ *has minimum degree at most*

$$6 + \frac{\omega^2 - 5\omega - 7}{|V(G)|}.$$

Proof. By the definition of ω, the complete graph $K_{\omega+1}$ cannot be embedded in Σ. Thus if $\Sigma = \mathbb{S}_g$ then $g = \frac{1}{2}(2-\chi) \le \lceil \frac{1}{12}(\omega-2)(\omega-3) \rceil - 1$, and if $\Sigma = \mathbb{N}_h$ then $h = 2-\chi \le \lceil \frac{1}{6}(\omega-2)(\omega-3) \rceil - 1$. In each case, it follows that $2-\chi \le \frac{1}{6}(\omega-2)(\omega-3) - \frac{1}{6}$. That is,

$$-6\chi \le \omega^2 - 5\omega - 7. \tag{1}$$

Say G has minimum degree d. It follows from Euler's Formula that $|E(G)| \le 3|V(G)| - 3\chi$. By (1),

$$d \le \frac{2|E(G)|}{|V(G)|} \le \frac{6|V(G)| - 6\chi}{|V(G)|} \le 6 + \frac{\omega^2 - 5\omega - 7}{|V(G)|}. \quad \square$$

For graphs in which the number of vertices is slightly more than ω, Lemma 2 can be reinterpreted as follows.

Lemma 3. *Assume $\Sigma \ne \mathbb{S}_0$. Let $s := \lceil \sqrt{\omega+11} - 3 \rceil \ge 1$. Let G be a graph embeddable in Σ. If G has at most $\omega+1$ vertices, then G has minimum degree at most $\omega-1$. If G has at least $\omega+j$ vertices, where $j \in [2, s]$, then G has minimum degree at most $\omega-j+1$.*

Proof. Say G has minimum degree d. If $|V(G)| \le \omega$, then trivially $d \le \omega-1$. If $|V(G)| = \omega+1$, then G is not complete (by the definition of ω), again implying that $d \le \omega-1$. Now assume $|V(G)| \ge \omega+j$ for some $j \in [2, s]$. By Lemma 2,

$$d \le 6 + \frac{\omega^2 - 5\omega - 7}{\omega+j} = \omega - j + 1 + \frac{j^2 + 5j - 7}{\omega+j}.$$

Since $j \le s < \sqrt{\omega+11} - 2$, we have $j^2 + 5j - 7 \le s^2 + 4s - 7 + j < \omega + j$. It follows that $d \le \omega - j + 1$. $\square$

Now we prove our first upper bound on the number of cliques.

Lemma 4. *Assume $\Sigma \ne \mathbb{S}_0$. Let $s := \lceil \sqrt{\omega+11} - 3 \rceil \ge 1$. Let G be an n-vertex graph embeddable in Σ. Then*

$$c(G) \le \begin{cases} \frac{5}{2} 2^{\omega} & \text{if } n \le \omega + s, \\ \frac{5}{2} 2^{\omega} + (n-\omega-s)2^{\omega-s+1} & \text{otherwise}. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of G such that v_i has minimum degree in the subgraph $G_i := G - \{v_1, \ldots, v_{i-1}\}$. Let d_i be the degree of v_i in G_i (which equals the minimum degree of G_i). Charge each non-empty clique C in G to the vertex $v_i \in C$ with i minimum. Charge the clique $\emptyset$ to v_n.

We distinguish three types of vertices. Vertex v_i is type-1 if $i \in [1, n-\omega-s]$. Vertex v_i is type-2 if $i \in [n-\omega-s+1, n-\omega]$. Vertex v_i is type-3 if $i \in [n-\omega+1, n]$.

Each clique charged to a type-3 vertex is contained in $\{v_{n-\omega+1}, \ldots, v_n\}$, and there are at most 2^{ω} such cliques.

Say C is a clique charged to a type-1 or type-2 vertex v_i. Then $C - \{v_i\}$ is contained in $N_{G_i}(v_i)$, which consists of d_i vertices. Thus the number of cliques charged to v_i is at most 2^{d_i}. Recall that d_i equals the minimum degree of G_i, which has $n-i+1$ vertices.

