THICKNESS AND ANTITHICKNESS OF GRAPHS

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Abstract. This paper studies questions about duality between crossings and non-crossings in graph drawings via the notions of thickness and antithickness. The *thickness* of a graph G is the minimum integer k such that in some drawing of G, the edges can be partitioned into k noncrossing subgraphs. The *antithickness* of a graph G is the minimum integer k such that in some drawing of G, the edges can be partitioned into k thrackles, where a *thrackle* is a set of edges, each pair of which intersect exactly once. (Here edges with a common endvertex v are considered to intersect at v.) So thickness is a measure of how close a graph is to being planar, whereas antithickness is a measure of how close a graph is to being a thrackle. This paper explores the relationship between the thickness and antithickness of a graph, under various graph drawing models, with an emphasis on extremal questions.

1 Introduction

This paper studies questions about duality between crossings and non-crossings in graph drawings. This idea is best illustrated by an example. A graph is *planar* if it has a drawing with no crossings, while a *thrackle* is a graph drawing in which every pair of edges intersect exactly once. So in some sense, thrackles are the graph drawings with the most crossings. Yet thracklehood and planarity appear to be related. In particular, a widely believed conjecture would imply that every thrackleable graph is planar. Loosely speaking, this says that a graph that can be drawn with the maximum number of crossings has another drawing with no crossings. This paper explores this seemingly counterintuitive idea through the notions of thickness and antithickness. First we introduce the essential definitions.

A (*topological*) *drawing* of a graph[1] G is a function that maps each vertex of G to a distinct point in the plane, and maps each edge of G to a simple closed curve between the images of its end-vertices, such that:

- the only vertex images that an edge image intersects are the images of its own end-vertices (that is, an edge does not 'pass through' a vertex),
- the images of two edges are not tangential at a common interior point (that is, edges cross 'properly').

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[1] We consider undirected, finite, simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. Let $G[S]$ denote the subgraph of G induced by a set of vertices $S \subseteq V(G)$. Let $G - S := G[V(G) \setminus S]$ and $G - v := G \setminus \{v\}$.

Where there is no confusion we henceforth do not distinguish between a graph element and its image in a drawing. Two edges with a common end-vertex are *adjacent*. Two edges in a drawing *cross* if they intersect at some point other than a common end-vertex. Two edges that do not intersect in a drawing are *disjoint*. A drawing of a graph is *noncrossing* if no two edges cross. A graph is *planar* if it has a noncrossing drawing.

In the 1960s John Conway introduced the following definition. A drawing of a graph is a *thrackle* if every pair of edges intersect exactly once (either at a common endvertex or at a crossing point). A graph is *thrackeable* if it has a drawing that is a thrackle; see [5, 12–15, 17, 22, 48, 50, 53, 73, 76, 84, 85, 89, 90, 93]. Note that in this definition, it is important that every pair of edges intersect *exactly* once since every graph has a drawing in which every pair of edges intersect at least once[2].

A drawing is *geometric* if every edge is a straight line segment. A geometric drawing is *convex* if every vertex is on the convex hull of the set of vertices. A 2*-track drawing* is a convex drawing of a bipartite graph in which the two colour classes are separated in the ordering of the vertices around the convex hull. For the purposes of this paper, we can assume that the two colour classes in a 2-track drawing are on two parallel lines (called *tracks*). The notion of a *convex thrackle* is closely related to that of *outerplanar thrackle*, which was independently introduced by Cairns and Nikolayevsky [17].

1.1 Thickness and Antithickness

The *thickness* of a graph G is the minimum $k \in \mathbb{N}$ such that the edge set $E(G)$ can be partitioned into k planar subgraphs. Thickness is a widely studied parameter; see the surveys [61, 80]. The *thickness* of a graph drawing is the minimum $k \in \mathbb{N}$ such that the edges of the drawing can be partitioned into k noncrossing subgraphs. Equivalently, each edge is assigned one of k colours such that crossing edges receive distinct colours.

Every planar graph can be drawn with its vertices at prespecified locations [57, 65, 86]. It follows that a graph has thickness k if and only if it has a drawing with thickness k [57, 65]. However, in such a representation the edges might be highly curved[3]. The minimum integer k such that a graph G has a geometric / convex / 2-track drawing with thickness k is called the *geometric* / *book* / 2*-track thickness* of G. Book thickness is also called *pagenumber* and *stacknumber* in the literature; see the surveys [7, 34][4]

The following results are well known for every graph G:

- G has geometric thickness 1 if and only if G is planar [43, 98].

[2]*Proof*: Let $V(G) = \{v_1, \dots, v_n\}$. Position each vertex v_i at $(i, 0)$. Define a relation $\prec$ on $E(G)$ where $v_iv_j \prec v_pv_q$ if and only if $i < j \leqslant p < q$. Observe that $\preceq$ is a partial order of $E(G)$. Let $E(G) = \{e_1, \dots, e_m\}$, where $e_i \prec e_j$ implies that $j < i$. Draw each edge $e_i = v_pv_q$ as the 1-bend polyline $(p, 0)(i, 1)(q, 0)$. Then every pair of edges intersect at least once.

[3]In fact, Pach and Wenger [86] proved that for every planar graph G that contains a matching of n edges, if the vertices of G are randomly assigned prespecified locations on a circle, then $\Omega(n)$ edges of G have $\Omega(n)$ bends in every polyline drawing of G.

[4]In the context of this paper it would make sense to refer to book thickness as *convex thickness*, and to refer to thickness as *topological thickness*, although we refrain from the temptation of introducing further terminology.

- G has book thickness 1 if and only if G is outerplanar [65].
- G has book thickness at most 2 if and only if G is a subgraph of a Hamiltonian planar graph [65].
- G has 2-track thickness 1 if and only if G is a forest of caterpillars [58].

The *antithickness* of a graph G is the minimum $k \in \mathbb{N}$ such that $E(G)$ can be partitioned into k thrackeable subgraphs. The *antithickness* of a graph drawing is the minimum $k \in \mathbb{N}$ such that the edges of the drawing can be partitioned into k thrackles. Equivalently, each edge is assigned one of k colours such that disjoint edges receive distinct colours. The minimum $k \in \mathbb{N}$ such that a graph G has a topological / geometric / convex / 2-track drawing with antithickness k is called the *topological* / *geometric* / *convex* / *2-track antithickness* of G. Thus a graph is thrackeable if and only if it has antithickness 1.

Lemma 1. *Every thrackeable graph G has a thrackled drawing with each vertex at a prespecified location.*

Proof. Consider a thrackled drawing of G. Replace each crossing point by a dummy vertex. Let H be the planar graph obtained. Let $p(v)$ be a distinct prespecified point in the plane for each vertex v of G. For each vertex $x \in V(H) - V(G)$ choose a distinct point $p(x) \in \mathbb{R}^2 \setminus \{p(v) : v \in V(G)\}$. Every planar graph can be drawn with its vertices at prespecified locations [57, 65, 86]. Thus H can be drawn planar with each vertex x of H at $p(x)$. This drawing defines a thrackled drawing of G with each vertex v of G at $p(v)$, as desired. □

Corollary 2. *A graph has antithickness k if and only if it has a drawing with antithickness k.*

Every graph G satisfies

$$\text{thickness}(G) \leqslant \text{geometric thickness}(G) \leqslant \text{book thickness}(G), \text{ and}$$
$$\text{antithickness}(G) \leqslant \text{geometric antithickness}(G) \leqslant \text{convex antithickness}(G).$$

Moreover, if G is bipartite, then

$$\text{book thickness}(G) \leqslant \text{2-track thickness}(G) = \text{2-track antithickness}(G), \text{ and}$$
$$\text{convex antithickness}(G) \leqslant \text{2-track thickness}(G) = \text{2-track antithickness}(G).$$

For the final equality, observe that a 2-track layout of G with antithickness k is obtained from a 2-track layout of G with thickness k by simply reversing one track, and vice versa.

1.2 An Example: Trees

Consider the thickness of a tree. Every tree is planar, and thus has thickness 1 and geometric thickness 1. It is well known that every tree T has 2-track thickness at most 2. *Proof:* Orient the edges away from some vertex r. Properly 2-colour the vertices of T *black* and *white*. Place each colour class on its own track, ordered according to a breadth-first search of T

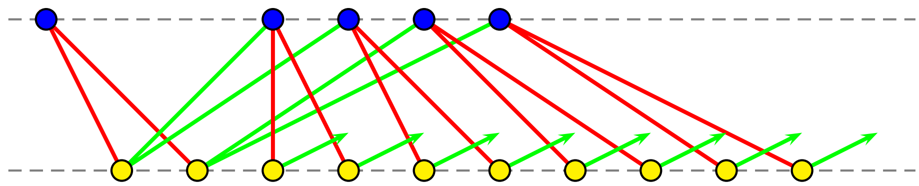

Figure 1: A 2-track drawing of a tree with thickness 2.

starting at r. Colour each edge according to whether it is oriented from a black to a white vertex, or from a white to a black vertex. It is easily seen that no two monochromatic edges cross, as illustrated in Figure 1.

The *2-claw* is the tree with vertex set $\{r, v_1, v_2, v_3, w_1, w_2, w_3\}$ and edge set $\{rv_1, rv_2, rv_3, v_1w_1, v_2w_2, v_3w_3\}$, as illustrated in Figure 2(a). The upper bound of 2 on the 2-track thickness of trees is best possible since Harary and Schwenk [58] proved that the 2-claw has 2-track thickness exactly 2, as illustrated in Figure 2(b).

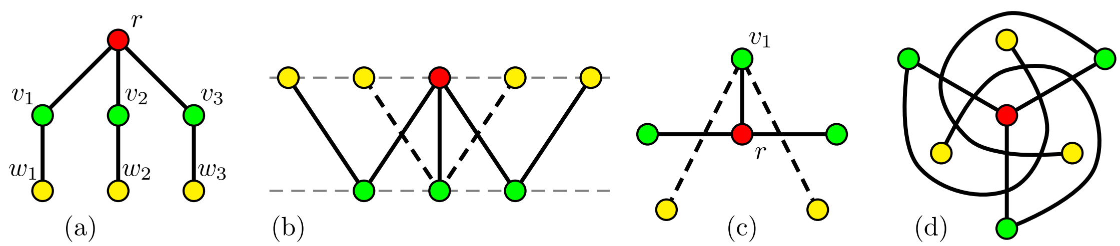

Figure 2: (a) The 2-claw. (b) The 2-claw has 2-track thickness 2. (c) The 2-claw is not a geometric thrackle. (d) The 2-claw drawn as a thrackle.

