

Upward Three-Dimensional Grid Drawings of Graphs

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Abstract A *three-dimensional grid drawing* of a graph is a placement of the vertices at distinct points with integer coordinates, such that the straight line segments representing the edges do not cross. Our aim is to produce three-dimensional grid drawings with small bounding box volume. Our first main result is that every n -vertex graph with bounded degeneracy has a three-dimensional grid drawing with $\mathcal{O}(n^{3/2})$ volume. This is the largest known class of graphs that have such drawings. A three-dimensional grid drawing of a directed acyclic graph (*dag*) is *upward* if every arc points up in the z -direction. We prove that every dag has an upward three-dimensional grid drawing with $\mathcal{O}(n^3)$ volume, which is tight for the complete dag. The previous best upper bound was $\mathcal{O}(n^4)$. Our main result concerning upward drawings is that every c -colourable dag (c constant) has an upward three-dimensional grid drawing with $\mathcal{O}(n^2)$ volume. This result matches the bound in the undirected case, and improves the best known bound from $\mathcal{O}(n^3)$ for many classes of dags, including planar, series parallel, and outerplanar. Improved bounds are also obtained for tree dags. We prove a strong relationship between upward three-dimensional grid drawings, upward track layouts, and upward queue layouts. Finally, we study upward three-dimensional grid drawings with bends in the edges.

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1. Introduction

Graph drawing is the study of aesthetically pleasing geometric representations of graphs. Graph drawing in the plane is well-studied; see [7, 28]. Motivated by experimental evidence suggesting that displaying a graph in three dimensions is better than in two [35, 36], and applications including information visualisation [35], VLSI circuit design [29], and software engineering [37], there is a growing body of research in three-dimensional graph drawing.

A *three-dimensional straight line grid drawing* of a graph, henceforth called a *3D drawing*, is a placement of the vertices at distinct points in \mathbb{Z}^3 (called *gridpoints*), such that the straight line segments representing the edges are pairwise non-crossing. That is, distinct edges only intersect at common endpoints, and each edge only intersects a vertex that is an endpoint of that edge. The coordinates of a vertex v are denoted by $(x(v), y(v), z(v))$. It is well known that every graph has a 3D drawing. We are therefore interested in optimising certain measures of the aesthetic quality of such drawings.

The *bounding box* of a 3D drawing is the minimum axis-aligned box that contains the drawing. If the bounding box has side lengths $X - 1$, $Y - 1$ and $Z - 1$, then we speak of an $X \times Y \times Z$ *drawing* with *width* X , *depth* Y , *height* Z , and *volume* $X \cdot Y \cdot Z$. That is, the volume of a 3D drawing is the number of gridpoints in the bounding box. This definition is formulated so that 2D drawings have positive volume. We are interested in 3D drawings with small volume, which are widely studied [3–5, 8, 10, 11, 14, 15, 18, 22, 32, 34].

3D drawings have been generalised in a number of ways. Multi-dimensional grid drawings have been studied [33, 38], as have 3D *polyline* grid drawings, where edges are allowed to bend at gridpoints [6, 15, 30]. The focus of this paper is upward 3D drawings of directed graphs, which have previously been investigated by Poranen [34] and Di Giacomo et al. [9]. A 3D drawing of a directed graph G is *upward* if $z(v) < z(w)$ for every arc \vec{vw} of G . Obviously an upward 3D drawing can only exist if G is acyclic (a *dag*). Upward two-dimensional drawings have been widely studied; see [2, 20, 27] for example.

As described in Table I, our main results are improved upper bounds on the volume of upward 3D drawings of dags. These results are presented in Sections 3 and 4, and in Section 9 in the case of trees. In addition, we prove that (undirected) graphs with bounded degeneracy have 3D drawings with $\mathcal{O}(n^{3/2})$ volume. This is the largest known class of graphs that have such drawings.

Other results in this paper include the following. In Section 5 we study upward track layouts, and show how they can be used to produce upward 3D drawings with small volume. These results are used in Section 9 to produce upward 3D drawings of trees. In Section 7 we explore the relationship between upward track layouts and upward queue layouts, which is a structure introduced by Heath et al. [24, 25].

Table 1 Upper bounds on the volume of 3D drawings of graphs and upward 3D drawings of dags with n vertices, m edges, chromatic number χ , and degeneracy d .

Graph family	Undirected		Upward dags			
			Previous best		This paper	
Arbitrary	$\Theta(n^3)$	[5]	$\mathcal{O}(n^4)$	[9]	$\Theta(n^3)$	§3
Arbitrary	$\mathcal{O}(m^{4/3}n)$	[14]				
Arbitrary	$\mathcal{O}(dmn)$	§6				
Arbitrary	$\mathcal{O}(\chi^2 n^2)$	[32]			$\mathcal{O}(\chi^4 n^2)$	§4
Constant χ	$\Theta(n^2)$	[32]			$\Theta(n^2)$	§4
Constant χ	$\mathcal{O}(m^{2/3}n)$	[14]			$\mathcal{O}(n^2)$	§4
Minor-closed	$\mathcal{O}(n^{3/2})$	[14]			$\mathcal{O}(n^2)$	§4
Constant d	$\mathcal{O}(n^{3/2})$	§6			$\mathcal{O}(n^2)$	§4
Planar	$\mathcal{O}(n^{3/2})$	[14]	$\mathcal{O}(n^3)$	[9]	$\mathcal{O}(n^2)$	§4
Constant treewidth	$\mathcal{O}(n)$	[11]	$\mathcal{O}(n^3)$	[9]	$\mathcal{O}(n^2)$	§4
Series parallel	$\mathcal{O}(n)$	[11]	$\mathcal{O}(n^3)$	[34]	$\mathcal{O}(n^2)$	§4
Outerplanar	$2 \times 2 \times n$	[18]	$\mathcal{O}(n^3)$	[9]	$\mathcal{O}(n^2)$	§4
Trees	$2 \times 2 \times n$	[18]	$7 \times 7 \times 7n$	[9]	$4 \times 4 \times \frac{7}{5}n$	§9
Caterpillars	$2 \times 2 \times n$	[18]			$2 \times 2 \times n$	§9

In Section 8 we describe an outerplanar graph that highlights the key differences between 3D drawings and upward 3D drawings. Finally in Section 10 we study upward layouts of graph subdivisions, and conclude with some bounds on the volume of upward 3D polyline drawings.

2. Preliminaries

The following notation is used throughout the paper. We consider finite simple graphs G with vertex set $V(G)$. If G is undirected then its edge set is denoted by $E(G)$. If G is directed then its arc set is denoted by $A(G)$. A *vertex ordering* of G is a bijection $\sigma : V(G) \rightarrow \{1, 2, \dots, n\}$, sometimes written as $\sigma = (v_1, v_2, \dots, v_n)$ where $\sigma(v_i) = i$. A vertex ordering σ of a directed graph G is *topological* if $\sigma(v) < \sigma(w)$ for every arc $\vec{vw} \in A(G)$. It is well known that a directed graph is acyclic if and only if it has a topological vertex ordering.

A *(vertex) c -colouring* of a graph G is a partition $\{V_i : i \in I\}$ of $V(G)$, such that $|I| = c$, and for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. Each $i \in I$ is a *colour*, each set V_i is a *colour class*, and if $v \in V_i$ then v is *coloured i* . If G has a vertex c -colouring then G is *c -colourable*. The *chromatic number* of G , denoted by $\chi(G)$, is the minimum integer c such that G is c -colourable.

A graph G is *d -degenerate* if every subgraph of G has a vertex of degree at most d . The *degeneracy* of G is the minimum integer d such that G is d -degenerate. A d -degenerate graph is $(d + 1)$ -colourable by a greedy algorithm. For example, every forest is 1-degenerate, every outerplanar graph is 2-degenerate, and every planar graph is 5-degenerate. In general, a d -degenerate n -vertex graph has less than dn edges.

3. Arbitrary Digraphs

Cohen et al. [5] proved that every graph has a 3D drawing with $\mathcal{O}(n^3)$ volume. The proof generalises for upward 3D drawings as follows.

