# PROGRESS ON DIRAC'S CONJECTURE

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Abstract. In 1951, Gabriel Dirac conjectured that every set $P$ of $n$ non-collinear points in the plane contains a point in at least $\frac{n}{2} - c$ lines determined by $P$, for some constant $c$. The following weakening was proved by Beck and Szemeredi–Trotter: every set of $n$ non-collinear points contains a point in at least $\frac{n}{c}$ lines determined by $P$, for some large unspecified constant $c$. We prove that every set $P$ of $n$ non-collinear points contains a point in at least $\frac{n-3}{76}$ lines determined by $P$.

## 1. Introduction

Let $P$ be a finite set of points in the plane. A line that contains at least two points in $P$ is said to be *determined* by $P$. In 1951, Dirac [6] made the following conjecture, which remains unresolved:

**Conjecture 1** (Dirac Conjecture)**.** Every set $P$ of $n$ non-collinear points contains a point in at least $\frac{n}{2} - c$ lines determined by $P$, for some constant $c$.

See reference [3] for examples showing that the $\frac{n}{2}$ bound would be tight. In 1961, Erdős [7] proposed the following weakened conjecture.

**Conjecture 2** (Weak Dirac Conjecture)**.** Every set $P$ of $n$ non-collinear points contains a point in at least $\frac{n}{c}$ lines determined by $P$, for some constant $c$.

In 1983, the Weak Dirac Conjecture was proved indepedently by Beck [4] and Szemerédi and Trotter [18], in both cases with $c$ unspecified and very large. We prove the Weak Dirac Conjecture with $c$ much smaller. (See references [8, 9, 11, 16] for more on Dirac's Conjecture.)

**Theorem 3.** *Every set $P$ of $n$ non-collinear points contains a point in at least $\frac{n-3}{76}$ lines determined by $P$.*

The following idea is the key to the proof of Theorem 3. The *visibility graph* $G$ of a point set $P$ has vertex set $P$, where $vw \in E(G)$ whenever the line segment $vw$ contains no other point in $P$ (that is, $v$ and $w$ are consecutive on a line determined by $P$). If $v$ is a

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vertex with degree $d$ in $G$, then $v$ is in at least $\frac{d}{2}$ lines determined by $P$. Thus, to prove the Weak Dirac Conjecture, it suffices to show that $G$ contains a vertex of degree $\Omega(n)$, which is implied if $G$ contains $\Omega(n^2)$ edges. Hence, we are left with the natural extremal question: what is the minimum number of edges in a visibility graph with $n$ vertices? Of course, the answer is only $n-1$ if the points are collinear, and if $P$ has $n-1$ collinear points with one point off the line, then the visibility graph of $P$ only has $2n-3$ edges. More generally, if $P$ contains $n-o(n)$ collinear points, then the visibility graph of $P$ has $o(n^2)$ edges. Thus to conclude an $\Omega(n^2)$ lower bound on the number of edges in a visibility graph, it is necessary to assume that at most a constant fraction of the points are collinear. In this case, Beck's Theorem implies that $P$ determines $\Omega(n^2)$ lines, which immediately implies that $G$ contains $\Omega(n^2)$ edges. However, the hidden constant here is very small. We prove this result with a reasonable constant.

**Theorem 4.** *For every set $P$ of $n$ points in the plane with at most $\frac{n}{76}$ collinear points, the visibility graph of $P$ has at least $\frac{n(n-3)}{76}$ edges.*

Theorem 4 enables the proof of Theorem 3.

*Proof of Theorem 3.* Let $P$ be a set of $n$ non-collinar points in the plane. If $P$ contains at least $\frac{n}{76}$ collinear points, then every other point is in at least $\frac{n}{76}$ lines determined by $P$ (one through each of the collinear points). Otherwise, by Theorem 4, the visibility graph $G$ of $P$ has at least $\frac{n(n-3)}{76}$ edges. Say $G$ has maximum degree $d$. Thus $\frac{dn}{2} \geqslant |E(G)| \geqslant \frac{n(n-3)}{76}$, implying $\frac{d}{2} \geqslant \frac{n-3}{76}$. Therefore, every vertex with degree $d$ is in at least $\frac{n-3}{76}$ lines determined by $P$. $\square$

The proof of Theorem 4 takes inspiration from the well known proof of Beck's Theorem [5] as a corollary of the Szemeredi–Trotter Theorem [18], and also from the simple proof of the Szemeredi–Trotter Theorem due to Székely [17], which in turn is based on the Crossing Lemma. A *drawing* of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such that the only vertices an edge intersects are its own endpoints, and no three edges intersect at a common point (except at a common endpoint). A *crossing* is a point of intersection between two edges (other than a common endpoint). The *crossing number* of a graph $G$, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings in a drawing of $G$. The following lower bound on $\mathrm{cr}(G)$ was first proved by Ajtai et al. [2] and Leighton [12] (with worse constants). Part (a) follows from the folklore probabilistic proof of the Crossing Lemma [1]. Part (b) is a special case of (a). Part (c) is due to Pach et al. [15].