What about the antithickness of a tree? Since every tree has 2-track thickness at most 2, by reversing one track, every tree has 2-track antithickness at most 2. And again the 2-claw shows that this bound is tight. In fact:

Lemma 3. *The 2-claw is not a geometric thrackle.*

Proof. Suppose to the contrary that the 2-claw is a geometric thrackle, as illustrated in Figure 2(c). For at least one of the three edges incident to r, say rv_1, the other two vertices adjacent to r are on distinct sides of the line through rv_1. Thus v_1w_1 can only intersect one of rv_2 and rv_3, which is the desired contradiction. □

This lemma shows that 2 is a tight upper bound on the geometric antithickness of trees. On the other hand, if we allow curved edges, Woodall [100] proved that every tree is thrackleable, and thus has antithickness 1, as illustrated in Figure 2(d) in the case of a 2-claw.

1.3 Main Results and Conjectures

A *graph parameter* is a function β that assigns to every graph G a non-negative integer $\beta(G)$. Examples that we have seen already include thickness, geometric thickness, book thickness, antithickness, geometric antithickness, and convex antithickness. Let $\mathcal{F}$ be a class of graphs. Let $\beta(\mathcal{F})$ denote the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n)$ is the maximum of $\beta(G)$, taken over all n-vertex graphs $G \in \mathcal{F}$. We say $\mathcal{F}$ has *bounded* β if $\beta(\mathcal{F}) \in \mathcal{O}(1)$ (where n is the hidden variable in $\mathcal{O}(1)$).

A graph parameter β is *bounded by* a graph parameter γ (for some class $\mathcal{F}$), if there exists a *binding* function g such that $\beta(G) \leqslant g(\gamma(G))$ for every graph G (in $\mathcal{F}$). If β is bounded by γ (in $\mathcal{F}$) and γ is bounded by β (in $\mathcal{F}$) then β and γ are *tied* (in $\mathcal{F}$). If β and γ are tied, then a graph family $\mathcal{F}$ has bounded β if and only if $\mathcal{F}$ has bounded γ. This definition is due to Ding and Oporowski [28] and Reed [91]. Note that 'tied' is a transitive relation. For β and γ to be not tied means that for some class of graphs, β is bounded but γ is unbounded (or vice versa). In this case, β and γ are *separated*, which is terminology introduced by Eppstein [38, 39].

The central questions of this paper ask which thickness/antithickness parameters are tied. In Section 4 we prove that thickness and antithickness are tied—in fact we prove that these parameters are both tied to arboricity, and thus only depend on the maximum density of the graph's subgraphs. (See Section 4 for the definition of arboricity.)

Eppstein [38] proved that book thickness and geometric thickness are separated. In particular, for every t, there exists a graph with geometric thickness 2 and book thickness at least t; see [8, 9] for a similar result. Thus book thickness is not bounded by geometric thickness. The example used here is K'_n, which is the graph obtained from K_n by subdividing each edge exactly once. In Lemma 17 we prove that K'_n has geometric antithickness 2. At the end of Section 5 we prove that K'_n has convex antithickness at least $\sqrt{n/6}$ (which is unbounded). Thus convex antithickness is not bounded by geometric antithickness, implying that convex antithickness and geometric antithickness are separated.

Eppstein [39] also proved that geometric thickness and thickness are separated. In particular, for every t, there exists a graph with thickness 3 and geometric thickness at least t. Thus geometric thickness is not bounded by thickness. (Note that it is open whether every graph with thickness 2 has bounded geometric thickness.) Eppstein [39] used the following graph to establish this result. Let G_n be the graph having as its $n+\binom{n}{3}$ vertices the singleton and tripleton subsets of an n-element set, with an edge between two subsets when one is contained in the other. (Note that K'_n can be analogously defined—just replace tripleton by doubleton.) Then G_n has thickness 3, and for all t there is an n for which G_n has geometric thickness at least t. We expect that an analogous separation result holds for antithickness and geometric antithickness. Since $E(G_n)$ has an edge-partition into three star-forests, G_n has antithickness 3. We conjecture that for all t there is an n for which G_n has geometric antithickness at least t. This would imply that geometric antithickness is not bounded by antithickness.

In the positive direction, we conjecture the following dualities:

Conjecture 4. *Geometric thickness and geometric antithickness are tied.*

Conjecture 5. *Book thickness and convex antithickness are tied.*

In Theorem 15 we prove that convex antithickness and queue-number (defined in Section 2) are tied. Thus the truth of Conjecture 5 would imply that book thickness and queue-number are tied. This would imply, since planar graphs have bounded book thickness [11, 101], that planar graphs have bounded queue-number, which is an open problem due to Heath et al. [59, 60]; see [23, 29, 31] for recent progress. Thus a seemingly easier open problem is to decide whether planar graphs have bounded geometric antithickness.

Lovász et al. [73] proved two related results. First they proved that every bipartite thrackleable graph is planar. And more generally, they proved that a bipartite graph has a drawing in which every pair of edges intersect an odd number of times if and only if the graph is planar. In their construction, non-adjacent edges cross once, and adjacent edges intersect three times.

1.4 Other Contributions

In addition to the results discussed above, this paper makes the following contributions. In Section 2 we prove that convex antithickness is tied to queue-number and track-number. Several interesting results follow from this theorem. Section 3 surveys the literature on the problem of determining the thickness or antithickness of a given (uncoloured) drawing of a graph. Sections 4 and 5 respectively prove two results discussed above, namely that thickness and antithickness are tied, and that convex antithickness and geometric antithickness are separated. Section 6 studies natural extremal questions for all of the above parameters. Finally, Section 7 considers the various antithickness parameters for a complete graph.

2 Stack, Queue and Track Layouts

This section introduces track and queue layouts, which are well studied graph layout models. We show that they are closely related to convex antithickness.

A *vertex ordering* of an n-vertex graph G is a bijection $\pi : V(G) \to \{1, 2, \dots, n\}$. We write $v <_\pi w$ to mean that $\pi(v) < \pi(w)$. Thus $\leqslant_\pi$ is a total order on $V(G)$. We say G or $V(G)$ is *ordered by* $<_\pi$. Let $L(e)$ and $R(e)$ denote the end-vertices of each edge $e \in E(G)$ such that $L(e) <_\pi R(e)$. At times, it will be convenient to express π by the list $(v_1, v_2, \dots, v_n)$, where $\pi(v_i) = i$. These notions extend to subsets of vertices in the natural way. Suppose that $V_1, V_2, \dots, V_k$ are disjoint sets of vertices, such that each V_i is ordered by $<_i$. Then $(V_1, V_2, \dots, V_k)$ denotes the vertex ordering π such that $v <_\pi w$ whenever $v \in V_i$ and $w \in V_j$ with $i < j$, or $v \in V_i$, $w \in V_i$, and $v <_i w$. We write $V_1 <_\pi V_2 <_\pi \cdots <_\pi V_k$.

Let π be a vertex ordering of a graph G. Consider two edges $e, f \in E(G)$ with no common end-vertex. There are the following three possibilities for the relative positions of the end-vertices of e and f in π. Without loss of generality $L(e) <_\pi L(f)$.

- e and f *cross*: $L(e) <_\pi L(f) <_\pi R(e) <_\pi R(f)$.
- e and f *nest* and f is *nested inside* e: $L(e) <_\pi L(f) <_\pi R(f) <_\pi R(e)$

- e and f are *disjoint*: $L(e) <_\pi R(e) <_\pi L(f) <_\pi R(f)$

A *stack* (respectively, *queue*) in π is a set of edges $F \subseteq E(G)$ such that no two edges in F are crossing (nested) in π. Observe that when traversing π, edges in a stack (queue) appear in LIFO (FIFO) order—hence the names.

A *linear layout* of a graph G is a pair $(\pi, \{E_1, E_2, \dots, E_k\})$ where π is a vertex ordering of G, and $\{E_1, E_2, \dots, E_k\}$ is a partition of $E(G)$. A *k-stack* (*k-queue*) *layout* of G is a linear layout $(\pi, \{E_1, E_2, \dots, E_k\})$ such that each E_i is a *stack* (*queue*) in π. At times we write $\mathsf{stack}(e) = \ell$ (or $\mathsf{queue}(e) = \ell$) if $e \in E_\ell$.

A graph admitting a k-stack (k-queue) layout is called a *k-stack* (*k-queue*) *graph*. The *stack-number* of a graph G, denoted by $\mathsf{sn}(G)$, is the minimum k such that G is a k-stack graph. The *queue-number* of G, denoted by $\mathsf{qn}(G)$, is the minimum k such that G is a k-queue graph. See [34] for a summary of results and references on stack and queue layouts.

A k-stack layout of a graph G defines a convex drawing of G with thickness k, and vice versa. Thus the stack-number of G equals the book thickness of G.

Lemma 6. *For every graph G, the queue-number of G is at most the convex antithickness of G.*

Proof. Consider a convex drawing of a graph G with convex antithickness k. Let $(v_1, \dots, v_n)$ be the underlying circular ordering and let $E_1, \dots, E_k$ be the corresponding edge-partition. Any two edges in E_i cross or intersect at a common end-vertex with respect to the vertex ordering $(v_1, \dots, v_n)$. Thus each E_i is a queue, and G has queue-number at most k. $\square$

We now set out to prove a converse to Lemma 6. A key tool will be track layouts, which generalise the notion of 2-track drawings, and have been previously studied by several authors [24–26, 29–32, 35, 77–79].

A *vertex $|I|$-colouring* of a graph G is a partition $\{V_i : i \in I\}$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. The elements of I are *colours*, and each set V_i is a *colour class*. Suppose that $<_i$ is a total order on each colour class V_i. Then each pair $(V_i, <_i)$ is a *track*, and $\{(V_i, <_i) : i \in I\}$ is an *$|I|$-track assignment* of G. To ease the notation we denote track assignments by $\{V_i : i \in I\}$ when the ordering on each colour class is implicit.