THEOREM 3.1. *Every dag G on n vertices has an upward $2n \times 2n \times n$ drawing with $4n^3$ volume. Moreover, the bounding box of every upward 3D drawing of the complete dag on n vertices is at least $\frac{n}{4} \times \frac{n}{4} \times n$, and thus has $\Omega(n^3)$ volume.*

Proof. Let (v_1, v_2, \dots, v_n) be a topological vertex ordering of G . By Bertrand's Postulate there is a prime number p such that $n < p \leq 2n$. Position each vertex v_i at $(i^3 \bmod p, i^2 \bmod p, i)$. Cohen et al. [5] proved that no two arcs cross (with the x - and z -coordinates switched). Clearly every arc is upward. The bounding box is at most $2n \times 2n \times n$. Cohen et al. [5] observed that the bounding box in every 3D drawing of K_n is at least $\frac{n}{4} \times \frac{n}{4} \times \frac{n}{4}$ (since at most four vertices can lie in a single gridplane). The same lower bound holds for upward 3D drawings of the n -vertex complete dag. In addition, the height is at least n , since the complete dag contains a Hamiltonian directed path; see Observation 4.4 below. \square

4. Coloured Digraphs

Pach et al. [32] proved that every c -colourable graph has a $\mathcal{O}(c) \times \mathcal{O}(n) \times \mathcal{O}(cn)$ drawing with $\mathcal{O}(c^2n^2)$ volume. The proof implicitly relied on the following constructions.

LEMMA 4.1. ([32]). *Let $\{V_i : 0 \leq i \leq c-1\}$ be a c -colouring of a graph G . Let $p \geq 2c-1$ be a prime number. Place each vertex in V_i at a distinct gridpoint (i, t, it) , where $t \equiv i^2 \pmod{p}$. Then a (crossing-free) 3D drawing of G is obtained.*

LEMMA 4.2. ([32]). *Let $\{V_i : 0 \leq i \leq c-1\}$ be a c -colouring of an n -vertex graph G . Then G has a $\mathcal{O}(c) \times \mathcal{O}(n) \times \mathcal{O}(cn)$ drawing, such that $\mathbf{x}(v) < \mathbf{x}(w)$ for all vertices $v \in V_i$ and $w \in V_j$ with $i < j$.*

The result of Pach et al. [32] generalises for upward 3D drawings as follows.

THEOREM 4.3. *Every n -vertex c -colourable dag G has an upward $c \times 4c^2n \times 4cn$ drawing with volume $\mathcal{O}(c^4n^2)$.*

Proof. Let p be a prime number with $2c-1 \leq p < 4c$. Let $\{V_i : 0 \leq i \leq c-1\}$ be a c -colouring of G . Let (v_1, v_2, \dots, v_n) be a topological ordering of G . Position each vertex $v_j \in V_i$ at $(x_j, y_j, z_j) \in \mathbb{Z}^3$, where $x_j := i$ and $y_j := i \cdot z_j$. It remains to compute the z_j . If $v_1 \in V_i$, then set $z_1 := i^2 \bmod p$. Now for all $j = 2, 3, \dots, n$, let z_j be the integer in $\{z_{j-1} + 1, z_{j-1} + 2, \dots, z_{j-1} + p\}$ such that $z_j \equiv i^2 \pmod{p}$. Thus $z_{j-1} < z_j$. Hence arcs are upward, and no two vertices are mapped to the same point. By Lemma 4.1 with the y - and z -coordinates switched, the drawing is crossing-free. Since $0 \leq x_j \leq c-1$, the width is c . Since $z_j \leq z_{j-1} + p$, the height is at most $pn \leq 4cn$. Since $y_j < c \cdot z_j$, the depth is less than $cpn \leq 4c^2n$. \square

Many dags are c -colourable, for some constant c . These include dags whose underlying undirected graph is outerplanar, is series parallel, is planar, or more generally, is from a proper minor-closed class, or has bounded degeneracy. Theorem 4.3 implies that all such dags have upward 3D drawings with $\mathcal{O}(n^2)$ volume.

Pach et al. [32] proved that the complete bipartite graph $K_{n,n}$ requires $\Omega(n^2)$ volume in every 3D drawing. Thus every acyclic orientation of $K_{n,n}$ requires $\Omega(n^2)$ volume in every upward 3D drawing. Hence Theorem 4.3 is tight for constant c . More generally, Bose et al. [3] proved that every 3D drawing of every n -vertex m -edge graph has at least $\frac{1}{8}(n+m)$ volume.

4.1. Long Paths

We have the following lower bound, since every vertex in a directed path must be assigned a distinct z -coordinate in an upward 3D drawing.

OBSERVATION 4.4. *Let G be a dag that contains a directed path on ℓ vertices. Then the height of every upward 3D drawing of G is at least ℓ .*

Conversely, we have the following upper bound.

THEOREM 4.5. *Every n -vertex dag G with no directed path on ℓ vertices, has an upward $\mathcal{O}(\ell n) \times \mathcal{O}(n) \times \mathcal{O}(\ell)$ drawing with $\mathcal{O}(\ell^2 n^2)$ volume.*

Proof. Colour each vertex $v \in V(G)$ by the number of vertices in the longest directed path ending at v . (This is well defined since G is a dag.) The number of colours is at most ℓ . Consider an arc $\vec{vw} \in A(G)$ such that v is coloured i . Thus there is an i -vertex path P ending at v . Moreover, $w \notin P$ as otherwise G would contain a directed cycle. Hence (P, \vec{vw}) is an $(i+1)$ -vertex path ending at w . Thus the colour of w is at least $i+1$. In particular, we have a proper ℓ -colouring of G . The result follows from Lemma 4.2 with the x - and z -coordinates switched. \square

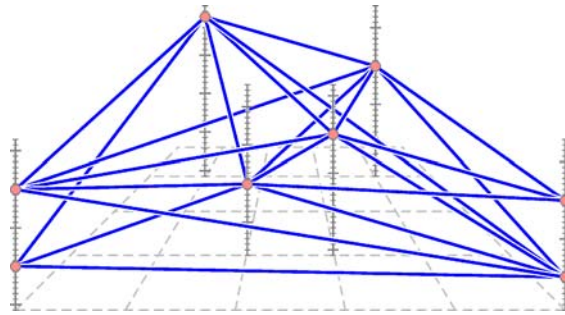
Theorem 4.5 is an improvement over Theorem 4.3 whenever $\ell < \chi(G)^2$.

5. Upward Track Layouts

Let $\{V_i : i \in I\}$ be a t -colouring of a graph G . Let $<_i$ be a total order on each colour class V_i . Then each pair $(V_i, <_i)$ is a *track*, and $\{(V_i, <_i) : i \in I\}$ is a *t -track assignment* of G . To ease the notation we denote track assignments by $\{V_i : i \in I\}$ when the ordering on each colour class is implicit. An *X -crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct colours i and j . A *t -track layout* of G is a t -track assignment of G with no X -crossing. The *track-number* of G , denoted by $\text{tn}(G)$, is the minimum integer t such that G has a t -track layout.¹

¹ Some authors [8–10] use a slightly different definition of track layout, in which *intra-track* edges are allowed between consecutive vertices in a track. In keeping with the terminology of Dujmović et al. [11] and for consistency with the notion of an *improper* colouring, we call this structure an *improper* track layout, and use *improper track-number* for the minimum number of tracks in this

Figure 1 A 3D drawing produced from a 6-track layout.



Track layouts and track-number were introduced by Dujmović et al. [11] although they are implicit in many previous works [18, 23, 26]. Track layouts and 3D drawings are closely related, as illustrated by the following results by Dujmović et al. [11, 14].

THEOREM 5.1. ([11, 14]). *Let G be an n -vertex graph with chromatic number $\chi(G) \leq c$ and track-number $\text{tn}(G) \leq t$. Then:*

- (a) G has a $\mathcal{O}(t) \times \mathcal{O}(t) \times \mathcal{O}(n)$ drawing with $\mathcal{O}(t^2n)$ volume (see Figure 1), and
- (b) G has a $\mathcal{O}(c) \times \mathcal{O}(c^2t) \times \mathcal{O}(c^4n)$ drawing with $\mathcal{O}(c^7tn)$ volume.