If v_i is type-2 then, by Lemma 3 with $j = n-\omega-i+1 \in [1, s]$, we have $d_i \le \omega - j + 1$, and $d_i \le \omega - j$ if $j = 1$. Thus the number of cliques charged to type-2 vertices is at most

$$2^{\omega-1} + \sum_{j=2}^{s} 2^{\omega-j+1} \le 2^{\omega-1} + \sum_{j=1}^{\omega-1} 2^{j} < \frac{3}{2} 2^{\omega}.$$

If v_i is type-1 then G_i has more than $\omega+s$ vertices, and thus $d_i \le \omega - s + 1$ by Lemma 3 with $j = s$. Thus the number of cliques charged to type-1 vertices is at most $(n-\omega-s)2^{\omega-s+1}$. $\square$

We now prove the main result of this section; it provides lower and upper bounds on the maximum number of cliques in a graph embeddable in Σ.

Theorem 2. *Every n-vertex graph embeddable in Σ contains at most $8n+\frac{5}{2}2^{\omega}+o(2^{\omega})$ cliques. Moreover, for each $n \geq \omega$, there is an n-vertex graph embeddable in Σ with $8(n-\omega)+2^{\omega}$ cliques.*

Proof. To prove the upper bound, we may assume that $\Sigma \neq \mathbb{S}_0$, and by Theorem 1, we need only consider n-vertex irreducible triangulations of Σ. Joret and Wood [7] proved that, in this case, $n \leq 22-13\chi$. By Eq. (1),

$$n \leq 22-13\chi \leq 22+\frac{13}{6}(\omega^2-5\omega-7) < 3\omega^2.$$

If $n \leq \omega+s$ then $c(G) \leq \frac{5}{2}2^{\omega}$ by Lemma 4. If $n > \omega+s$ then by the same lemma,

$$c(G) \leq \frac{5}{2}2^{\omega}+(3\omega^2-\omega-s)2^{\omega-s+1} < \frac{5}{2}2^{\omega}+3\omega^2 2^{\omega-s+1} < \frac{5}{2}2^{\omega}+2^{\omega-s+2\log\omega+3}.$$

Since $s \in \Theta(\sqrt{\omega})$, we have $c(G) \leq \frac{5}{2}2^{\omega}+o(2^{\omega})$.

To prove the lower bound, start with K_ω embedded in Σ (which has 2^{ω} cliques). Now, while there are less than n vertices, insert a new vertex adjacent to each vertex of a single face. Each new vertex adds at least 8 new cliques. Thus we obtain an n-vertex graph embedded in Σ with at least $8(n-\omega)+2^{\omega}$ cliques. □

5. Concluding conjectures

We conjecture that the upper bound in Theorem 2 can be improved to more closely match the lower bound.

Conjecture 1. *Every graph G embeddable in Σ has at most $8|V(G)|+2^{\omega}+o(2^{\omega})$ cliques.*

If K_ω triangulates Σ, then we conjecture the following exact answer.

Conjecture 2. *Suppose that K_ω triangulates Σ. Then every graph G embeddable in Σ has at most $8(|V(G)|-\omega)+2^{\omega}$ cliques, with equality if and only if G is obtained from K_ω by repeatedly splitting triangles.*

By Theorem 1, this conjecture is equivalent to the following.

Conjecture 3. *Suppose that K_ω triangulates Σ. Then K_ω is the only irreducible triangulation of Σ with maximum excess.*

The results in Section 3 confirm Conjectures 2 and 3 for $\mathbb{S}_0$, $\mathbb{S}_1$ and $\mathbb{N}_1$.

Now consider surfaces possibly with no complete graph triangulation. Then the bound $c(G) \leq 8(|V(G)|-\omega)+2^{\omega}$ (in Conjecture 2) is false for $\mathbb{S}_2$, $\mathbb{N}_2$, $\mathbb{N}_3$ and $\mathbb{N}_4$. Loosely speaking, this is because these surfaces have 'small' ω compared to χ. In particular, $\omega = \left\lfloor \frac{1}{2}(7+\sqrt{49-24\chi}) \right\rfloor$ except for $\mathbb{S}_0$ and $\mathbb{N}_2$, and $\omega = \frac{1}{2}(7+\sqrt{49-24\chi})$ if and only if K_ω triangulates $\Sigma \neq \mathbb{S}_0$. This phenomenon motivates the following conjecture.