An *X-crossing* in a track assignment consists of two edges vw and xy such that $v <_i x$ and $y <_j w$, for distinct colours i and j. An *edge k-colouring* of G is simply a partition $\{E_i : 1 \leqslant i \leqslant k\}$ of $E(G)$. A *(k,t)-track layout* of G consists of a t-track assignment of G and an edge k-colouring of G with no monochromatic X-crossing. A graph admitting a (k,t)-track layout is called a *(k,t)-track graph*. The *track-number* of a graph G is the minimum t such that G is a $(1,t)$-track graph.

The next two lemmas give a method that constructs a convex drawing from a track layout.

Lemma 7. *Suppose that K_t has a convex drawing with antithickness p, in which each thrackle is a matching. Then every (k,t)-track graph G has convex antithickness at most kp.*

Proof. In the given convex drawing of K_t, say the vertices are ordered $1, 2, \dots, t$ around a circle, and $\{T_1, T_2, \dots, T_p\}$ is an edge-partition into thrackled matchings. Let $\{(V_i, <_i) : 1 \leqslant i \leqslant t\}$ be the track assignment and $\{E_\ell : 1 \leqslant \ell \leqslant k\}$ be the edge colouring in a (k,t)-track layout of G. Let $\pi = (V_1, V_2, \dots, V_t)$ be a circular vertex ordering of G. For each $\ell \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, p\}$, let $E_{\ell,j} = \{vw \in E_\ell : v \in V_{i_1}, w \in V_{i_2}, i_1 i_2 \in T_j\}$. We now show that each set $E_{\ell,j}$ is a convex thrackle in π, as illustrated in Figure 3.

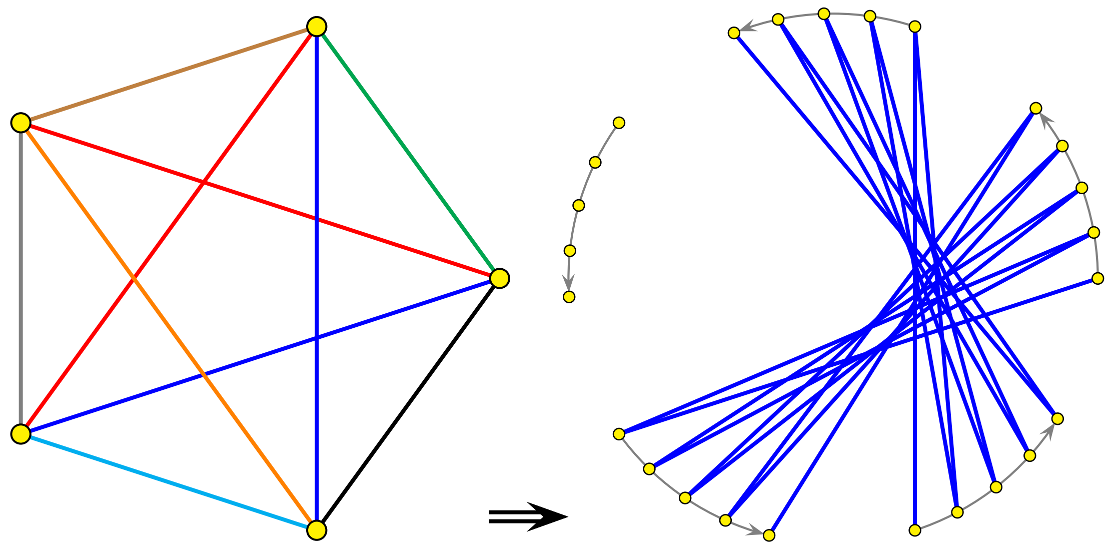

Figure 3: In the proof of Lemma 7, starting from a 5-track graph G, and a convex drawing of K_5 with antithickness 8, in which each thrackle is a matching, replace the vertices of K_5 by the tracks to produce a convex drawing of G with convex antithickness 8.

Consider two edges $e, f \in E_{\ell,j}$ with no common endvertex. If the endvertices of e and f belong to four distinct tracks, then e and f cross in π, since the edges in T_j pairwise cross. The endvertices of e and f do not belong to three distinct tracks, since T_j is a matching. If the endvertices of e and f belong to two distinct tracks, then e and f cross in π, as otherwise e and f form a monochromatic crossing in G. Thus $E_{\ell,j}$ is a convex thrackle in π, and G has convex antithickness at most kp. □

The following results show that (k,t)-track graphs have bounded convex antithickness (for bounded k and t). We start by considering small values of t.

Lemma 8. (a) *Every $(k,3)$-track graph G has convex antithickness at most $3k$.*
(b) *Every $(k,4)$-track graph G has convex antithickness at most $5k$.*
(c) *Every $(k,5)$-track graph G has convex antithickness at most $8k$.*

Proof. By Lemma 7, it is enough to show that K_3, K_4 and K_5 admit convex drawings with their edges partitioned into 3, 5, and 8 thrackled matchings, respectively. The first claim is trivial. For the second claim, position the vertices of K_4 around a circle. Colour the two crossing edges blue. Colour each of the four other edges by a distinct colour. We obtain a convex drawing of K_4 with its edges partitioned into five thrackled matchings. Finally, say $V(K_5) = \{1, 2, 3, 4, 5\}$. Position the vertices of K_5 around a circle in the order $1, 2, 3, 4, 5$. Then $\{13, 24\}, \{25, 14\}, \{35\}, \{12\}, \{23\}, \{34\}, \{45\}, \{15\}$ is a partition of $E(K_5)$ into 8 thrackled matchings. □

Dujmović et al. [32] proved that every outerplanar graph has a 5-track layout. Thus Lemma 8(c) with $k = 1$ implies:

Corollary 9. *Every outerplanar graph G has convex antithickness at most* 8.

Let $\ln n$ be the natural logarithm of n. Let $H(n) := \sum_{i=1}^{n} \frac{1}{i}$ denote the n-th harmonic number. It is well-known that

$$\ln n + \gamma \leqslant H(n) < \ln n + \gamma + \frac{1}{2n}, \tag{1}$$

where $\gamma = 0.577\ldots$ is the Euler–Mascheroni constant; see [21, 52].

The constructions in the proof of Lemma 8 generalise as follows.

Lemma 10. *The complete graph K_n has a convex drawing with antithickness p in which every thrackle is a matching, for some integer $p < n\ln(2n)$. That is, there is an edge p-colouring, such that edges that are disjoint or have a vertex in common receive distinct colours.*

Proof. By the above constructions, we may assume that $n \geqslant 6$. Let $(v_0, v_1, \ldots, v_{n-1})$ be the vertices of K_n in order around a circle. For each $\ell \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and $j \in \{0, 1, \ldots, \lceil n/\ell \rceil - 1\}$, let $E_{\ell,j}$ be the set of edges

$$E_{\ell,j} := \{v_i v_{(i+\ell) \bmod n} : j\ell \leqslant i \leqslant (j+1)\ell - 1\}.$$

As illustrated in Figure 4, $E_{\ell,j}$ is a thrackle and a matching.

Now

$$p \;=\; \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} \lceil n/\ell \rceil \;\leqslant\; \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n+\ell-1}{\ell} \;=\; (n-1)\cdot H(\lfloor \tfrac{n}{2} \rfloor) + \lfloor \tfrac{n}{2} \rfloor.$$

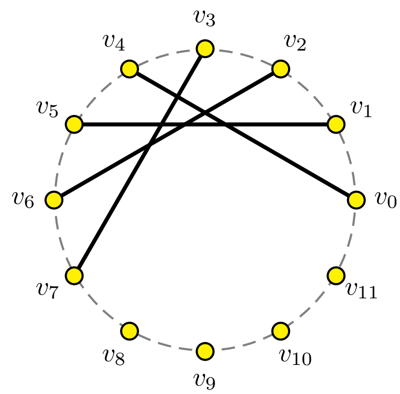

Figure 4: The set of edges $E_{\ell,j}$ in Lemma 10 with $\ell = 4$ and $j = 0$.

By (1),

$$\begin{aligned}
p &< (n-1)\left(\ln\lfloor\tfrac{n}{2}\rfloor + \gamma + \frac{1}{2\lfloor\frac{n}{2}\rfloor}\right) + \lfloor\tfrac{n}{2}\rfloor \\
&\leqslant n\ln(n) - n\ln(2) + \gamma n + \frac{n-1}{2\lfloor\frac{n}{2}\rfloor} + \tfrac{n}{2} \\
&\leqslant n\ln(n) + n\left(-\ln(2) + \gamma + \tfrac{1}{2}\right) + \tfrac{1}{4} \\
&< n\ln(n) + \tfrac{2n}{5} + \tfrac{1}{4} \\
&< n\ln(n) + \tfrac{n}{2} \\
&< n\ln(2n).
\end{aligned}$$

□

Lemmas 7 and 10 imply:

Theorem 11. *Every (k,t)-track graph G has convex antithickness at most $kt\ln(2t)$.*

A similar result was proved by Dujmović et al. [32], who showed that a (k,t)-track graph has geometric thickness at most $k\lceil\frac{t}{2}\rceil\lfloor\frac{t}{2}\rfloor$. It is interesting that track layouts can be used to produce graph drawings with small geometric thickness, and can be used to produce graph drawings with small convex antithickness.

Dujmović et al. [32] proved that every k-queue c-colourable graph has a $(2k,c)$-track layout. Thus Theorem 11 implies:

Corollary 12. *Every k-queue c-colourable graph G has convex antithickness at most $2kc\ln(2c)$.*

Dujmović and Wood [34] proved that every k-queue graph G is $4k$-colourable, and thus has a $(2k,4k)$-track layout. Thus Theorem 11 implies:

Corollary 13. *Every k-queue graph G has convex antithickness at most $8k^2\ln(8k)$.*

A graph is *series parallel* if it has no K_4-minor.

Theorem 14. *Every series parallel graph G has convex antithickness at most* 18.

