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Three-Dimensional Grid Drawings with Sub-Quadratic Volume

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ABSTRACT. A three-dimensional grid drawing of a graph is a placement of the vertices at distinct points with integer coordinates, such that the straight line-segments representing the edges are pairwise non-crossing. A $\mathcal{O}(n^{3/2})$ volume bound is proved for three-dimensional grid drawings of graphs with bounded degree, graphs with bounded genus, and graphs with no bounded complete graph as a minor. The previous best bound for these graph families was $\mathcal{O}(n^2)$. These results (partially) solve open problems due to Pach, Thiele, and Tóth (1997) and Felsner, Liotta, and Wismath (2001).

1. Introduction

A *three-dimensional straight-line grid drawing* of a graph, henceforth called a *3D drawing*, is a placement of the vertices at distinct points in $\mathbb{Z}^3$ (called *gridpoints*), such that the straight line-segments representing the edges are pairwise non-crossing. That is, distinct edges only intersect at common endpoints, and each edge only intersects a vertex that is an endpoint of that edge. In contrast to the case in the plane, it is well known that every graph has a 3D drawing. We are therefore interested in optimising certain measures of the aesthetic quality of such drawings.

The *bounding box* of a 3D drawing is the minimum axis-aligned box containing the drawing. If the bounding box has side lengths $X-1$, $Y-1$ and $Z-1$, then we speak of an $X \times Y \times Z$ drawing with *volume* $X \cdot Y \cdot Z$. That is, the volume of a 3D drawing is the number of gridpoints in the bounding box. This definition is formulated so that 2D drawings have positive volume. We are interested in 3D drawings with small volume, which is a widely studied problem [**3, 4, 5, 6, 9, 10, 11, 14, 24, 25, 27**]. Three-dimensional graph drawings in which the vertices are allowed real coordinates have also been studied (see the references in [**10**]). The authors have also established bounds on the volume of three-dimensional *polyline* grid drawings, where bends in the edges are also at gridpoints [**10**]. Table 1

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summarises the best known upper bounds on the volume of 3D drawings, including those established in this paper.

TABLE 1. Upper bounds on the volume of 3D drawings of graphs with n vertices and m edges.

graph family	volume	reference
arbitrary	$\mathcal{O}(n^3)$	Cohen *et al.* [**5**]
arbitrary	$\mathcal{O}(m^{4/3}n)$	Theorem 4
maximum degree Δ	$\mathcal{O}(\Delta mn)$	Theorem 3
constant maximum degree	$\mathcal{O}(n^{3/2})$	Theorem 10
constant chromatic number	$\mathcal{O}(n^2)$	Pach *et al.* [**24**]
constant chromatic number	$\mathcal{O}(m^{2/3}n)$	Theorem 6
no K_h-minor (h constant)	$\mathcal{O}(n^{3/2})$	Theorem 9
constant genus	$\mathcal{O}(n^{3/2})$	Theorem 8
constant tree-width	$\mathcal{O}(n)$	Dujmović and Wood [**11**]

Cohen *et al.* [**5**] proved that every graph has a 3D drawing with $\mathcal{O}(n^3)$ volume, and that this bound is asymptotically optimal for complete graphs K_n. Our edge-sensitive bounds of $\mathcal{O}(m^{4/3}n)$ and $\mathcal{O}(\Delta mn)$ are greater than $\mathcal{O}(n^3)$ in the worst case. It is an open problem whether there are edge-sensitive bounds that match the $\mathcal{O}(n^3)$ bound in the case of complete graphs.

Pach *et al.* [**24**] proved that graphs with constant chromatic number have 3D drawings with $\mathcal{O}(n^2)$ volume. For c-colourable graphs the actual bound is $\mathcal{O}(c^2n^2)$. Our edge-sensitive volume bound of $\mathcal{O}(m^{2/3}n)$ is an improvement on this result for graphs with constant chromatic number and $o(n^{3/2})$ edges. Pach *et al.* [**24**] also proved an $\Omega(n^2)$ lower bound for the volume of 3D drawings of the complete bipartite graph $K_{n,n}$. This lower bound was generalised to all graphs by Bose *et al.* [**3**], who proved that every 3D drawing has volume at least $\frac{1}{8}(n+m)$.

Graphs with constant maximum degree have constant chromatic number, and thus, by the result of Pach *et al.* [**24**], have 3D drawings with $\mathcal{O}(n^2)$ volume. Pach *et al.* [**24**] conjectured that graphs with constant maximum degree have 3D drawings with $o(n^2)$ volume. We verify this conjecture by proving that graphs with constant maximum degree have 3D drawings with $\mathcal{O}(n^{3/2})$ volume.

The first $\mathcal{O}(n)$ upper bound on the volume of 3D drawings was established by Felsner *et al.* [**14**] for outerplanar graphs. This result was generalised by the authors for graphs with constant tree-width [**11**]. Felsner *et al.* [**14**] proposed the following inviting open problem: does every planar graph have a 3D drawing with $\mathcal{O}(n)$ volume? In this paper we provide a partial solution to this question, by proving that planar graphs have 3D drawings with $\mathcal{O}(n^{3/2})$ volume. Note that $\mathcal{O}(n^2)$ is the optimal area for plane 2D grid drawings, and $\mathcal{O}(n^2)$ was the previous best upper bound on the volume of 3D drawings of planar graphs.