Conversely, if a graph G has an $X \times Y \times Z$ drawing, then G has track-number $\text{tn}(G) \leq 2XY$ (and improper track-number at most XY).

The proof of Theorem 5.1(a) implicitly used the following lemma.

LEMMA 5.2. ([11]). *Let $\{(V_i, <_i) : 1 \leq i \leq t\}$ be a t -track layout of a graph G . Let $p > t$ be a prime number. Let d_v be an integer for each vertex $v \in V(G)$, such that $d_v < d_w$ for all vertices $v, w \in V_i$ with $v <_i w$. If each vertex v is placed at $(i, i^2 \bmod p, p \cdot d_v + i^3 \bmod p)$, then we obtain a (crossing-free) 3D drawing of G .*

Di Giacomo et al. [9] extended the definition of track layouts to dags as follows.² An *upward track layout* of a dag G is a track layout of the underlying undirected graph of G , such that if G^+ is the directed graph obtained from G by adding an arc from each vertex v to the successor vertex in the track that contains v (if it exists), then G^+ is still acyclic. The *upward track-number* of G , denoted by $\text{utn}(G)$, is the minimum integer t such that G has an upward t -track layout. Di Giacomo et al. [9] observed that the following analogue of Theorem 5.1(a) follows from Lemma 5.2 by Dujmović et al. [11].

THEOREM 5.3. ([9, 11]). *Let G be an n -vertex graph with upward track-number $\text{utn}(G) \leq t$. Then G has an upward $\mathcal{O}(t) \times \mathcal{O}(t) \times \mathcal{O}(tn)$ drawing with $\mathcal{O}(t^3n)$ volume.*

setting. The improper track-number is at most the track-number, and the track-number is at most twice the improper track-number [11]. Moreover, for every graph class \mathcal{G} that includes all series parallel graphs, every graph in \mathcal{G} has track-number at most some constant t if and only if every graph in \mathcal{G} has improper track-number at most t [11].

² Di Giacomo et al. [9] allow intra-track arcs in their definition of upward track layout.

Conversely, if a dag G has an upward $X \times Y \times Z$ drawing then G has upward track-number $\text{utn}(G) \leq 2XY$ (and improper upward track-number at most XY).

Proof. Let p be a prime number with $t < p \leq 2t$. For each vertex $v \in V(G)$, let d_v be the maximum number of vertices in a directed path in G^+ that ends at v . Since each track induces a directed path in G^+ , we have $d_v < d_w$ for all vertices v and w with $v < w$ in a single track. For each vertex v in the i -th track, position v at $(i, i^2 \bmod p, p \cdot d_v + i^3 \bmod p)$. Draw the arcs straight. By Lemma 5.2, we obtain a crossing-free drawing. As in Theorem 4.5, $d_v < d_w$ for every arc $\vec{vw} \in A(G)$. Thus the drawing is upward. The claimed volume bound holds since $d_v \leq n$. The converse results are proved in the same way as the converse results in Theorem 5.1. \square

For small values of t , the constants in Theorem 5.3 can be greatly improved.

LEMMA 5.4. *Every n -vertex dag G that has an upward 3-track layout $\{V_1, V_2, V_3\}$ has an upward $2 \times 2 \times n$ drawing with $4n$ volume.*

Proof. Put the i -th vertex v in a topological ordering of G^+ at $(0, 0, i)$ if $v \in V_1$, at $(1, 0, i)$ if $v \in V_2$, and at $(0, 1, i)$ if $v \in V_3$. Draw each arc straight. Clearly we obtain an upward crossing-free drawing of G . \square

LEMMA 5.5. *Every n -vertex dag G that has an upward 4-track layout $\{V_1, V_2, V_3, V_4\}$ has an upward $2 \times 2 \times 2n$ drawing with $8n$ volume.*

Proof. Put the i -th vertex v in a topological ordering of G^+ at $(0, 0, 2i)$ if $v \in V_1$, at $(1, 0, 2i)$ if $v \in V_2$, at $(0, 1, 2i)$ if $v \in V_3$, and at $(1, 1, 2i - 1)$ if $v \in V_4$. Draw each arc straight. Every arc is upward, and the bounding box is at most $2 \times 2 \times 2n$. Dujmović and Wood [15, Lemma 36] proved that no two edges cross. \square

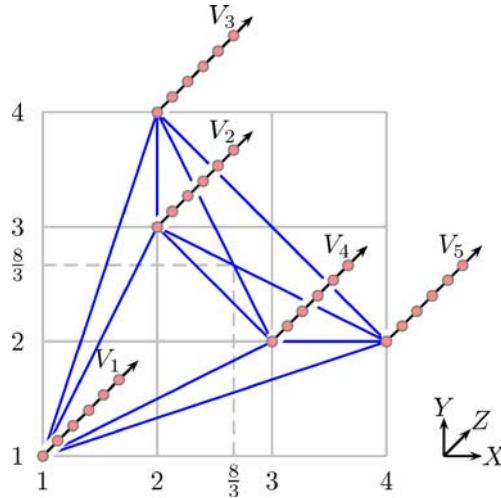
LEMMA 5.6. *Every n -vertex dag G that has an upward 5-track layout $\{V_1, V_2, V_3, V_4, V_5\}$ has an upward $4 \times 4 \times \frac{7}{5}n$ drawing with volume $22.4n$.*

Proof. Without loss of generality, $|V_3| + |V_5| \leq \frac{2}{3}n$. Clearly we can assign distinct z -coordinates to the vertices such that

- $z(v) < z(w)$ for every arc $\vec{vw} \in A(G)$,
- $z(v)$ is odd for every vertex $v \in V_3$,
- $z(v)$ is even for every vertex $v \in V_5$, and
- $1 \leq z(v) \leq n + |V_3| + |V_5| \leq \frac{7}{5}n$ for every vertex $v \in V(G)$.

As illustrated in Figure 2, put each vertex v at $(1, 1, z(v))$ if $v \in V_1$, at $(2, 3, z(v))$ if $v \in V_2$, at $(2, 4, z(v))$ if $v \in V_3$, at $(3, 2, z(v))$ if $v \in V_4$, and at $(4, 2, z(v))$ if $v \in V_5$. Draw each arc straight. Every arc is upward, and the bounding box is at most $4 \times 4 \times \frac{7}{5}n$. Suppose that edges vw and pq cross. Since there is no X -crossing in the track layout, vw and pq do not run between the same pair of tracks. The projection of the drawing onto the xy -plane is a subgraph of K_5 drawn with one crossing at $(\frac{8}{3}, \frac{8}{3})$. The crossing is between the pairs of tracks V_2V_5 and V_3V_4 . Thus without loss of generality $v \in V_2, w \in V_5, p \in V_3$, and $q \in V_4$. Now vw and pq intersect the line $\{(\frac{8}{3}, \frac{8}{3}, t) : t \in \mathbb{R}\}$, respectively, at $(\frac{8}{3}, \frac{8}{3}, \frac{1}{3}(2z(v) + z(w)))$ and $(\frac{8}{3}, \frac{8}{3}, \frac{1}{3}(2z(q) + z(p)))$. Thus $2z(v) + z(w) = 2z(q) + z(p)$. This is a contradiction since $z(w)$ is even and $z(p)$ is odd. Hence the drawing is crossing-free. \square

Figure 2 Construction of a 3D drawing from a 5-track layout.



6. Strong Star Colourings

Dujmović and Wood [14] defined a vertex colouring to be a *strong star colouring* if between each pair of colour classes, all edges (if any) are incident to a single vertex. That is, each bichromatic subgraph consists of a star and possibly some isolated vertices. The *strong star chromatic number* of a graph G , denoted by $\chi_{sst}(G)$, is the minimum number of colours in a strong star colouring of G . *Star colourings*, in which each bichromatic subgraph is a star forest, have also been studied; see [1, 19, 31] for example.