Proof. It is well known that G is 3-colourable. Rengarajan and Veni Madhavan [92] proved that G has a 3-queue layout; see [33] for an alternative proof. By the above-mentioned result of Dujmović et al. [32], G has a $(6, 3)$-track layout. The result follows from Lemma 8(a). □

Dujmović et al. [32] proved that queue-number and track-number are tied. Thus Lemma 6 and Corollary 13 imply:

Theorem 15. *Queue-number, track-number and convex antithickness are tied.*

Several upper bounds on convex antithickness immediately follow from known upper bounds on queue-number. In particular, Dujmović et al. [30] proved that graphs with bounded treewidth have bounded track-number (see [24, 99] for quantitative improvements to the bounds). Thus Theorem 11 with $k = 1$ implies that graphs with bounded treewidth have bounded convex antithickness. Improving on a breakthrough by Di Battista et al. [23], Dujmović [29] proved that n-vertex planar graphs have $O(\log n)$ queue-number. More generally, Dujmović et al. [31] proved that n-vertex graphs with Euler genus g have $O(g + \log n)$ queue-number. Since such graphs are $O(\sqrt{g})$-colourable, Corollary 12 implies a $O(\sqrt{g}(\log g)(g + \log n))$ upper bound on the convex antithickness. Most generally, for fixed H, Dujmović et al. [31] proved that H-minor-free graphs have $\log^{O(1)} n$ queue-number, which implies a $\log^{O(1)} n$ bound on the convex antithickness (since such graphs are $O(1)$-colourable).

3 Thickness and Antithickness of a Drawing

This section considers the problem of determining the thickness or antithickness of a given drawing of a graph. We employ the following standard terminology. For a graph G, a *clique* of G is a set of pairwise adjacent vertices of G. The *clique number* of G, denoted by $\omega(G)$, is the maximum number of vertices in a clique of G. The *clique-covering number* of G, denoted by $\sigma(G)$ is the minimum number of cliques that partition $V(G)$. An *independent set* of G is a set of pairwise nonadjacent vertices of G. The *independence number* of G, denoted by $\alpha(G)$, is the maximum number of vertices in an independent set of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of independent sets that partition $V(G)$. Obviously $\chi(G) \geqslant \omega(G)$ and $\sigma(G) \geqslant \alpha(G)$ for every graph G. Let $\mathcal{F}$ be a family of graphs. $\mathcal{F}$ is *χ-bounded* if χ is bounded by ω in $\mathcal{F}$, and $\mathcal{F}$ is *σ-bounded* if σ is bounded by α in $\mathcal{F}$.

Now let D be a drawing of a graph G. Let k be the maximum number of pairwise crossing edges in D, and let ℓ be the maximum number of pairwise disjoint edges in D. Then k is a lower bound on the thickness of D, and ℓ is a lower bound on the antithickness of D. Our interest is when the thickness of D is bounded from above by a function of k, or the antithickness of D is bounded from above by a function of ℓ.

Let H be the graph with $V(H) = E(G)$ such that two vertices of H are adjacent if and only if the corresponding edges cross in D. Let H^+ be the graph with $V(H) = E(G)$

such that two vertices of H^+ are adjacent if and only if the corresponding edges cross in D or have an endvertex in common. Note that H is a spanning subgraph of H^+. By definition, the thickness of D equals $\chi(H)$, and the antithickness of D equals $\sigma(H^+)$.

A *string graph* is the intersection graph of a family of simple curves in the plane; see [45–47, 64, 75, 94] for example. If we consider edges in D as curves, then H^+ is a string graph. And deleting a small disc around each vertex in D, we see that H is also a string graph. Moreover, if D is geometric, then both H and H^+ are intersection graphs of sets of segments in the plane. If D is convex, then both H and H^+ are intersection graphs of sets of chords of a circle, which is called a *circle graph*; see [56, 66] for example. If D is a 2-track drawing, then both H and H^+ are permutation graphs, which are perfect; see [51] for example.

Whether the thickness / antithickness of an (unrestricted) drawing is bounded by the maximum number of pairwise crossing / disjoint edges is equivalent to whether string graphs are χ-bounded / ω-bounded. Whether the thickness / antithickness of a geometric drawing is bounded by the maximum number of pairwise crossing / disjoint edges is equivalent to whether intersection graphs of segments are χ-bounded / ω-bounded. For many years both these were open; see [69, 70]. However, in a recent breakthrough, Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [87, 88] constructed set of segments in the plane, whose intersection graph is triangle-free and with unbounded chromatic number. Thus the thickness of a drawing is not bounded by any function of the maximum number of pairwise crossing edges, and this remains true in the geometric setting.

For convex drawings, more positive results are known. Gyárfás [54, 55] proved that the family of circle graphs is χ-bounded, the best known bound being $\chi(H) < 21 \cdot 2^{\omega(H)}$ due to Černý [20] (slightly improving an earlier bound by Kostochka and Kratochvíl [67]). This implies that a convex drawing with at most k pairwise crossing edges has thickness less than $21 \cdot 2^k$. For small values of k much better bounds are known [1, 2]. Kostochka [68] proved that $\sigma(H) \leqslant (1 + o(1))\alpha(H) \log \alpha(H)$ for every circle graph H; also see [67]. Thus a convex drawing with at most k pairwise disjoint edges has antithickness at most $(1 + o(1))k \log k$.

Now consider a 2-track drawing D. Then H and H^+ are permutation graphs, which are perfect. Thus $\chi(H) = \omega(H)$ and $\sigma(H^+) = \alpha(H^+)$. This says that if D has at most k pairwise crossing edges, and at most ℓ pairwise disjoint edges, then D has thickness at most k and antithickness at most ℓ. There is a very simple algorithm for computing these partitions. First we compute the partition into ℓ 2-track thrackles. For each edge vw, if $\{x_1y_1, x_2y_2, \dots, x_iy_i\}$ is a set of maximum size of pairwise disjoint edges such that $x_1 < x_2 < \cdots < x_i < v$ in one layer and $y_1 < y_2 < \cdots < y_i < w$ in the other layer, then assign vw to the $(i + 1)$-th set. Consider two disjoint edges v_1w_1 and v_2w_2. Without loss of generality, $v_1 < v_2$ and $w_1 < w_2$. Suppose that by the above rule v_1w_1 is assigned to the $(i + 1)$-th set and v_2w_2 is assigned to the $(j + 1)$-th set. Let $\{x_1y_1, \dots, x_iy_i\}$ be a set of maximum size of pairwise disjoint edges such that $x_1 < x_2 < \cdots < x_i < v_1$ in one layer and $y_1 < y_2 < \cdots < y_i < w_1$ in the other layer. Then $\{x_1y_1, \dots, x_iy_i, v_1w_1\}$ is a set of pairwise disjoint edges such that $x_1 < x_2 < \cdots < x_i < v_1 < v_2$ in one layer and $y_1 < y_2 < \cdots < y_i < w_1 < w_2$ in the other layer. Thus $j \geqslant i + 1$. That is, two edges that are both assigned to the same set are not disjoint, and each such set is a 2-track thrackle. In the above rule, $i \leqslant \ell - 1$. Thus this procedure partitions the edges into ℓ 2-track thrackles. To partition the edges into k 2-track noncrossing subdrawings, simply reverse the order of the vertices in one track, and apply the above procedure.

Consider the analogous question for queue layouts: Given a fixed vertex ordering π of a graph G, determine the minimum value k such that π admits a k-queue layout of G. We can again construct an auxillary graph H with $V(H) = E(G)$, where two vertices are adjacent if and only if the corresponding edges of G are nested in π. Then π admits a k-queue layout of G if and only if $\chi(H) \leqslant k$. A classical result by Dushnik and Miller [37] implies that H is a permutation graph, and is thus perfect. Hence $\chi(H) = \omega(H)$. A clique in H corresponds to a set of edges of G that are pairwise nested in π, called a *rainbow*. Hence π admits a k-queue layout of G if and only if π has no $(k+1)$-edge rainbow, which was also proved by Heath and Rosenberg [60]. Dujmović and Wood [34] observed the following simple way to assign edges to queues: if the maximum number of edges that are pairwise nested inside an edge e is i, then assign e to the $(i+1)$-th queue.

This procedure can also be used to prove that a convex drawing with at most k pairwise disjoint edges has antithickness at most $(1 + o(1))k \log k$. This is equivalent to the result of Kostochka [68] mentioned above. Let π be any vertex ordering obtained from the order of the vertices around the convex hull. Then π has no $(k+1)$-edge rainbow. Assign edges to k queues as described at the end of the previous paragraph. Partition the i-th queue into sets of pairwise non-disjoint edges as follows. For each edge e in the i-th queue, if the maximum number of pairwise disjoint edges with both end-vertices to the left of the left end-vertex of e is j, then assign e to the $(j+1)$-th set. Thus two edges in the i-th queue that are both assigned to the same set are not disjoint. Let S be a maximum set of pairwise disjoint edges in the i-th queue. Then $j \leqslant |S| - 1$. Thus the i-th queue can be partitioned into $|S|$ sets of pairwise non-disjoint edges. Now we bound $|S|$. Under each edge in S is an $(i-1)$-edge rainbow. This gives a set of $|S| \cdot i$ edges that are pairwise disjoint. Thus $|S| \cdot i \leqslant k$ and $|S| \leqslant \lfloor k/i \rfloor$. Thus we can partition the i-th queue into at most $\lfloor k/i \rfloor$ sets of pairwise non-disjoint edges. In total we have at most $\sum_{i=1}^{k} \lfloor k/i \rfloor$ sets, each with no two disjoint edges, which is less than $k(1 + \ln k)$. Loosely speaking, this proof shows that a convex drawing and an associated edge-partition into convex thrackles can be thought of as a combination of a queue layout and an arch layout; see [34] for the definition of an arch layout.

4 Thickness and Antithickness are Tied

The *arboricity* of a graph G is the minimum number of forests that partition $E(G)$. Nash-Williams [81] proved that the arboricity of G equals

$$\max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil . \tag{2}$$

We have the following connection between thickness, antithickness, and arboricity.

Theorem 16. *Thickness, antithickness, and arboricity are pairwise tied. In particular, for* *every graph G with thickness t, antithickness k, and arboricity ℓ,*

$$k \leqslant \ell \text{ and } \frac{k}{3} \leqslant t \leqslant \ell \leqslant \left\lceil \frac{3k}{2} \right\rceil .$$

Proof. Every forest is planar. Thus a partition of G into ℓ forests is also a partition of G into ℓ planar subgraphs. Thus $t \leqslant \ell$.

Woodall [100] proved that every forest is thrackeable. Thus a partition of G into ℓ forests is also a partition of G into ℓ thrackeable subgraphs. Thus $k \leqslant \ell$.