A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. The *genus* of a graph G is the minimum γ such that G can be embedded in the orientable surface with γ handles. Of course, planar graphs have genus 0 and no K_5-minor. A generalisation of our result for planar graphs is that every graph with constant genus or with no K_h-minor for constant h has a 3D drawing with $\mathcal{O}(n^{3/2})$ volume.

2. Track Layouts

We consider undirected, finite, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. A *vertex c-colouring* of G is a partition $\{V_i : 1 \leq i \leq c\}$ of $V(G)$, such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. Each $i \in \{1, 2, \ldots, c\}$ is a *colour*, and each set V_i is a *colour class*. At times it will be convenient to write $\mathsf{col}(v) = i$ rather than $v \in V_i$. If G has a vertex c-colouring then G is *c-colourable*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum c such that G is c-colourable.

Let $\{V_i : 1 \leq i \leq c\}$ be a vertex c-colouring of a graph G. Let $<_i$ be a total order on each colour class V_i. Then each pair $(V_i, <_i)$ is a *track*, and $\{(V_i, <_i) : 1 \leq i \leq t\}$ is a *t-track assignment* of G. To ease the notation we denote track assignments by $\{V_i : 1 \leq i \leq c\}$ when the ordering on each colour class is implicit. An *X-crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct colours i and j. A *t-track layout* of G consists of a t-track assignment of G with no X-crossing. The *track-number* of G, denoted by $\mathsf{tn}(G)$, is the minimum t such that G has a t-track layout[1].

Track layouts were introduced in [**9, 11**] although they are implicit in many previous works [**14, 18, 19**]. Track layouts and 3D drawings are closely related, as illustrated by the following result by Dujmović *et al.* [**9**]. Also note that there is a tight relationship between track layouts and another type of graph layout called a *queue layout* [**27**], which is a dual structure to a book embedding introduced by Heath *et al.* [**18, 19**].

THEOREM 1 ([**9**]). *Every n-vertex graph G with track-number $\mathsf{tn}(G) \leq t$ has a $2t \times 4t \times 4t\lceil \frac{n}{t} \rceil$ drawing with $\mathcal{O}(t^2 n)$ volume. Conversely, if a graph G has an $X \times Y \times Z$ drawing then G has track-number $\mathsf{tn}(G) \leq 2XY$.*

We have the following upper bounds on the track-number.

LEMMA 1. *Let G be a graph with n vertices, maximum degree Δ, path-width p, tree-width w, genus γ, and with no K_h-minor. Then the track-number of G satisfies:* (a) $\mathsf{tn}(G) \leq p + 1$, (b) $\mathsf{tn}(G) \leq \mathcal{O}(6^{4^w})$, (c) $\mathsf{tn}(G) \leq 72w\Delta$, (d) $\mathsf{tn}(G) \in \mathcal{O}(\gamma^{1/2} n^{1/2})$, (e) $\mathsf{tn}(G) \in \mathcal{O}(h^{3/2} n^{1/2})$.

PROOF. Part (a) is by Dujmović *et al.* [**9**]. Parts (b) and (c) are by the authors [**11**]. Gilbert *et al.* [**16**] and Djidjev [**8**] independently proved that G has a $\mathcal{O}(\gamma^{1/2} n^{1/2})$-separator, and thus has $\mathcal{O}(\gamma^{1/2} n^{1/2})$ path-width (see Bodlaender [**2**, Theorem 20(iii)]). Hence (d) follows from (a). Similarly (e) follows from the result by Alon *et al.* [**1**] that G has a $\mathcal{O}(h^{3/2} n^{1/2})$-separator. □

The next result is the fundamental contribution of this section.

THEOREM 2. *Every graph G with m edges and maximum degree Δ has track-number $\mathsf{tn}(G) \leq 14\sqrt{\Delta m}$.*

To prove Theorem 2 we introduce the following concept. A vertex colouring is a *strong star colouring* if between every pair of colour classes, all edges (if any) are incident to a single vertex. That is, each bichromatic subgraph consists of a star and possibly some isolated vertices. The *strong star chromatic number* of a

[1] Note that this definition of *track-number* is unrelated to that of Gyárfás and West [**17**].

graph G, denoted by $\chi_{\text{sst}}(G)$, is the minimum number of colours in a strong star colouring of G. Note that *star colourings*, in which each bichromatic subgraph is a forest of stars, have also been studied (see [**15, 23**] for example). The *star chromatic number* of a graph G, denoted by $\chi_{\text{st}}(G)$, is the minimum number of colours in a star-colouring of G.

With an arbitrary order on each colour class in a strong star colouring, there is no X-crossing. Thus track-number $\mathsf{tn}(G) \leq \chi_{\text{sst}}(G)$ for every graph G, and Theorem 2 is an immediate corollary of the next lemma.