With an arbitrary order of each colour class in a strong star colouring, there is no X-crossing. Thus track-number $tn(G) \leq \chi_{sst}(G)$, as observed by Dujmović and Wood [14]. Moreover, for a dag, each track can be ordered by a topological vertex ordering, to obtain an upward track layout. Thus $utn(G) \leq \chi_{sst}(G)$, as observed by Di Giacomo et al. [9].

Dujmović and Wood [14] proved that $\chi_{sst}(G) \leq 14\sqrt{\Delta m}$ and $\chi_{sst}(G) \leq 15m^{2/3}$ for every graph G with maximum degree Δ and m edges.³ In what follows we improve these bounds, by essentially replacing Δ by the weaker notion of degeneracy. The following concept will be useful. A colouring is *harmonious* if every bichromatic subgraph has at most one edge; see [16]. The *harmonious chromatic number* of G , denoted by $h(G)$, is the minimum number of colours in a harmonious colouring of G . Edwards and McDiarmid [17] proved the following upper bound on $h(G)$.

LEMMA 6.1. ([17]). *Let G be a d -degenerate graph with m edges and maximum degree Δ . Then G has harmonious chromatic number $h(G) \leq 2\sqrt{2dm} + (2d - 1)\Delta$.*

³ Patrice Ossona de Mendez [personal communication] and Jean-Sebastien Sereni and Stéphan Thomassé [personal communication] independently observed that if H is the graph consisting of k copies of K_k (which has $m = \Theta(k^3)$ edges), then $\chi_{sst}(H) = \Theta(k^2) = \Theta(m^{2/3})$. Thus the general upper bound $\chi_{sst}(G) \leq \mathcal{O}(m^{2/3})$ is best possible.

Lemma 6.1 implies the following upper bound on the strong star chromatic number.

LEMMA 6.2. *Let G be a d -degenerate graph G with m edges. Then the strong star chromatic number of G satisfies $\chi_{\text{sst}}(G) \leq 5\sqrt{2dm}$ and $\chi_{\text{sst}}(G) \leq (4 + 2\sqrt{2})m^{2/3}$.*

Proof. For the first bound, let A be the set of vertices of G with degree at least $\sqrt{2m/d}$. Then $|A| \leq \sqrt{2dm}$. Now $G \setminus A$ has maximum degree at most $\sqrt{2m/d}$. Thus $h(G \setminus A) \leq 2\sqrt{2dm} + (2d - 1)\sqrt{2m/d} < 4\sqrt{2dm}$ by Lemma 6.1. Using one colour for each vertex in A , we obtain a strong star colouring of G with $5\sqrt{2dm}$ colours. The second proof is similar, where A is the set of vertices with degree at least $m^{1/3}$. \square

Since $\text{tn}(G) \leq \chi_{\text{sst}}(G)$ and $\text{utn}(G) \leq \chi_{\text{sst}}(G)$ we have the following corollary of Lemma 6.2.

COROLLARY 6.3. *Let G be a d -degenerate graph with n vertices and m edges. Then the track-number of G satisfies $\text{tn}(G) \leq 5\sqrt{2dm} < 5d\sqrt{2n}$ and $\text{tn}(G) \leq (4 + 2\sqrt{2})m^{2/3}$. The same bounds hold for the upward track-number of every acyclic orientation of G .*

Corollary 6.3 does not give better bounds on the volume of upward 3D drawings than Theorem 4.3 because of the cubic dependence on the upward track-number in Theorem 5.3. However, for 3D drawings of undirected graphs, Theorem 5.1 and Corollary 6.3 imply the following.

THEOREM 6.4. *Let G be an n -vertex graph with degeneracy d . Then G has $\mathcal{O}(\sqrt{dm}) \times \mathcal{O}(\sqrt{dm}) \times \mathcal{O}(n)$ drawing with $\mathcal{O}(dnm)$ volume. If d is bounded, then G has a $\mathcal{O}(1) \times \mathcal{O}(\sqrt{n}) \times \mathcal{O}(n)$ drawing with $\mathcal{O}(n^{3/2})$ volume.*

A number of notes on Theorem 6.4 are in order.

- The above-mentioned bounds on the strong star chromatic number due to Dujmović and Wood [14] were non-constructive (since the proof was based on the Lovász Local Lemma). On the other hand, the proof of Lemma 6.1 is deterministic, and is easily seen to lead to polynomial time algorithms for computing the colouring in Lemma 6.2 and the drawing in Theorem 6.4.
- The $\mathcal{O}(dnm)$ volume bound in Theorem 6.4 represents a qualitative improvement over the best previous comparable bound of $\mathcal{O}(\Delta nm)$ in [14].
- Proper minor-closed graph families were the largest class of graphs for which a $\mathcal{O}(n^{3/2})$ volume bound was previously known [14]. The second part of Theorem 6.4 is strictly stronger, since there are graph classes with bounded degeneracy but with unbounded clique minors. For example, the graph K'_n obtained from K_n by subdividing every edge once has degeneracy 2, yet contains a K_n minor.

It is unknown what is the best possible bound on the track-number of graphs with bounded degeneracy. The graph K'_n seems to be an important example.

PROPOSITION 6.5. $\text{tn}(K'_n) = \Theta(n^{2/3})$.

Proof. First we prove the lower bound. Say K'_n has a t -track layout. Some track contains a set S of at least $p := \lceil n/t \rceil$ ‘original’ vertices of K_n . We can assume that p is even. Say S is ordered v_1, v_2, \dots, v_p in this track. Let T be the set of edges $v_i v_j$ of K_n

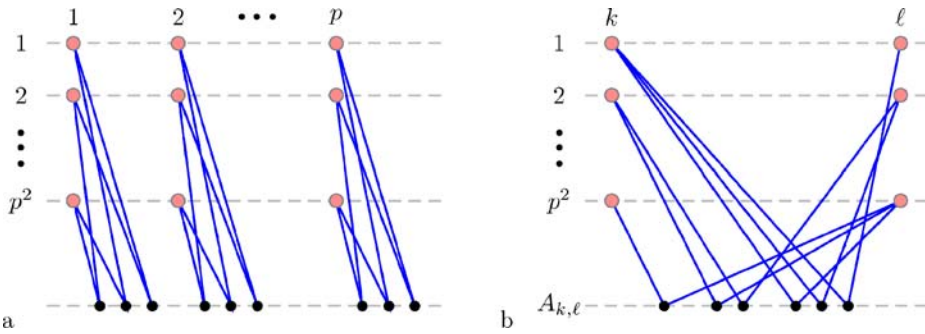


Figure 3 Construction of a track layout of K'_n .

such that $1 \leq i \leq p/2 < j \leq p$. Observe that $|T| = p^2/4$. For all edges e and f in T , the division vertex of K'_n that corresponds to e and f cannot be on the same track, as otherwise there will be an X-crossing. Thus the number of tracks $t \geq |T| \geq (n/t)^2/4$. Hence $t \geq (n/2)^{2/3}$.