**Theorem 5** (Crossing Lemma)**.** *For every graph $G$ with $n$ vertices and $m$ edges,*

(a) $$\forall\delta > 0 \quad m \geqslant (3+\delta)n \quad \Longrightarrow \quad \mathrm{cr}(G) \geqslant \frac{\delta\, m^3}{(3+\delta)^3\, n^2}\ ,$$

(b) $$m \geqslant 4n \quad \Longrightarrow \quad \mathrm{cr}(G) \geqslant \frac{m^3}{64n^2}\ ,$$

(c) $$m \geqslant \frac{103}{16}n \quad \Longrightarrow \quad \mathrm{cr}(G) \geqslant \frac{1024\, m^3}{31827\, n^2}\ .$$

## 2. Main Proof

Theorem 4 follows from Theorem 5(c) and the following general result (with $\epsilon = \frac{1}{76}$, $\alpha = \frac{103}{16}$, $\beta = \frac{31827}{1024}$ and $c = 32$, in which case $dn^2 - \frac{n}{c} = \frac{n^2}{75.831...} - \frac{n}{32} \geqslant \frac{n(n-3)}{76}$).

**Theorem 6.** *Let $\alpha$ and $\beta$ be positive constants such that every graph $H$ with $n$ vertices and $m \geqslant \alpha n$ edges satisfies*

$$\operatorname{cr}(H) \geqslant \frac{m^3}{\beta n^2} \quad .$$

*Fix $\epsilon$ such that $0 < \epsilon < \frac{1}{\alpha}$. Fix a positive integer $c$. Let*

$$d := \frac{1}{c} - \frac{\epsilon\alpha}{c} - \frac{\beta}{2c}\sum_{i\geqslant c}\frac{1}{i^2} \quad .$$

*Then for every set $P$ of $n$ points in the plane with at most $\epsilon n$ collinear points, the visibility graph $G$ of $P$ has at least $dn^2 - \frac{n}{c}$ edges.*

*Note.* Given $\alpha, \beta, \epsilon$ and $c$, the number $d$ is easily computed (approximately) since $\sum_{i\geqslant 1}\frac{1}{i^2} = \frac{\pi^2}{6} \approx 1.66493$. By choosing $c$ large enough so that $\sum_{i\geqslant c}\frac{1}{i^2} < \frac{2(1-\epsilon\alpha)}{\beta}$, we have $d > 0$.

*Proof.* Let $I := \{2, 3, \dots, \lfloor \epsilon n \rfloor\}$. For $i \in I$, an *$i$-line* is a line determined by $P$ with exactly $i$ points in $P$. Let $s_i$ be the number of $i$-lines. An *$i$-pair* is a pair of points in an $i$-line. An *$i$-edge* is an edge of $G$ contained in some $i$-line (that is, between a pair of consecutive points in some $i$-line). Since each $i$-line determines $\binom{i}{2}$ pairs,

$$\sum_{i\in I}\binom{i}{2}s_i = \binom{n}{2} \quad .$$

Since each $i$-line contributes $i-1$ edges to $G$,

$$|E(G)| = \sum_{i\in I}(i-1)s_i \quad .$$

For $i \in I$, let $G_i$ be the spanning subgraph of the visibility graph of $P$ consisting of all edges in $\ell$-lines where $\ell \geqslant i$; see Figure 1 for an example. Note that

$$G = G_2 \supseteq G_3 \supseteq G_4 \supseteq \cdots \supseteq G_{\lfloor \epsilon n \rfloor} \quad .$$

Let $k$ be the minimum integer such that $|E(G_k)| \leqslant \alpha n$. If there is no such $k$ then let $k := \infty$.

An integer $i \in I$ is *large* if $i \geqslant k$, and is *small* if $i \leqslant c$. An integer in $I$ that is neither large nor small is *medium*. A *small pair* is an $i$-pair for some large $i$. Define *medium pairs* and *small pairs* analogously. A *small edge* is an $i$-edge for some small $i$. Define *medium edges* and *large edges* analogously.