LEMMA 2. *Every graph G with m edges and maximum degree $\Delta \geq 1$ has strong star chromatic number $\chi_{\text{sst}}(G) \leq 14\sqrt{\Delta m}$.*

The proof of Lemma 2 uses the weighted version of the Lovász Local Lemma [**13**].

LEMMA 3 ([**22**, p. 221]). *Let $\mathcal{E} = \{A_1, \dots, A_n\}$ be a set of 'bad' events. Let $0 \leq p \leq \frac{1}{4}$ be a real number, and let $t_1, \dots, t_n \geq 1$ be integers. Suppose that for all $A_i \in \mathcal{E}$,*

(a) *the probability $\mathbf{P}(A_i) \leq p^{t_i}$,*
(b) *A_i is mutually independent of $\mathcal{E} \setminus (\{A_i\} \cup D_i)$ for some $D_i \subseteq \mathcal{E}$, and*
(c) $\displaystyle\sum_{A_j \in D_i} (2p)^{t_j} \leq \frac{t_i}{2}$.

Then with positive probability, no event in $\mathcal{E}$ occurs.

PROOF OF LEMMA 2. Let $c \geq 4$ be a positive integer to be specified later. Let $p = \frac{1}{c}$. Then $0 < p \leq \frac{1}{4}$. For each vertex $v \in V(G)$, randomly and independently choose $\mathsf{col}(v)$ from $\{1, 2, \dots, c\}$.

For each edge $vw \in E(G)$, let A_{vw} be the *type-I* event that $\mathsf{col}(v) = \mathsf{col}(w)$. Let E' be the set of arcs $E' = \{(v, w), (w, v) : vw \in E(G)\}$. For each pair of arcs $(v, w), (x, y) \in E'$ with no endpoint in common, let $B_{(v,w),(x,y)}$ be the *type-II* event that $\mathsf{col}(v) = \mathsf{col}(x)$ and $\mathsf{col}(w) = \mathsf{col}(y)$.

We will apply Lemma 3 to obtain a colour assignment such that no type-I event and no type-II event occurs. No type-I event implies that we have a (proper) vertex colouring. No type-II event implies that no two disjoint edges share the same pair of colours; that is, we have a strong star colouring.

For each type-I event A, $\mathbf{P}(A) = \frac{1}{c}$. Let $t_A = 1$. Then $\mathbf{P}(A) = p^{t_A}$. For each type-II event B, $\mathbf{P}(B) = \frac{1}{c^2}$. Let $t_B = 2$. Then $\mathbf{P}(B) = p^{t_B}$. Thus condition (a) of Lemma 3 is satisfied.

An event involving a particular set of vertices is dependent only on other events involving at least one of the vertices in that set. Each vertex is involved in at most Δ type-I events, and at most $2\Delta|E'| = 4\Delta m$ type-II events. A type-I event involves two vertices, and is thus mutually independent of all but at most 2Δ type-I events and at most $8\Delta m$ type-II events. A type-II event involves four vertices, and is thus mutually independent of all but at most 4Δ type-I events and at most $16\Delta m$ type-II events.

For condition (c) of Lemma 3 to hold we need $2\Delta(\frac{2}{c})^1 + 8\Delta m(\frac{2}{c})^2 \leq \frac{1}{2}$ for the type-I events, and $4\Delta(\frac{2}{c})^1 + 16\Delta m(\frac{2}{c})^2 \leq 1$ for the type-II events. It is a happy coincidence that these two equations are equivalent, and it is easily verified that $c = \lceil 4(\Delta + \sqrt{\Delta(1 + 4m)}\,)\rceil \geq 4$ is a solution.

Thus by Lemma 3, with positive probability no type-I event and no type-II event occurs. Thus for every vertex $v \in V(G)$, there exists $\mathsf{col}(v) \in \{1, \dots, c\}$ such that no type-I event and no type-II event occurs. As proved above such a colouring is a strong star colouring. Since $\Delta \leq \sqrt{\Delta m}$, the number of colours $c \leq \lceil 4(1+\sqrt{5})\sqrt{\Delta m}\rceil \leq 14\sqrt{\Delta m}$. □

Theorems 1 and 2 imply:

THEOREM 3. *Every graph with n vertices, m edges and maximum degree* Δ *has a* $\mathcal{O}((\Delta m)^{1/2}) \times \mathcal{O}((\Delta m)^{1/2}) \times \mathcal{O}(n)$ *drawing with* $\mathcal{O}(\Delta mn)$ *volume.* □

We have the following corollary of Lemma 2.

LEMMA 4. *Every graph G with m edges has strong star chromatic number* $\chi_{\mathrm{sst}}(G) \leq 15m^{2/3}$.