Now we prove the upper bound. We can suppose that $p := n^{1/3}$ is an integer. Partition the original vertices of K'_n into p^2 sets, each with p vertices. Place each set in its own track. Let $v_{i,k}$ be the k -th original vertex in the i -th track ($1 \leq i \leq p^2, 1 \leq k \leq p$). For each such k , let $S_k := \{v_{i,k} : 1 \leq i \leq p^2\}$. It remains to place the division vertices. As illustrated in Figure 3(a), the division vertices that correspond to edges with both endpoints in some S_k are placed in one track ordered by increasing k . Since every vertex in each S_k is in a distinct track, there is no X-crossing. For all $1 \leq k < \ell \leq p$, let $T_{k,\ell}$ be the set $\{v_{i,k}v_{j,\ell} : 1 \leq i, j \leq p^2\}$ of edges of K_n . Place the division vertices of the edges in $T_{k,\ell}$ on two tracks as follows. Let $A_{k,\ell} := \{v_{i,k}v_{j,\ell} \in T_{k,\ell} : 1 \leq i \leq j \leq p^2\}$ and $B_{k,\ell} := \{v_{i,k}v_{j,\ell} \in T_{k,\ell} : 1 \leq j < i \leq p^2\}$. As illustrated in Figure 3(b), the division vertices that correspond to the edges in $A_{k,\ell}$ are placed on one track ordered by non-increasing i , breaking ties by decreasing j . The division vertices that correspond to the edges in $B_{k,\ell}$ are placed on one track ordered by non-decreasing j , breaking ties by increasing i . It is easily seen that there is no X-crossing. In total we have $p^2 + 1 + 2\binom{p}{2} \sim 2p^2 \sim 2n^{2/3}$ tracks. (Note that the constant in this upper bound can be improved by combining tracks $A_{k,\ell}$ and $A_{k',\ell'}$ when $k < \ell < k' < \ell'$; we omit the details.) □

7. Upward Queue Layouts

A k -queue layout of a graph G consists of a vertex ordering σ of G , and a partition $\{E_1, E_2, \dots, E_k\}$ of $E(G)$, such that no two edges in each E_i are nested in σ . That is, for all edges $vw, xy \in E_i$, we do not have $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w)$. The queue-number of G , denoted by $qn(G)$, is the minimum integer k such that G has a k -queue layout. Queue layouts were introduced by Heath et al. [23, 26]; see [13] for references and results.

Heath et al. [24, 25] extended the definition of queue layouts to dags as follows. An upward k -queue layout of a dag G is a k -queue layout of the underlying undirected graph of G such that the vertex ordering σ is topological. For example, every tree has a 1-queue layout [26], and every tree dag has an upward 2-queue layout [25]. The

upward queue-number of G , denoted by $\text{uqn}(G)$, is the minimum integer k such that G has an upward k -queue layout.

Consider a vertex colouring $\{V_i : i \in \mathbb{Z}\}$ of a graph G in which the colours are integers. An edge $vw \in E(G)$ with $v \in V_i$ and $w \in V_j$ has *span* $|j - i|$. This definition naturally extends to track layouts $\{(V_i, <_i) : i \in \mathbb{Z}\}$. Let $(\dots, V_{-1}, V_0, V_1, \dots)$ denote the vertex ordering σ in which $\sigma(v) < \sigma(w)$ whenever $v \in V_i$ and $w \in V_j$ for some $i < j$, or $v <_i w$ within some V_i . Dujmović et al. [12, Lemma 19] proved a characterisation of 1-queue graphs in terms of track layouts with span 2 that immediately generalises for upward 1-queue layouts as follows.

LEMMA 7.1. *A dag G has an upward 1-queue layout if and only if G has a track layout $\{V_i : i \in \mathbb{Z}\}$, such that for every arc $\overrightarrow{vw} \in A(G)$ with $v \in V_i$ and $w \in V_j$, we have $i < j \leq i + 2$, and if $j = i + 2$ then w is the first vertex in V_j , and there is no arc \overrightarrow{xy} with $v < x \in V_i$ and $y \in V_{i+1}$. In particular, $(\dots, V_{-1}, V_0, V_1, \dots)$ defines an upward 1-queue layout of G .*

An important technique for ‘wrapping’ an undirected track layout is generalised for a particular type of upward track layout as follows.

LEMMA 7.2. *Let $\{V_i : i \in \mathbb{Z}\}$ be a track layout of a dag G , such that for every arc $\overrightarrow{vw} \in A(G)$ with $v \in V_i$ and $w \in V_j$, we have $i < j \leq i + s$.*

- (a) *Then $\{V_i : i \in \mathbb{Z}\}$ is an upward track layout.*
- (b) *The vertex ordering $\sigma = (\dots, V_{-1}, V_0, V_1, \dots)$ defines an upward s -queue layout of G , and G has upward queue-number $\text{uqn}(G) \leq s$.*
- (c) *For each $0 \leq i \leq 2s$, define*

$$W_i := (\dots, V_{i-2(2s+1)}, V_{i-(2s+1)}, V_i, V_{i+2s+1}, V_{i+2(2s+1)}, \dots).$$

Then $\{W_0, W_1, \dots, W_{2s}\}$ is an upward $(2s + 1)$ -track layout of G , and G has upward track-number $\text{utn}(G) \leq 2s + 1$.

Proof. Observe that σ is a topological ordering of G^+ . Thus G^+ is acyclic, and $\{V_i : i \in \mathbb{Z}\}$ is an upward track layout. This proves (a). Now we prove (b). Two arcs with the same span are not nested in σ [11, 26]. Thus we can partition the arcs into s queues in σ according to their span. Now we prove (c). The track assignment $\{W_0, W_1, \dots, W_{2s}\}$ is upward since the corresponding graph G^+ is acyclic. Dujmović et al. [11, Lemma 3.4] proved that there is no X-crossing in $\{W_0, W_1, \dots, W_{2s}\}$. □

A (k, t) -track layout of a graph G is a t -track assignment of G in which every edge is assigned one of k colours, such that there is no monochromatic X-crossing. By $\text{tn}_k(G)$ we denote the minimum integer t such that G has a (k, t) -track layout. Thus $\text{tn}(G) = \text{tn}_1(G)$. These definitions immediately generalise to the setting of upward (k, t) -track layouts.

Dujmović et al. [12] proved that queue-number and track-number are tied, in the sense that there is a function f such that for every graph G , we have $\text{tn}(G) \leq f(\text{qn}(G))$ and $\text{qn}(G) \leq f(\text{tn}(G))$. In one direction the proof is easy. Given a (k, t) -track layout $\{V_1, V_2, \dots, V_t\}$ of G , Dujmović et al. [12] proved that the vertex ordering (V_1, V_2, \dots, V_t) admits a $k(t - 1)$ -queue layout, and thus $\text{qn}(G) \leq k(\text{tn}_k(G) - 1)$.

Di Giacomo et al. [9] proved the case $k = 1$ of the following analogous relationship between upward track-number and upward queue-number.

LEMMA 7.3. For all $k \geq 1$ and for every graph G , $uqn(G) \leq k \cdot \binom{utn_k(G)}{2}$.

Proof. Let σ be a topological ordering of G^+ (defined with respect to a given (k, t) -track layout). Thus the order of each track is preserved in σ . Monochromatic arcs between each pair of tracks form a queue in σ [12, Lemma 14]. \square

LEMMA 7.4. Upward track-number is bounded by upward queue-number. In particular, every dag G with upward queue-number $uqn(G) \leq q$ has upward track-number $utn(G) \leq 4q \cdot 4^{q(2q-1)(4q-1)}$.

Proof. Dujmović et al. [12, Theorem 8] proved that every (undirected) graph G with queue-number $qn(G) \leq q$ has track-number $tn(G) \leq 4q \cdot 4^{q(2q-1)(4q-1)}$. In the proof, the ordering of $V(G)$ in the given q -queue layout is preserved in every track of the track layout. Thus the result also holds for upward track-number. \square

Lemmas 7.3 and 7.4 imply the following theorem.

THEOREM 7.5. Upward track-number and upward queue-number are tied.

8. An Example

As illustrated in Figure 4, let G_n be the dag with vertex set $\{u_i : 1 \leq i \leq 2n\}$ and arc set $\{\overrightarrow{u_i u_{i+1}} : 1 \leq i \leq 2n - 1\} \cup \{\overrightarrow{u_i u_{2n-i+1}} : 1 \leq i \leq n\}$.

Observe that G_n is outerplanar and has a Hamiltonian directed path $(u_1, u_2, \dots, u_{2n})$. Thus $(u_1, u_2, \dots, u_{2n})$ is the only topological ordering of G_n , in which the edges $\{u_i u_{2n-i+1} : 1 \leq i \leq n\}$ are pairwise nested. Thus $uqn(G_n) \geq n$; it is easily seen that in fact $uqn(G_n) = n$. These observations were made by Heath et al. [25]. Theorem 7.5 implies that G_n has unbounded upward track-number. Di Giacomo et al. [9] proved the same result with the stronger bound of $utn(G_n) \geq \sqrt{2n}$, which follows from Lemma 7.3 with $k = 1$ and since $uqn(G_n) \geq n$. An upper bound of $utn(G_n) \leq \mathcal{O}(\sqrt{n})$ follows from Corollary 6.3 (and since G_n has bounded degree, from the earlier bounds on track-number in [14]). Di Giacomo et al. [9] gave an elegant construction of an improper $\mathcal{O}(\sqrt{n})$ -track layout of G_n . Suppose that G_n has an upward $X \times Y \times Z$ drawing. Then $Z \geq 2n$ by Observation 4.4. The second part of Theorem 5.3 implies that $2XY \geq utn(G_n) \geq \sqrt{2n}$. Hence the volume is $\Omega(n^{3/2})$, as proved by Di Giacomo et al. [9].