The proof procedes by establishing an upper bound on the number of small pairs in terms of the number of small edges. Analogous bounds are proved for the number of medium

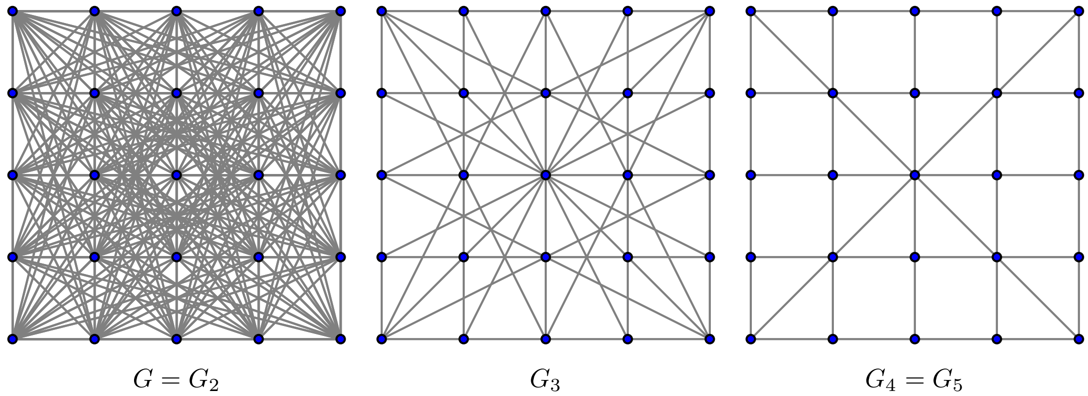


FIGURE 1. The graphs $G_2, G_3, G_4, G_5$ in the case of the $5 \times 5$ grid.

pairs, and the number of large pairs. Combining these results gives the desired lower bound on the total number of edges. First,

$$\text{(1)} \qquad (\text{\# small pairs}) = \sum_{i=2}^{c} \binom{i}{2} s_i \leqslant \frac{c}{2} \sum_{i=2}^{c} (i-1) s_i = \frac{c}{2} (\text{\# small edges}) \ .$$

We now bound the number of medium pairs. Consider a medium $i \in I$. Since $i$ is not large, $|E(G_i)| > \alpha n$. By the assumed Crossing Lemma applied to $G_i$,

$$\operatorname{cr}(G_i) \geqslant \frac{|E(G_i)|^3}{\beta n^2} = \frac{(\sum_{j \geqslant i} (j-1) s_j)^3}{\beta n^2} \geqslant \frac{(i-1)^2 (\sum_{j \geqslant i} s_j)^2 (\sum_{j \geqslant i} (j-1) s_j)}{\beta n^2} \ .$$

On the other hand, since two lines cross at most once,

$$\frac{(i-1)^2 (\sum_{j \geqslant i} s_j)^2 (\sum_{j \geqslant i} (j-1) s_j)}{\beta n^2} \leqslant \operatorname{cr}(G_i) \leqslant \binom{\sum_{j \geqslant i} s_j}{2} \leqslant \frac{1}{2} \Big( \sum_{j \geqslant i} s_j \Big)^2 \ .$$

Hence, for each medium $i \in I$,

$$\text{(2)} \qquad \sum_{j \geqslant i} (j-1) s_j \leqslant \frac{\beta\, n^2}{2(i-1)^2} \ ,$$

(This bound can be considered to be a slight strengthening of the Szemeredi-Trotter Theorem.) Given the $\frac{c}{2}$ factor in the bound on the number of small pairs in (1), we need to introduce the same factor in our bound on the number of medium pairs:

$$\begin{aligned}
(\text{\# medium pairs}) - \frac{c}{2} (\text{\# medium edges}) &= \left( \sum_{i=c+1}^{k-1} \binom{i}{2} s_i \right) - \frac{c}{2} \left( \sum_{i=c+1}^{k-1} (i-1) s_i \right) \\
&= \frac{1}{2} \sum_{i=c+1}^{k-1} (i-1)(i-c) s_i \\
&= \frac{1}{2} \sum_{i=c+1}^{k-1} \sum_{j=i}^{k-1} (j-1) s_j \ .
\end{aligned}$$

By (2),

$$\text{(3)} \qquad (\text{\# medium pairs}) - \frac{c}{2} (\text{\# medium edges}) \leqslant \frac{1}{2} \sum_{i \geqslant c+1} \frac{\beta\, n^2}{2(i-1)^2} = \frac{\beta\, n^2}{4} \sum_{i \geqslant c} \frac{1}{i^2} \ .$$

Finally, we bound the number of large pairs:

$$(\#\text{ large pairs}) = \sum_{i=k}^{\lfloor \epsilon n \rfloor} \binom{i}{2} s_i \leqslant \frac{\epsilon n}{2} \sum_{i \geqslant k} (i-1) s_i = \frac{\epsilon n}{2} |E(G_k)| \leqslant \frac{\epsilon \alpha \, n^2}{2} \quad . \tag{4}$$

Combining (1), (3) and (4),

$$\begin{aligned} \binom{n}{2} &= (\#\text{ small pairs}) + (\#\text{ medium pairs}) + (\#\text{ large pairs}) \\ &\leqslant \frac{c}{2}(\#\text{ small edges}) + \frac{c}{2}(\#\text{ medium edges}) + \left( \frac{\beta \, n^2}{4} \sum_{i \geqslant c} \frac{1}{i^2} \right) + \frac{\epsilon \alpha \, n^2}{2} \quad . \end{aligned}$$