PROOF. Let X be the set of vertices of G with degree greater than $\frac{1}{4}m^{1/3}$. Let H be the subgraph of G induced by $V(G)\setminus X$. Thus H has maximum degree at most $\frac{1}{4}m^{1/3}$. By Lemma 2, H has a strong star colouring with at most $14(\frac{1}{4}m^{1/3}m)^{1/2} = 7m^{2/3}$ colours. Now $|X| \leq 2m/(\frac{1}{4}m^{1/3}) = 8m^{2/3}$. By adding each vertex in X to its own colour class we obtain a strong star colouring of G with at most $15m^{2/3}$ colours. □

Since $\mathsf{tn}(G) \leq \chi_{\mathrm{sst}}(G)$, Lemma 4 implies that $\mathsf{tn}(G) \leq 15m^{2/3}$, and by Theorem 1 we have:

THEOREM 4. *Every graph with n vertices and m edges has a* $\mathcal{O}(m^{2/3}) \times \mathcal{O}(m^{2/3}) \times \mathcal{O}(n)$ *drawing with* $\mathcal{O}(m^{4/3}n)$ *volume.* □

3. Sub-Quadratic Volume Bounds

Vertex colourings [**24**] and track layouts [**9**] have previously been used to produce 3D drawings with small volume. In the following sequence of results we combine vertex colourings and track layouts to reduce the quadratic dependence on t in Theorem 1 to linear. This comes at the expense of a higher dependence on the chromatic number. However, in the intended applications, the chromatic number will be constant, or at least will be independent of the size of the graph. The proof of the next lemma is a further generalisation of the 'moment curve' method for three-dimensional graph drawing [**5, 9, 24**], which dates to the seminal construction by Erdös [**12**] for the no-three-in-line problem.

LEMMA 5. *Let G be a graph with a vertex c-colouring* $\{V_i : 0 \leq i \leq c-1\}$, *and a track layout* $\{T_{i,j} : 0 \leq i \leq c-1, 1 \leq j \leq t_i\}$, *such that each* $T_{i,j} \subseteq V_i$. *Then G has a* $\mathcal{O}(c) \times \mathcal{O}(c^2\, t) \times \mathcal{O}(c^5\, tn')$ *drawing, where* $t = \max_i t_i$ *and* $n' = \max_{i,j} |T_{i,j}|$.

PROOF. Let p be the minimum prime such that $p \geq c$. Then $p < 2c$ by Bertrand's postulate. Let $v(i,j,k)$ denote the k^{th} vertex in track $T_{i,j}$. Define

$$Y(i,j) = p(2\,i\,t + j) + (i^2 \bmod p), \text{ and}$$
$$Z(i,j,k) = p(20\,c\,i\,n' \cdot Y(i,j) + k) + (i^3 \bmod p)\ .$$

Position each vertex $v(i,j,k)$ at the gridpoint $(i, Y(i,j), Z(i,j,k))$, and draw each edge as a line-segment between its endpoints. Since $Y(i,j) \in \mathcal{O}(c^2\, t)$ and $Z(i,j,k) \in \mathcal{O}(c^3\, n' \cdot Y(i,j))$, the drawing is $\mathcal{O}(c) \times \mathcal{O}(c^2\, t) \times \mathcal{O}(c^5\, tn')$.

Observe that the tracks from a single colour class are within a distinct YZ-plane, each track occupies a distinct vertical line, and the Z-coordinates of the vertices within a track preserve the given ordering of that track. In addition, the Y-coordinates satisfy the following property.

Claim 1. For all distinct colours i_1 and i_2 and for all $1 \leq j_1, j_2 \leq t$, we have that $2c\,|\,(Y(i_1,j_1) - Y(i_2,j_2)\,|$ is greater than the Y-coordinate of any vertex.

Proof. Without loss of generality $i_1 > i_2$. Observe that every Y-coordinate is less than $p(2(c-1)t+t)+p \;=\; p(2\,c\,t - t + 1) \;\leq\; 2cpt$. Now $2c\,|\,Y(i_1,j_1) - Y(i_2,j_2)\,| \;>\; 2c\,|\,p(2\,i_1\,t+1) - p(2\,i_2\,t+t+1)\,| \;\geq\; 2cp\,|\,2(i_2+1)t - (2i_2t+t)\,| \;=\; 2cpt$. □

We first prove that the only vertices each edge intersects are its own endpoints. It suffices to prove that if three tracks are collinear in the XY-plane then they are all from the same colour class. Loosely speaking, an edge does not pass through any track. Clearly two tracks from the same colour class are not collinear (in the XY-plane) with a third track from a distinct colour class. Thus we need only consider tracks $\{T(i_\alpha, j_\alpha) : 1 \leq i_\alpha \leq 3\}$ from three distinct colour classes $\{i_1, i_2, i_3\}$. Let R be the determinant,

$$R = \begin{vmatrix} 1 & i_1 & Y(i_1,j_1) \\ 1 & i_2 & Y(i_2,j_2) \\ 1 & i_3 & Y(i_3,j_3) \end{vmatrix} \quad .$$

If the tracks $\{T(i_\alpha, j_\alpha) : 1 \leq i_\alpha \leq 3\}$ are collinear in the XY-plane then $R = 0$. However $Y(i,j) \equiv i^2 \pmod{p}$, and thus

$$R \equiv \begin{vmatrix} 1 & i_1 & i_1^2 \\ 1 & i_2 & i_2^2 \\ 1 & i_3 & i_3^2 \end{vmatrix} = \prod_{1 \leq \alpha < \beta \leq 3} (i_\alpha - i_\beta) \;\not\equiv 0 \pmod{p} \;,$$

since $i_\alpha \neq i_\beta$, and p is a prime greater than any $i_\alpha - i_\beta$. Thus $R \neq 0$, and the tracks $\{T(i_\alpha, j_\alpha) : 1 \leq i_\alpha \leq 3\}$ are not collinear in the XY-plane. Hence the only vertices that an edge intersects are its own endpoints.