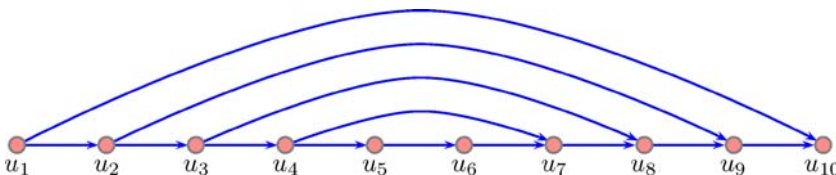


Figure 4 Illustration of G_5 .

This example highlights a substantial difference between 3D drawings of undirected graphs and upward 3D drawings of dags, since every (undirected) outerplanar graph has a 3D drawing with linear volume [18]. In the full version of their paper, Di Giacomo et al. [9] constructed an upward 3D drawing of G_n with $\mathcal{O}(n^{3/2})$ volume. It is unknown whether every n -vertex outerplanar dag has an upward 3D drawing with $\mathcal{O}(n^{3/2})$ volume.

9. Upward Layouts of Trees

A *caterpillar* is a tree, such that deleting the leaves gives a path (called the *spine*). A graph has a 2-track layout if and only if it is a caterpillar forest [21]. Felsner et al. [18] proved that every tree has a 3-track layout. Using this result, Di Giacomo et al. [9] proved that every tree dag has an upward 7-track layout, and that there exist tree dags with no upward 3-track layout. We improve this upper bound as follows.

THEOREM 9.1. *Every tree dag has an upward 5-track layout.*

Theorem 9.1 follows from Lemma 7.2(c) with $s = 2$ and the following lemma. Similarly, Lemma 7.2(b) with $s = 2$ and following lemma imply that every tree dag has upward queue-number at most 2, as proved by Heath et al. [25].

LEMMA 9.2. *Every tree dag T has a track layout $\{V_i : i \in \mathbb{Z}\}$, such that for every arc $\vec{vw} \in A(T)$ with $v \in V_i$ and $w \in V_j$, we have $i < j \leq i + 2$.*

Proof. Choose an arbitrary vertex r of T . Consider a vertex v . The *distance* $d(v)$ is the distance between v and r in the underlying undirected tree of T . Let $a(v)$ be the number of arcs on the path from v to r that are directed toward r . Let $b(v)$ be the number of arcs on the path from v to r that are directed away from r . Thus $d(v) = a(v) + b(v)$. As illustrated in Figure 5, put v in track $V_{2b(v)-a(v)}$. Within each track the vertices are ordered in non-decreasing order of distance from r . It remains

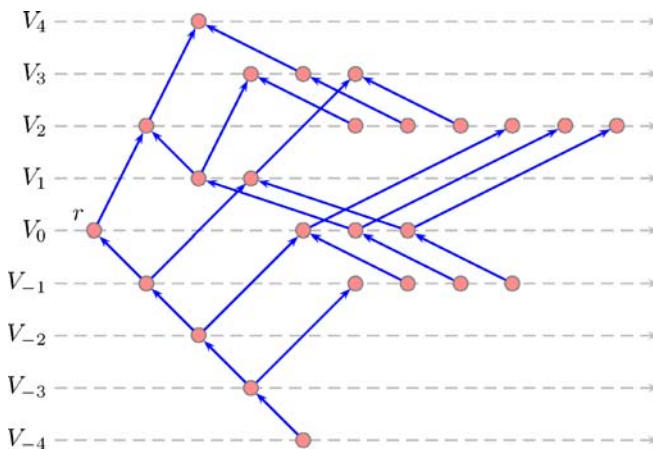


Figure 5 An upward track layout of a tree dag with span 2.

to order the vertices in a single track at the same distance. We do so inductively by increasing distance. Suppose the vertices at distance at most $d - 1$ are ordered. Consider two vertices v_1, v_2 in the same track and at distance d . Let w_1, w_2 be their respective parent vertices at distance $d - 1$. If $w_1 < w_2$ in the same track, then place $v_1 < v_2$. If w_1 and w_2 are in different tracks, then the relative order of v_1 and v_2 is irrelevant. This completes the construction.

Consider an arc $\vec{vw} \in A(T)$ with $v \in V_i$ and $w \in V_j$. First suppose that \vec{vw} is directed toward r . Then $b(w) = b(v)$ and $a(v) = a(w) + 1$. Thus $j = i + 1$ and \vec{vw} has span 1. Now suppose that \vec{vw} is directed away from r . Then $a(w) = a(v)$ and $b(w) = b(v) + 1$. Thus $j = i + 2$ and \vec{vw} has span 2. It remains to prove that there is no X-crossing. Consider arcs \vec{vw} and \vec{xy} between the same pair of tracks. Thus both arcs have the same span. First suppose their span is 1. Thus both \vec{vw} and \vec{xy} are directed towards r . Hence $d(v) = d(w) + 1$ and $d(x) = d(y) + 1$. Without loss of generality $v < x$ in their track. By construction $d(v) \leq d(w)$. If $d(v) < d(x)$ then $d(w) < d(y)$, and $w < y$ in their track since tracks are ordered by non-decreasing distance. If $d(v) = d(x)$ then $d(w) = d(y)$, and by construction $w < y$ in their track. In both cases, the arcs do not form an X-crossing. If \vec{vw} and \vec{xy} have span 2, then both \vec{vw} and \vec{xy} are directed away from r , and an analogous argument proves that the arcs do not form an X-crossing. \square

Di Giacomo et al. [9] proved that every tree dag has an upward $7 \times 7 \times 7n$ drawing. Lemma 5.6 and Theorem 9.1 imply the following improved bound.

THEOREM 9.3. *Every n -vertex tree dag has an upward $4 \times 4 \times \frac{7}{5}n$ drawing.*

The next result generalises and improves upon the result of Di Giacomo et al. [9] that every directed path has an improper upward 3-track layout.

THEOREM 9.4. *Every caterpillar dag has an upward 3-track layout and an upward 1-queue layout.*

Theorem 9.4 follows from the following lemma and Lemma 7.2 with $s = 1$.

LEMMA 9.5. *Every caterpillar dag C has a track layout $\{V_i : i \in \mathbb{Z}\}$ such that for every arc $\vec{vw} \in A(C)$, if $v \in V_i$ then $w \in V_{i+1}$.*

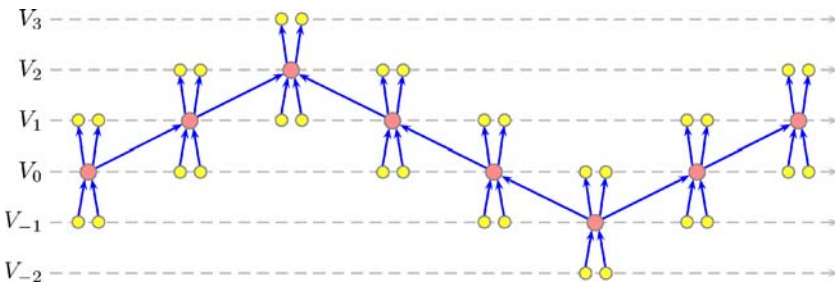


Figure 6 An upward track layout of a caterpillar dag with span 1.