Every planar graph G has arboricity at most 3 by (2) and since $|E(G)| \leqslant 3|V(G)|-6$. (Indeed, much more is known about edge-partitions of planar graphs into three forests [95].) Since every forest is thrackeable [100], every planar graph has antithickness at most 3. Thus a partition of G into t planar subgraphs gives a partition of G into $3t$ thrackeable subgraphs. Thus $k \leqslant 3t$.

It remains to prove that $\ell \leqslant \lceil \frac{3}{2}k \rceil$. By (2), it suffices to show that $\frac{m}{n-1} \leqslant \frac{3k}{2}$ for every subgraph H of G with n vertices and m edges. Cairns and Nikolayevsky [15] proved that every thrackle has at most $\frac{3}{2}(n-1)$ edges. Since every subgraph of a thrackle is a thrackle, H has antithickness at most k, and $m \leqslant \frac{3}{2}k(n-1)$, as desired. □

Theorem 16 with $k = 1$ implies that every thrackle has arboricity at most 2. It is an open problem whether every thrackle is planar. It follows from a result by Cairns and Nikolayevsky [16] that every thrackle has a crossing-free embedding in the projective plane. Also note that the constant $\frac{3}{2}$ in Theorem 16 can be improved to $\frac{167}{117}$ using the result of Fulek and Pach [48].

5 Separating Convex Antithickness and Geometric Thickness

As discussed in Section 1.3, the following lemma is a key step in showing that convex antithickness and geometric antithickness are separated. Recall that K'_n is the graph obtained from K_n by subdividing each edge exactly once.

Lemma 17. *K'_n has geometric antithickness* 2.

Proof. Let $v_1, \dots, v_n$ be the original vertices of K'_n. Position each v_i at $(2i, 0)$. For $1 \leqslant i < j \leqslant n$, let $x_{i,j}$ be the division vertex of the edge v_iv_j; colour the edge $v_ix_{i,j}$ *blue*, and colour the edge $v_jx_{i,j}$ *red*. Orient each edge of K'_n from the original endvertex to the division endvertex. This orientation enables us to speak of the order of crossings along an edge.

We now construct a geometric drawing of K'_n, such that every pair of blue edges cross, and every pair of red edges cross. Thus the drawing has antithickness 2. In addition, the following invariants are maintained for all $i \in [1, n-2]$ and $j \in [i+2, n]$:

(1) No blue edge crosses $v_ix_{i,i+1}$ after the crossing between $v_ix_{i,i+1}$ and $v_jx_{j-1,j}$.

(2) No red edge crosses $v_jx_{j-1,j}$ after the crossing between $v_ix_{i,i+1}$ and $v_jx_{j-1,j}$.

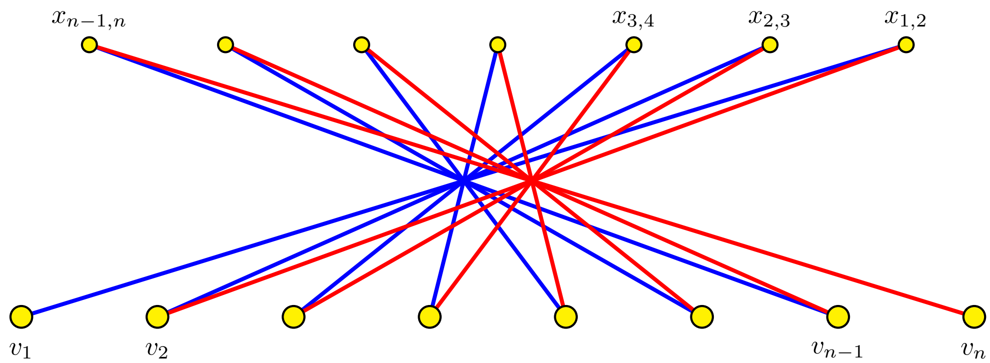

Figure 5: Initial vertex placement in a geometric drawing of K'_n with antithickness 2.

The drawing is constructed in two stages. First, for $i \in [n-1]$, position $x_{i,i+1}$ at $(2(n-i)+1, 1)$, as illustrated in Figure 5.

Observe that all the blue segments intersect at $(n+\frac{1}{2}, \frac{1}{2})$, and all the red segments intersect at $(n+\frac{3}{2}, \frac{1}{2})$. Thus the invariants hold in this subdrawing. Moreover, every blue edge intersects every red edge (although this property will not be maintained).

For $i \in [1, n-2]$ and $j \in [i+2, n]$ (in an arbitrary order) position $x_{i,j}$ as follows. The blue segment $v_i x_{i,i+1}$ and the red segment $v_j x_{j-1,j}$ were drawn in the first stage, and thus cross at some point c. In the arrangement formed by the drawing produced so far, let F be the face that contains c, such that the blue segment $v_i x_{i,i+1}$ is on the left of F, and the red segment $v_j x_{j-1,j}$ is on the right of F. Position $x_{i,j}$ in the interior of F, as illustrated in Figure 6.

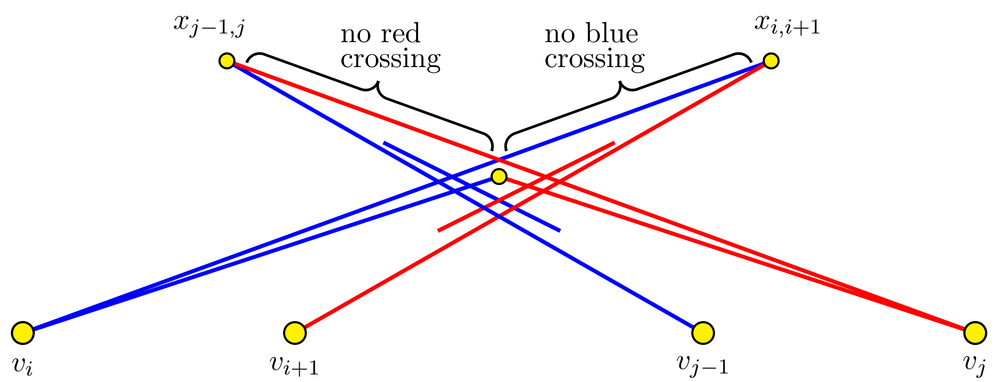

Figure 6: Placing $x_{i,j}$ where $i \in [1, n-2]$ and $j \in [i+2, n]$.

By invariant (1), no blue edge crosses $v_i x_{i,i+1}$ after the red edge $v_j x_{j-1,j}$. It follows that the new blue edge $v_i x_{i,j}$ crosses every blue edge already drawn, and invariant (1) is maintained. By invariant (2), no red edge crosses $v_j x_{j-1,j}$ after the blue edge $v_i x_{i,i+1}$. It follows that the new red edge $v_j x_{i,j}$ crosses every red edge already drawn, and invariant (2) is maintained. □

Dujmović and Wood [35, Lemma 10] proved that K_n' has queue-number at least $\sqrt{n/6}$. Since the queue-number of a graph is at most its convex antithickness (Lemma 6), K_n' has convex antithickness at least $\sqrt{n/6}$. This proves the claim in Section 1 that implies that convex antithickness is not bounded by geometric antithickness.

6 Extremal Questions

This section studies the maximum number of edges in an n-vertex graph with topological (or geometric or convex or 2-track) thickness (or antithickness) k. The results are summarised in Table 1. First we describe results from the literature, followed by our original results.

For book thickness and 2-track thickness the maximum number of edges is known. Bernhart and Kainen [6] proved that the maximum number of edges in an n-vertex graph with book thickness k equals $(k+1)n - 3k$. Dujmović and Wood [34] proved that the maximum number of edges in an n-vertex graph with 2-track thickness k equals $k(n-k)$.

Determining the maximum number of edges in a thrackle is a famous open problem proposed by John Conway, who conjectured that every thrackle on n vertices has at most n edges. Improving upon previous bounds by Lovász et al. [73] and Cairns and Nikolayevsky [15], Fulek and Pach [48] proved that every thrackle has at most $\frac{167}{117}n$ edges. Thus every graph with antithickness k has at most $\frac{167}{117}kn$ edges. For $n \geqslant 2k+1$, it is easy to construct an n-vertex graph consisting of k edge-disjoint copies of C_n. Thus this graph has antithickness k and kn edges.

Many authors have proved that every geometric thrackle has at most n edges [40, 62, 83, 100]. Thus every graph with geometric antithickness k has at most kn edges. For convex antithickness, Fabila-Monroy and Wood [42] improved this upper bound to $kn - \binom{k}{2}$, and Fabila-Monroy, Jonsson, Valtr, and Wood [41] established a matching lower bound. This lower bound is the best known lower bound in the geometric setting. It is an open problem to determine the maximum number of edges in an n-vertex graph with geometric antithickness k.

We also mention that many authors have considered graph drawings, with at most k pairwise crossing edges or at most k pairwise disjoint edges (instead of thickness k or antithickness k). These weaker assumptions allow for more edges. See [3, 18, 19, 44, 49, 49, 71, 72, 96, 97].

6.1 Thickness

Since every planar graph with $n \geqslant 3$ vertices has at most $3(n-2)$ edges, every graph with $n \geqslant 3$ vertices and thickness k has at most $3k(n-2)$ edges. We now prove a lower bound.

Theorem 18. *For all k and infinitely many n there is a graph with n vertices, thickness k, and exactly $3k(n-2)$ edges.*

Let G be a graph. Let f be a bijection of $V(G)$. Let G^f be the graph with vertex set $V(G^f) = V(G)$ and edge set $E(G^f) = \{f(vw) : vw \in E(G)\}$, where $f(vw)$ is an

Table 1: The maximum number of edges in an n-vertex graph with given parameter k.

parameter	lower bound	upper bound	reference
thickness k	$3k(n-2)$	$3k(n-2)$	Theorem 18
geometric thickness 1	$3n-6$	$3n-6$	
geometric thickness 2	$6n-20$	$6n-18$	[63]
geometric thickness k	$k(3n-4k-3)$	$k(3n-k-5)$	Theorem 20
book thickness k	$(k+1)n-3k$	$(k+1)n-3k$	[6]
2-track thickness k	$k(n-k)$	$k(n-k)$	[34]
antithickness 1	n	$\frac{167}{117}n$	[48]
antithickness k	kn	$\frac{167}{117}kn$	
geometric antithickness 1	n	n	[40, 62, 83, 100]
geometric antithickness k	$kn-\binom{k}{2}$	kn	
convex antithickness k	$kn-\binom{k}{2}$	$kn-\binom{k}{2}$	[41, 42]
2-track antithickness k	$k(n-k)$	$k(n-k)$	[34]

abbreviation for the edge $f(v)f(w)$. Bijections f_1 and f_2 of $V(G)$ are *compatible* if G^{f_1} and G^{f_2} are edge-disjoint. By taking a union, the next lemma implies Theorem 18.