It remains to prove that there are no edge crossings. Consider two edges e and e' with distinct endpoints $v(i_\alpha, j_\alpha, k_\alpha)$, $1 \leq \alpha \leq 4$. (Clearly edges with a common endpoint do not cross.) Let $Y_\alpha = Y(i_\alpha, j_\alpha)$. Consider the following determinant

$$D = \begin{vmatrix} 1 & i_1 & Y_1 & Z(i_1,j_1,k_1) \\ 1 & i_2 & Y_2 & Z(i_2,j_2,k_2) \\ 1 & i_3 & Y_3 & Z(i_3,j_3,k_3) \\ 1 & i_4 & Y_4 & Z(i_4,j_4,k_4) \end{vmatrix} \quad .$$

If e and e' cross then their endpoints are coplanar, and $D = 0$. Thus it suffices to prove that $D \neq 0$. We proceed by considering the number $N = |\{i_1, i_2, i_3, i_4\}|$ of distinct colours assigned to the four endpoints of e and e'. Clearly $N \in \{2, 3, 4\}$.

Case $N = 4$: Since $Y_\alpha \equiv i_\alpha^2 \pmod{p}$ and $Z(i_\alpha, j_\alpha, k_\alpha) \equiv i_\alpha^3 \pmod{p}$,

$$D \equiv \begin{vmatrix} 1 & i_1 & i_1^2 & i_1^3 \\ 1 & i_2 & i_2^2 & i_2^3 \\ 1 & i_3 & i_3^2 & i_3^3 \\ 1 & i_4 & i_4^2 & i_4^3 \end{vmatrix} = \prod_{1 \leq \alpha < \beta \leq 4} (i_\alpha - i_\beta) \;\not\equiv 0 \pmod{p} \;,$$

since $i_\alpha \neq i_\beta$, and p is a prime greater than any $i_\alpha - i_\beta$. Thus $D \neq 0$.

Case $N = 3$: Without loss of generality $i_1 = i_2$. It follows that $D = 5S_0 + S_1 + S_2 + S_3 + S_4$ where

$$\begin{aligned}
S_0 &= 4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4) \\
S_1 &= p(Y_2 - Y_1)\big(k_3(i_4 - i_1) \,-\, k_4(i_3 - i_1)\big) \\
S_2 &= p(i_4 - i_3)(k_2 Y_1 \,-\, k_1 Y_2) \\
S_3 &= p(k_2 - k_1)(Y_4(i_3 - i_1) - Y_3(i_4 - i_1)) \\
S_4 &= (Y_2 - Y_1)\big(\,(i_3 - i_4)(i_1^3 \bmod p) \,-\, (i_3 - i_1)(i_4^3 \bmod p) \,+\, (i_4 - i_1)(i_3^3 \bmod p)\,\big)\ .
\end{aligned}$$

If $Y_1 = Y_2$ then e and e' do not cross, since no three tracks from distinct colour classes are collinear in the XY-plane. Assume $Y_1 \neq Y_2$. If $i_3 < i_1 < i_4$ or $i_4 < i_1 < i_3$ then e and e' do not cross, simply by considering the projection in the XY-plane. Thus $i_1 < i_3, i_4$ or $i_1 > i_3, i_4$, which implies

$$(i_4 - i_1)(i_3 - i_1) \,>\, |i_4 - i_3|\ . \tag{1}$$

Claim 2. If $|S_0| \geq |S_1|$, $|S_0| \geq |S_2|$, $|S_0| \geq |S_3|$ and $|S_0| \geq |S_4|$ then $D \neq 0$.

Proof. To prove that $D = 5S_0 + S_1 + S_2 + S_3 + S_4$ is nonzero it suffices to show that $D' = \pm 5|S_0| \pm |S_1| \pm |S_2| \pm |S_3| \pm |S_4|$ is nonzero for all combinations of pluses and minuses. Consider $X = \pm|S_1| \pm |S_2| \pm |S_3| \pm |S_4|$ for some combination of pluses and minuses. Since $|S_1| \leq |S_0|, |S_2| \leq |S_0|, |S_3| \leq |S_0|$, and $|S_4| \leq |S_0|$, we have $-4|S_0| \leq X \leq 4|S_0|$. Since $S_0 \neq 0$, we have $5|S_0| + X \neq 0$ and $-5|S_0| + X \neq 0$. That is, all values of D' are nonzero. Therefore $D \neq 0$. □