Proof. Let $P = (v_1, v_2, \dots, v_n)$ be the spine of C . Put v_1 in track V_0 . For $j = 2, 3, \dots, n$, place v_j as follows. Say $v_{j-1} \in V_i$. If $\overrightarrow{v_{j-1}v_j} \in A(P)$ then put v_j rightmost in track V_{i+1} . Otherwise $\overrightarrow{v_jv_{j-1}} \in A(P)$, in which case put v_j rightmost in track V_{i-1} . Clearly $\{V_i : i \in \mathbb{Z}\}$ is a track layout of P . Consider a leaf vertex w adjacent to $v_j \in V_i$. If $\overrightarrow{wv_j} \in A(C)$ then put w in V_{i-1} . Otherwise $\overrightarrow{v_jw} \in A(C)$, in which case put w in V_{i+1} . Order each track as shown in Figure 6. Clearly $\{V_i : i \in \mathbb{Z}\}$ is a track layout of C . \square

Note the converse result.

PROPOSITION 9.6. *Suppose that every orientation of a tree T has an upward track layout $\{V_i : i \in \mathbb{Z}\}$ such that for every arc \overrightarrow{vw} , if $v \in V_i$ then $w \in V_{i+1}$. Then T is a caterpillar.*

Proof. Suppose that T is a tree that is not a caterpillar, and every orientation of T has the desired track layout. Then T contains a 2-claw [21], which is the tree with vertices r, u, v, w, x, y, z and edges ru, rv, rw, ux, vy, wz . Orient the edges $\overrightarrow{ru}, \overrightarrow{rv}, \overrightarrow{rw}, \overrightarrow{xu}, \overrightarrow{yv}, \overrightarrow{zw}$. Then this directed 2-claw has the desired upward track layout. Say $r \in V_i$. Then $u, v, w \in V_{i+1}$ and $x, y, z \in V_i$. Without loss of generality, v is between u and w in V_{i+1} . Thus \overrightarrow{yv} forms an X-crossing with either \overrightarrow{ru} or \overrightarrow{rw} , which is the desired contradiction. \square

Lemma 5.4 and Theorem 9.4 imply the following.

COROLLARY 9.7. *Every n -vertex caterpillar dag has a $2 \times 2 \times n$ upward 3D drawing.*

10. Upward Layouts of Subdivisions

A *subdivision* of a graph G is a graph obtained from G by replacing each edge $vw \in E(G)$ by a path with endpoints v and w . Each vertex of G is called an *original* vertex of the subdivision. Dujmović and Wood [15] proved that every graph G has a 2-queue subdivision with $\mathcal{O}(\log \varphi n(G))$ division vertices per edge, and that this bound is best possible. We now prove a similar result for upward queue layouts. A *subdivision* of a dag G is a graph obtained from G by replacing each arc $\overrightarrow{vw} \in A(G)$ by a path from v to w .

Let σ be a vertex ordering of a graph G . The *bandwidth* of σ is

$$\max\{|\sigma(w) - \sigma(v)| : vw \in E(G)\}.$$

The *bandwidth* of G is the minimum bandwidth of a vertex ordering of G . The *directed bandwidth* of a dag G is the minimum bandwidth of a topological vertex ordering of G .

THEOREM 10.1. *Every dag G with directed bandwidth b has an upward 2-queue subdivision with at most $\frac{1}{2}(b - 1)$ division vertices per edge.*

Proof. Let (v_1, v_2, \dots, v_n) be a topological ordering of G with bandwidth b . Consider each arc $\overrightarrow{v_iv_j} \in A(G)$. If $j - i$ is even, then replace $\overrightarrow{v_iv_j}$ by the directed path

$$(v_i, x(i, j, i + 2), x(i, j, i + 4), \dots, x(i, j, j - 4), x(i, j, j - 2), v_j)$$

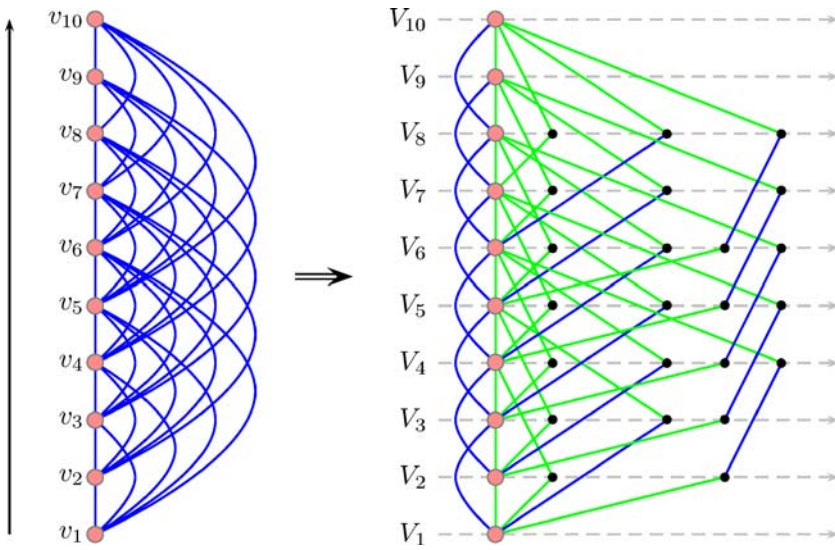


Figure 7 $(2, n)$ -track layout in the proof of Theorem 10.1.

with $\frac{1}{2}(j - i - 2)$ division vertices. If $j = i + 1$ then $\overrightarrow{v_i v_j}$ is not subdivided. If $j - i$ is odd and $j - i \geq 3$, then replace $\overrightarrow{v_i v_j}$ by the directed path

$$(v_i, x(i, j, i + 1), x(i, j, i + 3), \dots, x(i, j, j - 4), x(i, j, j - 2), v_j)$$

with $\frac{1}{2}(j - i - 1)$ division vertices.

First we construct an upward $(2, n)$ -track layout of this subdivision of G . Place each vertex v_i leftmost in track V_i . Position every vertex $x(i, j, \ell)$ in track V_ℓ . It remains to order the vertices within each track. Observe that $V_1 = \{v_1\}$. Order V_2 arbitrarily (with v_2 leftmost). Now order V_3 , then V_4 , and so on up to $V_n = \{v_n\}$. In track V_ℓ , except for the original vertex v_ℓ , each vertex has exactly one neighbour in $\cup\{V_i : 1 \leq i < \ell\}$. Thus we can order V_ℓ with v_ℓ leftmost, so that no two edges incident to vertices in $V_\ell \setminus \{v_\ell\}$ form an X-crossing. The only possible X-crossing involves an edge incident to v_ℓ . Thus we can colour all arcs $\overrightarrow{v_i v_{i+1}}$, $\overrightarrow{v_i x(i, j, i + 1)}$, and $\overrightarrow{x(i, \ell, \ell - 2) v_\ell}$ green, colour all other arcs blue, and there is no monochromatic X-crossing. Hence we have a $(2, n)$ -track layout as illustrated in Figure 7.

Observe that the green arcs satisfy Lemma 7.1. Thus the green arcs form a single queue in the vertex ordering $\sigma := (V_1, V_2, \dots, V_n)$. Every blue arc has span 2. Thus, as in Lemma 7.2(b), the blue arcs form a single queue in σ . Hence σ admits an upward 2-queue layout of the subdivision of G . \square

Dujmović and Wood [15] proved that every graph has a 4-track subdivision with $\mathcal{O}(\log \text{qn}(G))$ division vertices per edge, and that this bound is best possible. We have the following similar result for upward track layouts.