Lemma 19. *For each integer $k \geqslant 1$ there are infinitely many edge-maximal planar graphs that admits k pairwise compatible bijections.*

Proof. Let n be a prime number greater than $3k^2$. Let G be the graph with vertex set

$$V(G) = \{u\langle i\rangle, v\langle i\rangle, w\langle i\rangle : i \in [0, n-1]\}$$

and edge set $E(G) = A \cup B \cup C$, where

$$\begin{aligned} A &= \{u\langle i\rangle u\langle i+1\rangle, v\langle i\rangle v\langle i+1\rangle, w\langle i\rangle w\langle i+1\rangle : i \in [0, n-2]\}, \\ B &= \{u\langle i\rangle v\langle i\rangle, v\langle i\rangle w\langle i\rangle, w\langle i\rangle u\langle i\rangle : i \in [0, n-1]\}, \\ C &= \{u\langle i\rangle v\langle i+1\rangle, v\langle i\rangle w\langle i+1\rangle, w\langle i\rangle u\langle i+1\rangle : i \in [0, n-2]\}. \end{aligned}$$

Then G is edge-maximal planar, as illustrated in Figure 7.

For each $p \in [1, k]$, let $f_p : V(G) \to V(G)$ be the function defined by

$$\begin{aligned} f_p(u\langle i\rangle) &:= u\langle pi\rangle \\ f_p(v\langle i\rangle) &:= v\langle pi + p(k+1)\rangle \\ f_p(w\langle i\rangle) &:= w\langle pi + 2p(k+1)\rangle, \end{aligned}$$

where vertex indices are always in the cyclic group $\mathbb{Z}_n$. Thus f_p is a bijection.

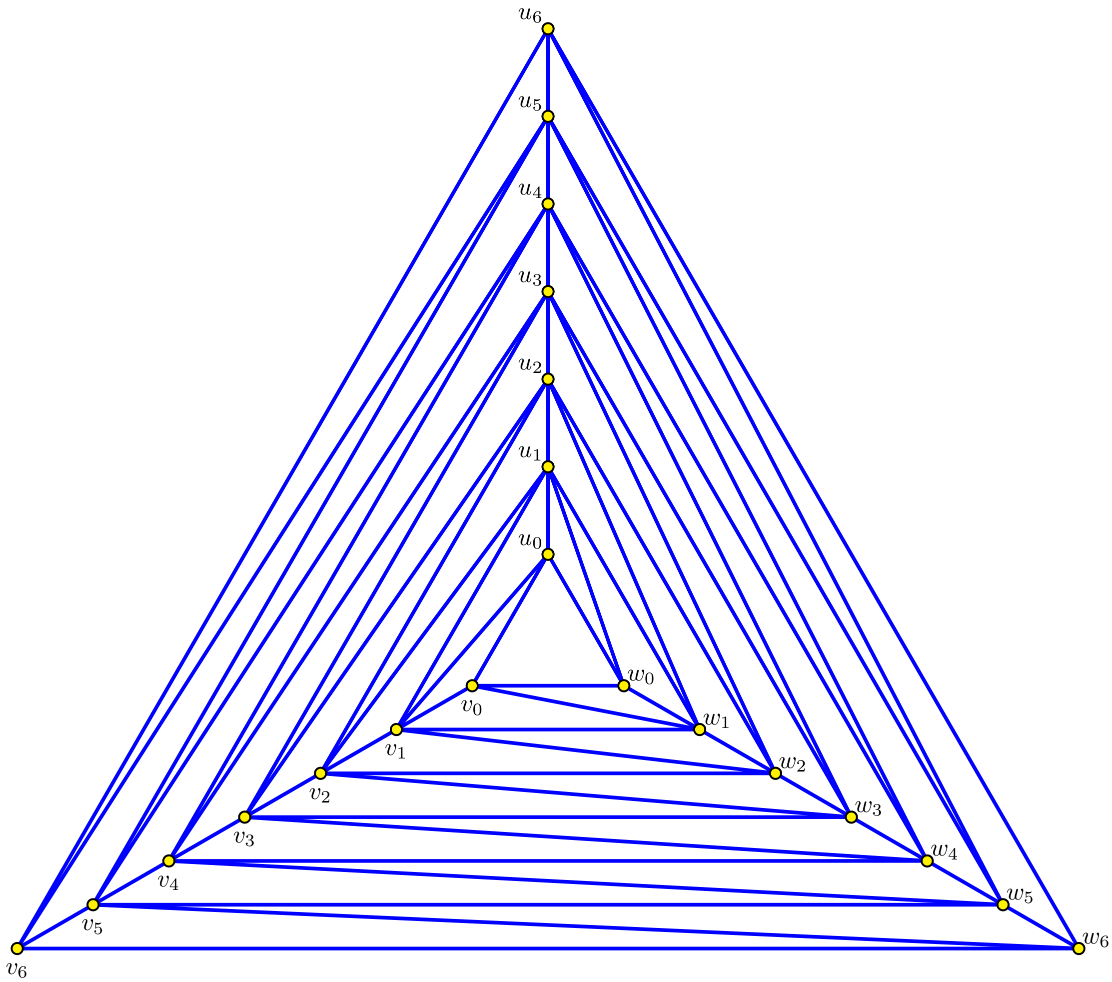

Figure 7: The nested triangles graph.

Suppose to the contrary that G^{f_p} and G^{f_q} have an edge e in common, for some distinct $p, q \in [1, k]$. Since $\{u\langle 0\rangle, \ldots, u\langle n-1\rangle\}$ is mapped to $\{u\langle 0\rangle, \ldots, u\langle n-1\rangle\}$ by both f_p and f_q, and similarly for the $v\langle i\rangle$ and $w\langle i\rangle$, the following cases suffice. All congruences are modulo n.

Case 1. e is from A in both G^{f_p} and G^{f_q}:

Case 1a. $e = f_p(u\langle i\rangle u\langle i+1\rangle) = f_q(u\langle j\rangle u\langle j+1\rangle)$ for some i, j: Thus $u\langle pi\rangle u\langle p(i+1)\rangle = u\langle qj\rangle u\langle q(j+1)\rangle$. Then $pi \equiv qj$ and $pi + p \equiv qj + q$ (implying $p \equiv q$), or $pi \equiv qj + q$ and $pi + p \equiv qj$ (implying $pi - qj \equiv q \equiv -p$), which is a contradiction since $n > 2k \geqslant p + q$.

Case 1b. $e = f_p(v\langle i\rangle v\langle i+1\rangle) = f_q(v\langle j\rangle v\langle j+1\rangle)$ for some i, j: Thus $v\langle pi + p(k+1)\rangle v\langle p(i+1) + p(k+1)\rangle = v\langle qj + q(k+1)\rangle v\langle q(j+1) + q(k+1)\rangle$. If $pi + p(k+1) \equiv qj + q(k+1)$ and $p(i+1) + p(k+1) \equiv q(j+1) + q(k+1)$, then $p \equiv q$, which is a contradiction since $n > k \geqslant p, q$. Otherwise $pi + p(k+1) \equiv q(j+1) + q(k+1)$ and $p(i+1) + p(k+1) \equiv qj + q(k+1)$, implying $p + q \equiv 0$, which is a contradiction since

$n > 2k \geqslant p + q$.

Case 1c. $e = f_p(w\langle i\rangle w\langle i+1\rangle) = f_q(w\langle j\rangle w\langle j+1\rangle)$ for some i, j: Thus $w\langle pi + 2p(k+1)\rangle w\langle p(i+1) + 2p(k+1)\rangle = w\langle qj + 2q(k+1)\rangle w\langle q(j+1) + 2q(k+1)\rangle$. If $pi + 2p(k+1) \equiv qj + 2q(k+1)$ and $p(i+1) + 2p(k+1) \equiv q(j+1) + 2q(k+1)$, then $p \equiv q$, which is a contradiction since $n > k \geqslant p, q$. Otherwise $pi + 2p(k+1) \equiv q(j+1) + 2q(k+1)$ and $p(i+1) + 2p(k+1) \equiv qj + 2q(k+1)$, implying $p + q \equiv 0$, which is a contradiction since $n > 2k \geqslant p + q$.

Case 2. e is from B in both G^{f_p} and G^{f_q}:

Case 2a. $e = f_p(u\langle i\rangle v\langle i\rangle) = f_q(u\langle j\rangle v\langle j\rangle)$ for some i, j. Thus $u\langle pi\rangle v\langle pi + p(k+1)\rangle = u\langle qj\rangle v\langle qj + q(k+1)\rangle$. Then $pi \equiv qj$ and $pi + p(k+1) \equiv qj + q(k+1)$. Hence $p(k+1) \equiv q(k+1)$ and $p \equiv q$ since n is prime, which is a contradiction since $n > k \geqslant p, q$.

Case 2b. $e = f_p(v\langle i\rangle w\langle i\rangle) = f_q(v\langle j\rangle w\langle j\rangle)$ for some i, j. Thus $v\langle pi + p(k+1)\rangle w\langle pi + 2p(k+1)\rangle = v\langle qj + q(k+1)\rangle w\langle qj + 2q(k+1)\rangle$. Then $pi+p(k+1) \equiv qj + q(k+1)$ and $pi + 2p(k+1) \equiv qj + 2q(k+1)$. Hence $p(k+1) \equiv q(k+1)$, implying $p \equiv q$ since n is prime, which is a contradiction since $n > k \geqslant p, q$.

Case 2c. $e = f_p(w\langle i\rangle u\langle i\rangle) = f_q(w\langle j\rangle u\langle j\rangle)$ for some i, j. Thus $w\langle pi + 2p(k+1)\rangle u\langle pi\rangle = w\langle qj + 2q(k+1)\rangle u\langle qj\rangle$. Then $pi + 2p(k+1) \equiv qj + 2q(k+1)$ and $pi \equiv qj$. Hence $2p(k+1) \equiv 2q(k+1)$, implying $p \equiv q$ since n is prime, which is a contradiction since $n > k \geqslant p, q$.