Therefore, to prove that $D \neq 0$ it suffices to show that $|S_0| \geq |S_1|$, $|S_0| \geq |S_2|$, $|S_0| \geq |S_3|$ and $|S_0| \geq |S_4|$. We will use the following elementary facts regarding absolute values:

$$\begin{aligned}
&\forall a_1, \dots, a_k \in \mathbb{R} \\
&\quad |a_1 a_2 \dots a_k| \;=\; |a_1||a_2| \cdots |a_k| \\
&\quad |a_1 + a_2 + \cdots + a_k| \;\leq\; |a_1| + |a_2| + \cdots + |a_k| \;\leq\; k \cdot \max\{|a_1|, |a_2|, \dots, |a_k|\}\ .
\end{aligned}$$

- First we prove that $|S_0| \geq |S_1|$. That is,

$$|4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \;\geq\; |p(Y_2 - Y_1)\big(k_3(i_4 - i_1) \,-\, k_4(i_3 - i_1)\big)|\ .$$

Hence,

$$\begin{aligned}
&|S_0| > |S_1| \\
&\Longleftarrow\; 2n'|i_3 - i_1||i_4 - i_1||Y_3 - Y_4| \;\geq\; |\,k_3(i_4 - i_1) \,-\, k_4(i_3 - i_1)|\ . \\
&\Longleftarrow\; 2n'|i_3 - i_1||i_4 - i_1||Y_3 - Y_4| \;\geq\; 2 \cdot \max\{|k_4(i_3 - i_1)|, |k_3(i_4 - i_1)|\}\ .
\end{aligned}$$

Since $n' \geq k_3, k_4$ and $|Y_3 - Y_4| \geq 1$,

$$|S_0| > |S_1| \;\Longleftarrow\; |i_3 - i_1||i_4 - i_1| \geq \max\{|i_3 - i_1|, |i_4 - i_1|\}\ .$$

Thus $|S_0| \geq |S_1|$ since $|i_3 - i_1| \geq 1$ and $|i_4 - i_1| \geq 1$.

- Now we prove that $|S_0| \geq |S_2|$. That is,

$$|4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \;\geq\; |p(i_4 - i_3)(k_2 Y_1 \,-\, k_1 Y_2)|\ .$$

By (1) and since $|Y_2 - Y_1| \geq 1$,

$$|S_0| \geq |S_2| \Longleftarrow |4cn'(Y_3 - Y_4)| \ \geq\ |k_2Y_1 \ -\ k_1Y_2| \ .$$
$$\Longleftarrow |2cn'(Y_3 - Y_4)| \ \geq\ \max\{|k_2Y_1|,\ |k_1Y_2|\} \ ,$$

which holds since $n' \geq k_1, k_2$ and $|2c(Y_3 - Y_4)| \geq \max\{Y_1, Y_2\}$ by Claim 1.

• Now we prove that $|S_0| \geq |S_3|$. That is,

$$|4cpn'(i_3-i_1)(i_4-i_1)(Y_2-Y_1)(Y_3-Y_4)| \ \geq\ |p(k_2-k_1)\big(Y_4(i_3-i_1) - Y_3(i_4-i_1)\big)| \ .$$

Since $n' \geq |k_2 - k_1|$ and since $|Y_2 - Y_1| \geq 1$,

$$|S_0| \geq |S_3|$$
$$\Longleftarrow\ |4c(i_3 - i_1)(i_4 - i_1)(Y_3 - Y_4)| \ \geq\ |\,Y_4(i_3 - i_1) \ -\ Y_3(i_4 - i_1)\,| \ .$$
$$\Longleftarrow\ |2c(i_3 - i_1)(i_4 - i_1)(Y_3 - Y_4)| \ \geq\ \max\{|\,Y_4(i_3 - i_1)|\,,\ |Y_3(i_4 - i_1)\,|\} \ ,$$

which holds since $|2c(Y_3 - Y_4)| \geq \max\{Y_1, Y_2\}$ by Claim 1.

• Finally we prove that $|S_0| \geq |S_4|$. That is,

$$|4cpn'(i_3 - i_1)(i_4 - i_1)(Y_2 - Y_1)(Y_3 - Y_4)| \ \geq$$
$$|(Y_2 - Y_1)\big(\,(i_3 - i_4)(i_1^3 \bmod p) \ -\ (i_3 - i_1)(i_4^3 \bmod p) \ +\ (i_4 - i_1)(i_3^3 \bmod p)\,\big)| \ .$$

Since $cn'|Y_3 - Y_4| \geq 1$,

$$|S_0| > |S_4|$$
$$\Longleftarrow\ |3p(i_3 - i_1)(i_4 - i_1)| \ \geq$$
$$|\,(i_3 - i_4)(i_1^3 \bmod p) \ -\ (i_3 - i_1)(i_4^3 \bmod p) \ +\ (i_4 - i_1)(i_3^3 \bmod p)\,|$$
$$\Longleftarrow\ |3p(i_3 - i_1)(i_4 - i_1)| \ \geq$$
$$3 \cdot \max\{|\,(i_3 - i_4)(i_1^3 \bmod p)|\,,\ |(i_3 - i_1)(i_4^3 \bmod p)|\,,\ |(i_4 - i_1)(i_3^3 \bmod p)\,|\}$$
$$\Longleftarrow\ |(i_3 - i_1)(i_4 - i_1)| \ \geq\ \max\{|\,i_3 - i_4|\,,\ |i_3 - i_1|\,,\ |i_4 - i_1\,|\} \ ,$$

which holds by (1).