Thus

$$\left( \frac{1}{2} - \frac{\epsilon \alpha}{2} - \frac{\beta}{4} \sum_{i \geqslant c} \frac{1}{i^2} \right) n^2 - \frac{n}{2} \leqslant \frac{c}{2}(\#\text{ small edges}) + \frac{c}{2}(\#\text{ medium edges}) \quad .$$

Hence

$$|E(G)| \geqslant \left( \frac{1}{c} - \frac{\epsilon \alpha}{c} - \frac{\beta}{2c} \sum_{i \geqslant c} \frac{1}{i^2} \right) n^2 - \frac{n}{c} \quad .$$

This completes the proof. □

## 3. Discussion

We now discuss a number of small improvements that can be made to the above proof. We omit these details for the sake of clarity and simplicity of the proof.

(1) The '-3' in Theorems 3 and 4 can be eliminated by counting large edges in the proof of Theorem 6.

(2) A number of inequalities involving the $s_i$ are known. For example, Melchior's inequality [14] states that $s_2 \geqslant 3 + \sum_{i \geqslant 4} (i-3) s_i$ (assuming $s_n = 0$). This inequality can be easily incorporated into the upper bound on the number of small pairs in the proof of Theorem 6 to slightly improve the result. A similar inequality by Hirzebruch [10] can also be used to make modest improvements.

(3) Given a set $P$ of points, let $G$ be the visibility graph of $P$. Let $G'$ be the multigraph obtained from $G$ as follows: for each line $L$ determined by $P$, add an edge $vw$ to $G'$ where $v$ and $w$ are the first and last vertex in $L \cap P$. Note that if $|L \cap P| = 2$ then $vw$ will appear twice in $G'$. Call $G'$ the *projective visibility multigraph* of $P$. Observe that if a vertex $v$ has degree $d$ in $G'$, then $v$ is in exactly $\frac{d}{2}$ lines determined by $P$. Thus to prove that some vertex is in at least $\frac{n}{c}$ lines determined by $P$, it suffices to show that some vertex has degree at least $\frac{2n}{c}$ in $G'$, which is implied if $|E(G')| \geqslant \frac{n^2}{c}$. The proof of Theorem 6 is easily adapted to count edges in $G'$. In particular, each $i$-line determines $i$ edges in $G'$. Note that the crossing lemma should be applied to $G$ not $G'$.

Our proof of Theorem 3 easily extends to the setting of pseudolines. However, the (strong) Dirac Conjecture does not hold in this setting, as recently shown by Lund et al. [13]. They constructed arrangements of $n$ pseudolines in which every point is in most $\frac{4n}{9}$ pseudolines.

Since planar graphs can have close to $3n$ edges, $\alpha > 3$ in the Crossing Lemma. Thus Theorem 6 requires $\epsilon < \frac{1}{\alpha} < \frac{1}{3}$. For $\epsilon$ close to $\frac{1}{3}$, by Theorem 5(a), we may apply Theorem 6 with $\alpha = 3 + \delta$ and $\beta = (3+\delta)^3/\delta$, where $\delta < \frac{1}{\epsilon} - 3$. However, for point sets that do have $\epsilon n$ collinear points, the following simple result is superior to Theorem 6 when $\epsilon$ is reasonably large.

**Proposition 7.** *Let $P$ be a set of $n$ points in the plane, such that some line $L$ contains exactly $\epsilon n$ points in $P$. Then the visibility graph $G$ of $P$ contains at least $\epsilon(1-\epsilon)n^2$ edges.*

*Proof.* For each point $v \in L \cap P$ and for each point $w \in P - L$, count the edge of $G$ incident to $w$ in the direction of $v$. Since $L$ is collinear and $w$ is not in $L$, no edge is counted twice. Thus $|E(G)| \geqslant |L \cap P| \cdot |P - L| = \epsilon(1-\epsilon)n^2$. □

Proposition 7 implies that for every set of $n$ points that contains a subset of *exactly* $\frac{n}{2}$ collinear points, the visibility graph has at least $\frac{n^2}{4}$ edges. We conjecture that the same bound holds for sets with *at most* $\frac{n}{2}$ collinear points. This would be tight (ignoring linear terms) for two parallel lines, each with $\frac{n}{2}$ points. Similarly, we conjecture that for every set of $n$ points with at most $\frac{n}{2}$ collinear points, the projective visibility multigraph has at least $\frac{n^2}{2}$ edges, which again would be tight for two parallel lines each with $\frac{n}{2}$ points. By the discussion above, this conjecture would imply Dirac's Conjecture.

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