Conjecture 4. *Every graph G embeddable in Σ has at most*

$$8|V(G)|-4(7+\sqrt{49-24\chi})+2^{(7+\sqrt{49-24\chi})/2}$$

cliques, with equality if and only if K_ω triangulates Σ and G is obtained from K_ω by repeatedly splitting triangles.

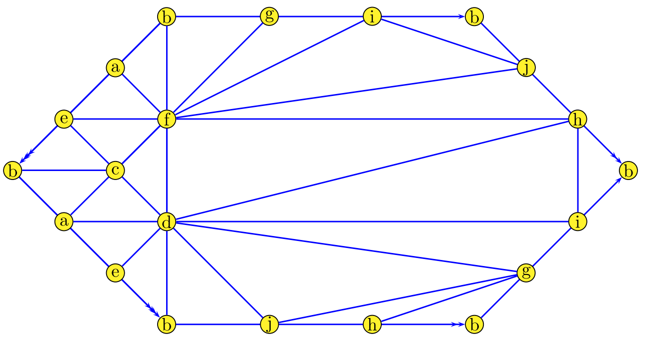

Fig. 4. Triangulation #2464 of $\mathbb{N}_3$.

There are two irreducible triangulations of $\mathbb{S}_2$ with maximum excess, there are three irreducible triangulations of $\mathbb{N}_2$ with maximum excess, there are 15 irreducible triangulations of $\mathbb{N}_3$ with maximum excess, and there are three irreducible triangulations of $\mathbb{N}_4$ with maximum excess. This suggests that for surfaces with no complete graph triangulation, a succinct characterisation of the extremal examples (as in Conjecture 3) might be difficult. Nevertheless, we conjecture the following strengthening of Conjecture 3 for all surfaces.

Conjecture 5. *Every irreducible triangulation of Σ with maximum excess contains K_ω as a subgraph.*

A triangulation of a surface Σ is *vertex-minimal* if it has the minimum number of vertices in a triangulation of Σ. Of course, every vertex-minimal triangulation is irreducible. Ringel [15] and Jungerman and Ringel [8] together proved that the order of a vertex-minimal triangulation is ω if K_ω triangulates Σ, is $\omega + 2$ if $\Sigma \in \{\mathbb{S}_2, \mathbb{N}_2, \mathbb{N}_3\}$, and is $\omega + 1$ for every other surface.

Triangulations #26 of $\mathbb{N}_2$ and #2464 of $\mathbb{N}_3$ are the only triangulations in Propositions 1–7 that are not vertex-minimal. Triangulation #26 of $\mathbb{N}_2$ is obtained from two embeddings of K_6 in $\mathbb{N}_1$ joined at the face *bdf* (see Fig. 3). Triangulation #2464 of $\mathbb{N}_3$ is obtained by joining an embedding of K_6 in $\mathbb{N}_1$ and an embedding of K_7 in $\mathbb{S}_1$ at the face *bdf* (see Fig. 4).

Every other triangulation in Propositions 1–7 is obtained from an embedding of K_ω by adding (at most two) vertices and edges until a vertex-minimal triangulation is obtained. This provides some evidence for our final conjecture.

Conjecture 6. *For every surface Σ, the maximum excess is attained by some vertex-minimal triangulation of Σ that contains K_ω as a subgraph. Moreover, if $\Sigma \notin \{\mathbb{N}_2, \mathbb{N}_3\}$ then every irreducible triangulation with maximum excess is vertex-minimal and contains K_ω as a subgraph.*

We have verified Conjectures 4–6 for $\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2, \mathbb{N}_1, \mathbb{N}_2, \mathbb{N}_3$ and $\mathbb{N}_4$.

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