Case $N = 2$: Without loss of generality $i_1 = i_2 \neq i_3 = i_4$. If $Y_1 = Y_2$ and $Y_3 = Y_4$ then e and e' do not cross as otherwise there would be an X-crossing in the track layout. If $Y_1 = Y_2$ and $Y_3 \neq Y_4$ (or $Y_1 \neq Y_2$ and $Y_3 = Y_4$) then e and e' do not cross, by considering the projection in the XY-plane. Thus we can assume that $Y_1 \neq Y_2$ and $Y_3 \neq Y_4$. It follows that

$$D \ =\ p(i_1 - i_3)\ \big(\ 5 \cdot 4cn'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1) \ +\ (k_1 - k_2)(Y_4 - Y_3) \ +$$
$$(k_4 - k_3)(Y_2 - Y_1)\ \big) \ .$$

As in Claim 2, to show that $D \neq 0$ it suffices to show that

(2) $$|\,4cn'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1)\,| \ \geq\ |\,(k_1 - k_2)(Y_4 - Y_3)\,| \ ,$$

and

(3) $$|\,4cn'(Y_2 - Y_1)(Y_4 - Y_3)(i_3 - i_1)\,| \ \geq\ |\,(k_4 - k_3)(Y_2 - Y_1)\,| \ .$$

Inequalities (2) and (3) hold since $n' > |k_1 - k_2|$ and $n' > |k_4 - k_3|$. □

Note that the constant 20 in the definition of $Z(i, j, k)$ in the proof of Lemma 5 is chosen to enable a relatively simple proof. It is easily seen that it can be reduced. The proof of the next lemma is based on an idea of Pach *et al.* [**24**] for balancing the size of the colour classes in a vertex colouring.

LEMMA 6. *Let G be an n-vertex graph with a c-colouring $\{V_i : 0 \le i \le c-1\}$ and a track layout $\{T_{i,j} : 0 \le i \le c-1, 1 \le j \le t_i\}$, such that each $T_{i,j} \subseteq V_i$. Let $k = \sum_i t_i$ be the total number of tracks. Then G has a $\mathcal{O}(c) \times \mathcal{O}(ck) \times \mathcal{O}(c^4 n)$ drawing.*

PROOF. Replace each track by tracks of size exactly $\lceil \frac{n}{k} \rceil$, except for at most one track of size at most $\lceil \frac{n}{k} \rceil$. Order the vertices within each track according to the original track, and consider the new tracks to belong to the same colour class as the original. Clearly no X-crossing is created. Within V_i there are now at most $t_i + |V_i|/\lceil \frac{n}{k} \rceil$ tracks. The total number of tracks is $\sum_i (t_i + |V_i|/\lceil \frac{n}{k} \rceil) \le 2k$. For each colour class V_i, partition the set of tracks in V_i into sets of size exactly $\lceil \frac{2k}{c} \rceil$, except for one set of size at most $\lceil \frac{2k}{c} \rceil$. Consider each set to correspond to a colour. The number of colours is now at most $c + 2k/\lceil \frac{2k}{c} \rceil \le 2c$. Applying Lemma 5 with $2c$ colours, $n' = \lceil \frac{n}{k} \rceil$, and $t = \lceil \frac{2k}{c} \rceil$, we obtain the desired drawing. □

THEOREM 5. *Every c-colourable graph G with n vertices and track-number $\mathsf{tn}(G) \le t$ has a $\mathcal{O}(c) \times \mathcal{O}(c^2 t) \times \mathcal{O}(c^4 n)$ drawing with $\mathcal{O}(c^7 tn)$ volume.*

PROOF. Let $\{V_i : 0 \le i \le c-1\}$ be a c-colouring of G. Let $\{T_j : 1 \le j \le t\}$ be a t-track layout of G. For all $0 \le i \le c-1$ and $1 \le j \le t$, let $T_{i,j} = V_i \cap T_j$. Then $\{V_i : 0 \le i \le c-1\}$ and $\{T_{i,j} : 0 \le i \le c-1, 1 \le j \le t\}$ satisfy Lemma 6 with $k = ct$. Thus G has the desired drawing. □

In the case of bipartite graphs we have a simple proof of Theorem 5 with improved constants.