THEOREM 10.2. *Every dag G with directed bandwidth b has an upward 4-track subdivision with at most b division vertices per arc.*

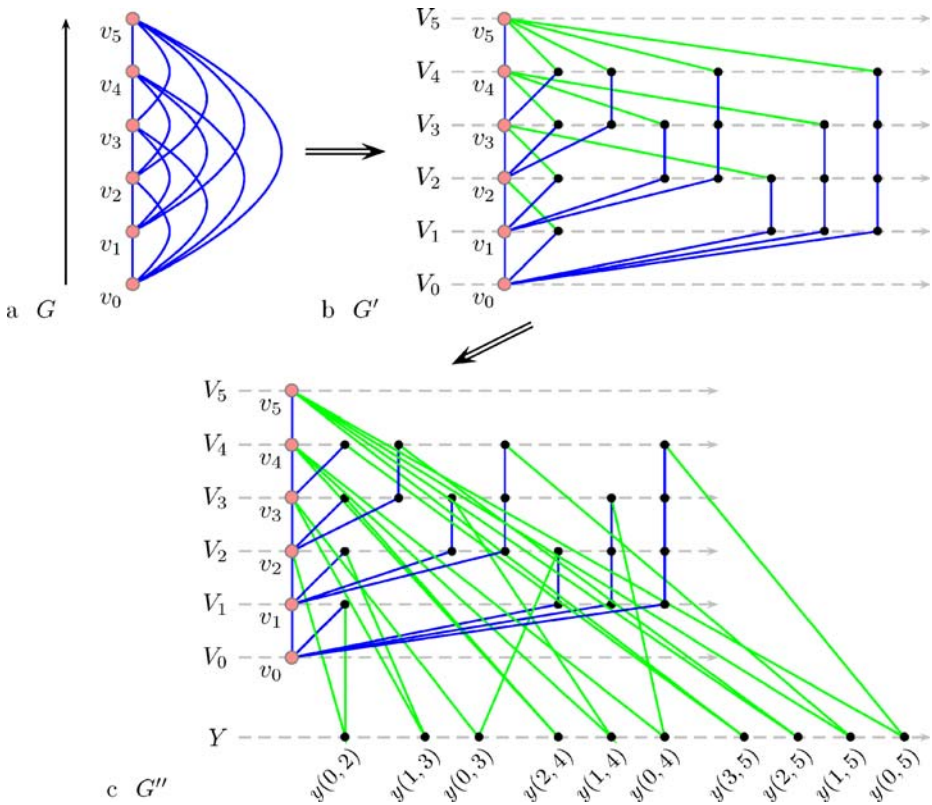


Figure 8 Track layouts in the proof of Theorem 10.2.

Proof. The proof proceeds in three steps. First, we construct an upward $(2, n)$ -track layout of a subdivision G' of G that has at most $b - 1$ division vertices per arc. Second, we construct an upward $(n + 1)$ -track layout of a subdivision G'' of G that has at most b division vertices per arc. Finally, we wrap n of the tracks into three tracks, to obtain an upward 4-track layout of G'' .

Let $(v_0, v_1, \dots, v_{n-1})$ be a topological ordering of G with bandwidth b . Let G' be the subdivision of G obtained by replacing each arc $\overrightarrow{v_i v_j} \in A(G)$ for which $j \geq i + 2$ by the directed path

$$(v_i, x(i, j, i + 1), x(i, j, i + 2), \dots, x(i, j, j - 1), v_j).$$

Note that G' has at most $b - 1$ division vertices per arc. Let $g(i, j)$ denote the final arc $(x(i, j, i - 1), v_j)$ in each such path. Colour $g(i, j)$ green, and colour the remaining arcs in G' blue (including the non-subdivided arcs $\overrightarrow{v_i v_{i+1}}$).

As illustrated in Figure 8(b), create a $(2, n)$ -track layout of G' as follows. Place each original vertex v_ℓ leftmost in track V_ℓ followed by the division vertices $x(i, j, \ell)$ for all arcs $\overrightarrow{v_i v_j} \in A(G)$ with $i < \ell < j$. Order the division vertices in V_ℓ by non-increasing i , breaking ties by increasing j . Clearly no two monochromatic edges form an X-crossing.

Order the green arcs $g(i, j)$ by non-decreasing j , breaking ties by decreasing i . Let π be the total order obtained. Consider two green arcs g_1 and g_2 with $g_1 <_{\pi} g_2$. Observe that if g_1 has an endpoint p in the same track as an endpoint q of g_2 , then $p \leq q$ in this track. Call this property (\star) .

For every arc $\overrightarrow{v_i v_j} \in A(G)$ with $j \geq i + 2$, subdivide the green arc $g(i, j)$ once to obtain a subdivision G'' of G with at most b division vertices per arc. Denote the division vertex by $y(i, j)$. As illustrated in Figure 8(c), place all the vertices $y(i, j)$ in a new track Y ordered by π . By property (\star) , we obtain an $(n + 1)$ -track layout of G'' with no X-crossing.

Now $\{V_1, V_2, \dots, V_n\}$ is an upward track layout of the subgraph of G'' consisting of the blue arcs, all of which have span 2. By Lemma 7.2(c), $\{V_1, V_2, \dots, V_n\}$ can be wrapped into an upward 3-track layout $\{X_0, X_1, X_2\}$, where $X_i = (V_i, V_{i+3}, V_{i+4}, \dots)$. Observe that property (\star) is maintained. Hence $\{X_0, X_1, X_2, Y\}$ is a 4-track layout of G'' . To see that $\{X_0, X_1, X_2, Y\}$ is upward, think of the $y(i, j)$ vertices as being in the middle of the corresponding green arcs in Figure 8(b); then all the arcs in $(G'')^+$ point up or to the right. Thus $(G'')^+$ is acyclic, and $\{X_0, X_1, X_2, Y\}$ is an upward 4-track layout of G'' . □

Theorems 10.1 and 10.2 are best possible in the sense that every subdivision H of a non-planar dag has $\text{uqn}(H) \geq 2$ and $\text{utn}(H) \geq 4$ (since 1-queue graphs and 3-track graphs are planar, and subdividing edges preserves planarity). In the undirected case, Dujmović and Wood [15] proved that planarity characterises those graphs with 1-queue or 3-track subdivisions. That is, a graph G is planar if and only if G has a 1-queue subdivision if and only if G has a 3-track subdivision. While we have not found such a characterisation for upward layouts, the proof of sufficiency by Dujmović and Wood [15, Lemma 32] generalises as follows. A dag G is *upward planar* if G has a crossing-free drawing in the plane, such that every arc $\overrightarrow{vw} \in A(G)$ is represented by a y -monotone curve with $y(v) < y(w)$.

PROPOSITION 10.3. *Every upward planar dag G has a subdivision that admits an upward 1-queue layout and an upward 3-track layout.*

Proof. Given an upward planar drawing of G , draw a horizontal line through every vertex. Now subdivide every edge whenever it crosses such a horizontal line. We obtain an upward track layout with span 1, which by Lemma 7.1 has an upward 1-queue layout and an upward 3-track layout. □

10.1. Upward Polyline Drawings

A b -bend 3D drawing of a graph G is a 3D drawing of a subdivision of G with at most b division vertices per edge. Lemma 5.5 and Theorem 10.2 imply the following result.

COROLLARY 10.4. *Let G be an n -vertex m -arc dag with directed bandwidth b . Then G has an upward b -bend $2 \times 2 \times (n + bm)$ drawing.*

The following theorem is a generalisation of a result by Dujmović and Wood [15].

THEOREM 10.5. *Every n -vertex dag G with upward queue-number $\text{uqn}(G) \leq k$ has an upward 2-bend $2k \times 2 \times 2n$ drawing.*

Proof. Let (v_1, v_2, \dots, v_n) be a topological vertex ordering of G that admits an upward k -queue layout. Number the queues $0, 1, \dots, k-1$. Put each vertex v_i at $(0, 0, 2i)$. Draw each arc $\overrightarrow{v_i v_{i+1}}$ straight. For all $j \geq i+2$, draw each arc $\overrightarrow{v_i v_j}$ in queue ℓ with the 2-bend polyline $(0, 0, 2i) \rightarrow (2\ell, 1, i+j) \rightarrow (2\ell+1, 1, i+j+1) \rightarrow (0, 0, 2j)$. Since $2i < i+j < i+j+1 < 2j$, the drawing is upward. No two arcs cross [15, Theorem 26]. \square

It follows from results of Heath and Rosenberg [26] that the upward queue-number of every n -vertex graph is at most $\lfloor \frac{n}{2} \rfloor$ (which is tight for the complete dag). Thus we have the following corollary of Theorem 10.5.

COROLLARY 10.6. *Every n -vertex dag G has an upward 2-bend $n \times 2 \times 2n$ drawing with volume $4n^2$.*

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