Case 3. e is from C in both G^{f_p} and G^{f_q}:

Case 3a. $e = f_p(u\langle i\rangle v\langle i+1\rangle) = f_q(u\langle j\rangle v\langle j+1\rangle)$ for some i, j. Thus $u\langle pi\rangle v\langle p(i+1) + p(k+1)\rangle = u\langle qj\rangle v\langle q(j+1) + q(k+1)\rangle$. Thus $pi \equiv qj$ and $p(i+1) + p(k+1) \equiv q(j+1) + q(k+1)$. Hence $p(k+2) \equiv q(k+2)$, implying $p \equiv q$ since n is prime, which is a contradiction since $n > k \geqslant p, q$.

Case 3b. $e = f_p(v\langle i\rangle w\langle i+1\rangle) = f_q(v\langle j\rangle w\langle j+1\rangle)$ for some i, j. Thus $v\langle pi + p(k+1)\rangle w\langle p(i+1) + 2p(k+1)\rangle = v\langle qj + q(k+1)\rangle w\langle q(j+1) + 2q(k+1)\rangle$. Thus $pi + p(k+1) \equiv qj + q(k+1)$ and $p(i+1) + 2p(k+1) \equiv q(j+1) + 2q(k+1)$. Hence $p(k+2) \equiv q(k+2)$, implying $p \equiv q$ since n is prime, which is a contradiction since $n > k \geqslant p, q$.

Case 3c. $e = f_p(w\langle i\rangle u\langle i+1\rangle) = f_q(w\langle j\rangle u\langle j+1\rangle)$ for some i, j. Thus $w\langle pi + 2p(k+1)\rangle u\langle p(i+1)\rangle = w\langle qj + 2q(k+1)\rangle u\langle q(j+1)\rangle$. Thus $pi + 2p(k+1) \equiv qj + 2q(k+1)$ and $p(i+1) \equiv q(j+1)$. Hence $p(2k+1) \equiv q(2k+1)$, implying $p \equiv q$ since n is prime, which is a contradiction since $n > k \geqslant p, q$.

Case 4. e is from B in G^{f_p} and from C in G^{f_q}:

Case 4a. $e = f_p(u\langle i\rangle v\langle i\rangle) = f_q(u\langle j\rangle v\langle j+1\rangle)$ for some i, j. Thus $u\langle pi\rangle v\langle pi + p(k+1)\rangle = u\langle qj\rangle v\langle q(j+1) + q(k+1)\rangle$. Then $pi \equiv qj$ and $pi + p(k+1) \equiv q(j+1) + q(k+1)$, implying $(p-q)(k+1) \equiv q$. If $p > q$ then $(p-q)(k+1) \in [k+1, k^2-1]$ is not congruent to $q \in [1, k]$ since $n > k^2$. Otherwise $p < q$, implying $(p-q)(k+1) \in [-(k+1), -(k^2-1)]$ is not congruent to $q \in [1, k]$ since $n > 2k^2$.

Case 4b. $e = f_p(v\langle i\rangle w\langle i\rangle) = f_q(v\langle j\rangle w\langle j+1\rangle)$ for some i, j. Thus $v\langle pi + p(k+1)\rangle w\langle pi + 2p(k+1)\rangle = v\langle qj + q(k+1)\rangle w\langle q(j+1) + 2q(k+1)\rangle$. Then $pi +$

$p(k+1) \equiv qj + q(k+1)$ and $pi + 2p(k+1) \equiv q(j+1) + 2q(k+1)$. Hence $(p-q)(k+1) \equiv q$. As in Case 3a, this is a contradiction.

Case 4c. $e = f_p(w\langle i\rangle u\langle i\rangle) = f_q(w\langle j\rangle u\langle j+1\rangle)$ for some i, j. Thus $w\langle pi + 2p(k+1)\rangle u\langle pi\rangle = w\langle qj + 2q(k+1)\rangle u\langle q(j+1)\rangle$. Then $pi+2p(k+1) \equiv qj+2q(k+1)$ and $pi \equiv q(j+1)$. Hence $pi - qj \equiv 2(q-p)(k+1) \equiv q$. If $q > p$ then $2(q-p)(k+1) \in [2k+2, 2k^2-2]$ is not congruent to $q \in [1,k]$ since $n > 3k^2$. Otherwise $q < p$, implying $2(q-p)(k+1) \in [-2k^2+2, -2k-2]$ is not congruent to $q \in [1,k]$ since $n > 3k^2$.

Therefore $f_1, \dots, f_k$ are pairwise compatible bijections of G. □

6.2 Geometric Thickness

Every graph with $n \geqslant 3$ vertices and geometric thickness k has at most $3k(n-2)$ edges. Of course, this bound is tight for $k = 1$. But for $k = 2$, Hutchinson et al. [63] improved this upper bound to $6n - 18$, and constructed a graph with geometric thickness 2 and $6n - 20$ edges. We have the following lower and upper bounds for general k. The proof is inspired by the proofs of lower and upper bounds on the geometric thickness of complete graphs due to Dillencourt, Eppstein, and Hirschberg [27].

Theorem 20. *For $k \geqslant 1$ and $n \geqslant \max\{2k, 3\}$, every graph with n vertices and geometric thickness k has at most $k(3n-k-5)$ edges. Conversely, for all such $n \equiv 0 \pmod{2k}$, there is an n-vertex graph with geometric thickness k and exactly $k(3n-4k-3)$ edges.*

Proof of Upper Bound. Let $T_1, \dots, T_k$ be triangulations of $V(G)$ such that $E(G)$ is contained in $T_1 \cup \cdots \cup T_k$. Assume that no two vertices have the same x-coordinate. Let A be the set of the k leftmost vertices. Let B be the set of the k rightmost vertices. Since $n \geqslant 2k$, we have $A \cap B = \emptyset$. For distinct vertices $v, w \in A$, the line segment vw crosses a number of triangular faces in T_i. The left sides of these faces form a vw-path in $T_i[A]$. Thus $T_i[A]$ is connected. Similarly $T_i[B]$ is connected. Thus $T_i[A]$ and $T_i[B]$ both have at least $k-1$ edges. Hence T_i contains at most $3n - 6 - 2(k-1) = 3n - 2k - 4$ edges with at most one end-vertex in A and at most one end-vertex in B. Thus $|E(G)| \leqslant |E(T_1 \cup \cdots \cup T_k)| \leqslant 2\binom{k}{2} + k(3n-2k-4) = k(3n-k-5)$. □

Proof of Lower Bound. Fix a positive integer s. We construct a geometric graph G with $n = 2sk$ vertices and geometric thickness k. The vertices are partitioned into levels $V_1, \dots, V_s$ each with $2k$ vertices, where $V_a := \{(a,i) : i \in [1,2k]\}$ for $a \in [1,s]$. The vertices in each level V_a are evenly spaced on a circle C_a of radius r_a centred at the origin, where $1 = r_1 < \cdots < r_s$ are specified below. The vertices in V_a are ordered $(a,1), \dots, (a,2k)$ clockwise around C_a. Thus (a,j) is opposite $(a,k+j)$, where the second coordinate is always modulo $2k$. All congruences below are modulo $2k$.

The first level V_1 induces a complete graph. For distinct $i, j \in [1,2k]$, the edge $(1,i)(1,j)$ is coloured by the $\ell \in [1,k]$ such that $i + j \equiv 2\ell$ or $i + j \equiv 2\ell - 1$. The edges coloured ℓ form a non-crossing path with end-vertices $(1,\ell)$ and $(1,k+\ell)$, as illustrated in Figure 8. Note that $(1, \ell + \lfloor \frac{k}{2} \rfloor)(1, \ell + \lfloor \frac{3k}{2} \rfloor)$ is the 'long' edge in this path. This is a well-known construction of a k-page book embedding of K_{2k}; see [10] for example. This contributes $\binom{2k}{2}$ edges to G.

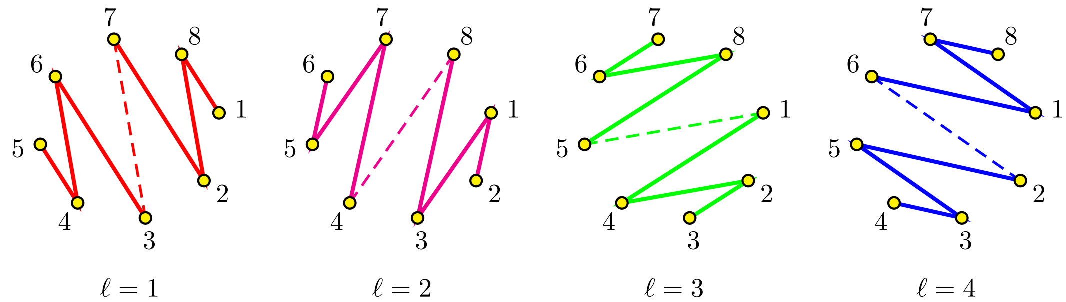

Figure 8: The edges between vertices in V_a with $k = 4$. The dashed edges are included only for $a = 1$.

Every other level V_a (where $a \in [2, s]$) induces a complete graph minus a perfect matching. We use a partition into non-crossing paths, analogous to that used in the $a = 1$ case, except that the 'long' edge in each path is not included. More precisely, for distinct $i, j \in [1, 2k]$ with $i \not\equiv k + j$, the edge $(a, i)(a, j)$ is coloured by the $\ell \in [1, k]$ such that $i + j \equiv 2\ell$ or $i + j \equiv 2\ell - 1$. The edges coloured ℓ form two non-crossing paths, one with end-vertices (a, ℓ) and $(a, \ell + \lfloor \frac{3k}{2} \rfloor)$, the other with end-vertices $(a, \ell + \lfloor \frac{k}{2} \rfloor)$ and $(a, k + \ell)$, as illustrated in Figure 8. This contributes $(s - 1)(\binom{2k}{2} - k)$ edges to G.

We now define the edges between the layers, as illustrated in Figure 9. For $a \in [2, s]$ and $i \in [1, k]$ and $j \in [1, 2k]$, the edges $(a, i + \lfloor \frac{k}{2} \rfloor)(a - 1, j)$ and $(a, i + \lfloor \frac{3k}{2} \rfloor)(a - 1, j)$ are present and are coloured i. This contributes $2(s-1)2k^2$ edges to G. Finally, for $a \in [2, s-1]$ and $i \in [1, k]$, the edges $(a+1, i+\lfloor \frac{k}{2} \rfloor)(a-1, k+i)$ and $(a+1, i+\lfloor \frac{3k}{2} \rfloor)(a-1, i)$ are present and are coloured i. This contributes $2(s - 2)k$ edges to G.