LEMMA 7. *Every n-vertex bipartite graph G with track-number $\mathsf{tn}(G) \le t$ has a $2 \times t \times n$ drawing.*

PROOF. Let $\{A, B\}$ be the bipartition of $V(G)$. Let $\{T_i : 1 \le i \le t\}$ be a t-track layout of G. For each $1 \le i \le t$, let $A_i = T_i \cap A$ and $B_i = T_i \cap B$. Order each A_i and B_i as in T_i. Place the j^{th} vertex in A_i at $(0, i, j + \sum_{k=1}^{i-1} |A_k|)$. Place the j^{th} vertex in B_i at $(1, t-i+1, j + \sum_{k=1}^{i-1} |B_k|)$. The drawing is thus $2 \times t \times n$. Let $A_i B_j$ be the set of edges with one endpoint in A_i and the other in B_j. There is no crossing between edges in $A_i B_j$ and $A_i B_j$ as otherwise there would be an X-crossing in the track layout. Clearly there is no crossing between edges in $A_i B_j$ and $A_i B_k$ for $j \ne k$. Suppose there is a crossing between edges in $A_i B_j$ and $A_k B_\ell$ with $i \ne k$ and $j \ne \ell$, and without loss of generality $i < k$. Then the projections of the edges in the XY-plane also cross, and thus $j < \ell$. This implies that the projections of the edges in the XZ-plane do not cross, and thus the edges do not cross. □

Lemma 4 with $\mathsf{tn}(G) \le \chi_{\text{sst}}(G)$ and Theorem 5 imply:

THEOREM 6. *Every c-colourable graph with n vertices and m edges has a $\mathcal{O}(c) \times \mathcal{O}(c^2 m^{2/3}) \times \mathcal{O}(c^4 n)$ drawing with $\mathcal{O}(c^6 m^{2/3} n)$ volume.* □

The next result is one of the main contributions of this paper.

THEOREM 7. *Every planar graph with n vertices has a $\mathcal{O}(1) \times \mathcal{O}(n^{1/2}) \times \mathcal{O}(n)$ drawing with $\mathcal{O}(n^{3/2})$ volume.*

PROOF. Planar graphs have $\mathcal{O}(n^{1/2})$ path-width [**2**], and thus have $\mathcal{O}(n^{1/2})$ track-number by Lemma 1(a). The result follows from Theorem 5 since planar graphs are 4-colourable. $\square$

The following generalisation of Theorem 7 for graphs G with genus γ follows from Lemma 1(d), Theorem 5, and the classical result of Heawood [**20**] that $\chi(G) \in \mathcal{O}(\gamma^{1/2})$.

THEOREM 8. *Every n-vertex graph with genus γ has a $\mathcal{O}(\gamma^{1/2}) \times \mathcal{O}(\gamma^{3/2} n^{1/2}) \times \mathcal{O}(\gamma^2 n)$ drawing with $\mathcal{O}(\gamma^4 n^{3/2})$ volume.* $\square$

The next generalisation of Theorem 7 for graphs with no K_h-minor follows from Lemma 1(e), Theorem 5, and the result independently due to Kostochka [**21**] and Thomason [**26**] that $\chi(G) \in \mathcal{O}(h \log^{1/2} h)$ (see [**7**]).

THEOREM 9. *Every n-vertex graph with no K_h-minor has a $\mathcal{O}(h \log^{1/2} h) \times \mathcal{O}(h^{7/2} \log h \cdot n^{1/2}) \times \mathcal{O}(h^4 \log^2 h \cdot n)$ drawing with volume $\mathcal{O}(h^{17/2} \log^{7/2} h \cdot n^{3/2})$.* $\square$

Finally we consider the maximum degree as a parameter. By the sequential greedy algorithm, G is $(\Delta + 1)$-colourable. Thus by Theorems 2 and 5 we have:

THEOREM 10. *Every graph with n vertices, m edges, and maximum degree Δ has a $\mathcal{O}(\Delta) \times \mathcal{O}(\Delta^{5/2} m^{1/2}) \times \mathcal{O}(\Delta^4 n)$ drawing with $\mathcal{O}(\Delta^{15/2} m^{1/2} n)$ volume.* $\square$

By Theorem 8, 9 and 10 and since graphs with constant maximum degree have $\mathcal{O}(n)$ edges we have:

COROLLARY 1. *Every n-vertex graph with constant genus, or with no K_h-minor for some constant h, or with constant maximum degree has a $\mathcal{O}(1) \times \mathcal{O}(n^{1/2}) \times \mathcal{O}(n)$ drawing with $\mathcal{O}(n^{3/2})$ volume.* $\square$

We conclude with the following open problems: Does every graph have a 3D drawing with $\mathcal{O}(nm)$ volume? Does every graph with constant chromatic number have a 3D drawing with $\mathcal{O}(n\sqrt{m})$ volume? These bounds match the lower bounds for K_n and $K_{n,n}$, and would make edge-sensitive improvements to the existing upper bounds of $\mathcal{O}(n^3)$ and $\mathcal{O}(n^2)$, respectively. These edge-sensitive bounds would be implied by Theorems 1 and 5 should every graph have $\mathcal{O}(\sqrt{m})$ track-number. In turn, this track-number bound would be implied should every graph have $\mathcal{O}(\sqrt{m})$ strong star chromatic number. As far as the authors are aware, a $\mathcal{O}(\sqrt{m})$ bound is not even known for the star chromatic number. The best known bound in this direction is $\chi_{\rm st}(G) \leq 11 m^{3/5}$, which can be proved in a similar fashion to Lemma 4, in conjunction with the result of Fertin *et al.* [**15**] that $\chi_{\rm st}(G) \leq \lceil 20\Delta^{3/2} \rceil$ (see [**10**]).

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