As illustrated in Figure 9, given a drawing of the first a layers (which are defined by $r_1, \dots, r_a$) there is a sufficiently large value of r_{a+1} such that the addition of the $(a + 1)$-th layer does not create any crossings between edges with the same colour.

In total, G contains $\binom{2k}{2} + (s - 1)(\binom{2k}{2} - k) + (s - 1)(\binom{2k}{2} - k) + 2(s - 2)k$ edges, which equals $6ks - 4k^2 - 3k = k(3n - 4k - 3)$. □

Examples of the construction in Theorem 20 are given in Figures 10 and 11.

7 Antithickness of Complete Graphs

Let $\mathsf{cat}(G)$ be the convex antithickness of a graph G. We now consider $\mathsf{cat}(K_n)$. Araujo, Dumitrescu, Hurtado, Noy, and Urrutia [4] proved[5] that

$$2 \left\lfloor \frac{n+1}{3} \right\rfloor - 1 \leqslant \mathsf{cat}(K_n) < n - \frac{1}{2} \lfloor \log n \rfloor . \quad (3)$$

[5]Araujo et al. [4] did not use the terminology of 'antithickness', but it is easily seen that their definition of $d_c(n)$ equals $\mathsf{cat}(K_n)$.

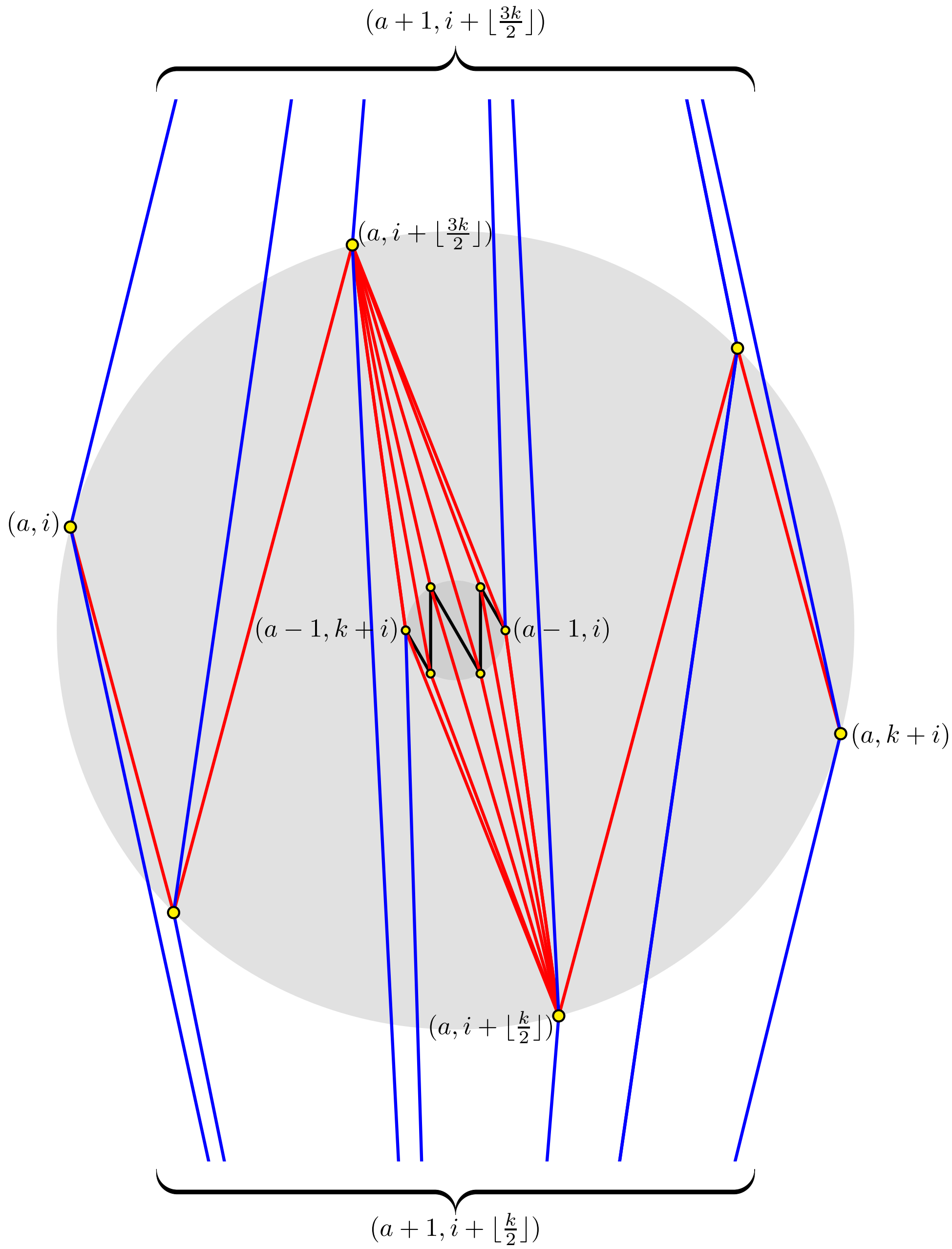

Figure 9: Edges coloured i between layers.

In an early version of this paper (cited in [41, 42]), we improved both the lower and upper bound to the following (see [36] for the proof):

$$\frac{3n-6}{4} \leqslant \mathsf{cat}(K_n) < n - \sqrt{\frac{n}{2}} - \frac{\ln n}{2} + 4. \tag{4}$$

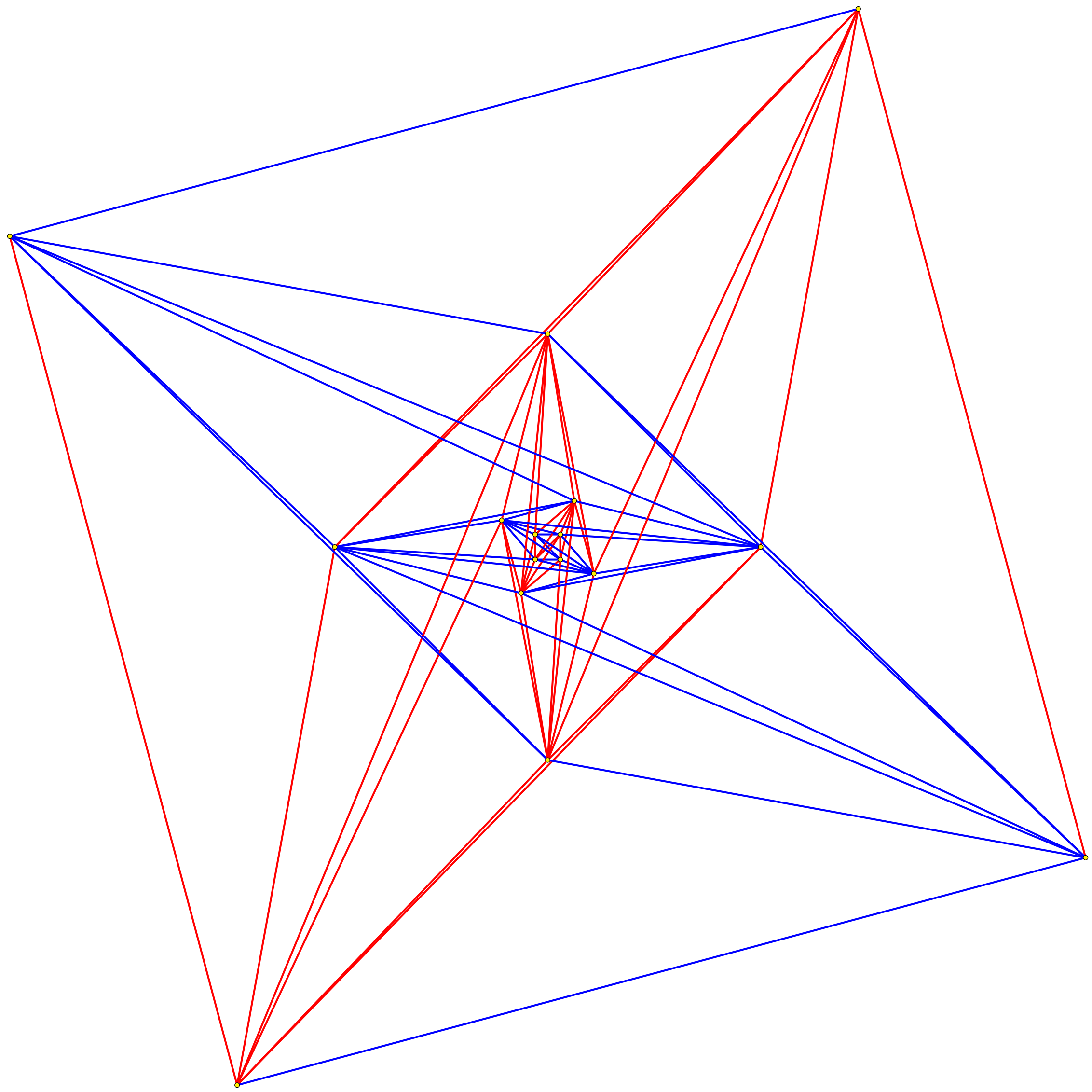

Figure 10: The construction in Theorem 20 with $k = 2$ and $s = 4$.

In 2007, we conjectured that $\mathsf{cat}(K_n) = n - o(n)$. This conjecture was subsequently verified by Fabila-Monroy and Wood [42] who proved that every n-vertex graph with convex antithickness k has at most $kn - \binom{k}{2}$ edges, which implies that

$$\mathsf{cat}(K_n) \geqslant n - \sqrt{2n + \tfrac{1}{4}} + \tfrac{1}{2}. \tag{5}$$

This is a significant improvement over the lower bound in (4). The upper bound in (4) has since been improved by Fabila-Monroy et al. [41] to match the lower bound in (5). Thus

$$\mathsf{cat}(K_n) = n - \left\lfloor \sqrt{2n + \tfrac{1}{4}} - \tfrac{1}{2} \right\rfloor .$$

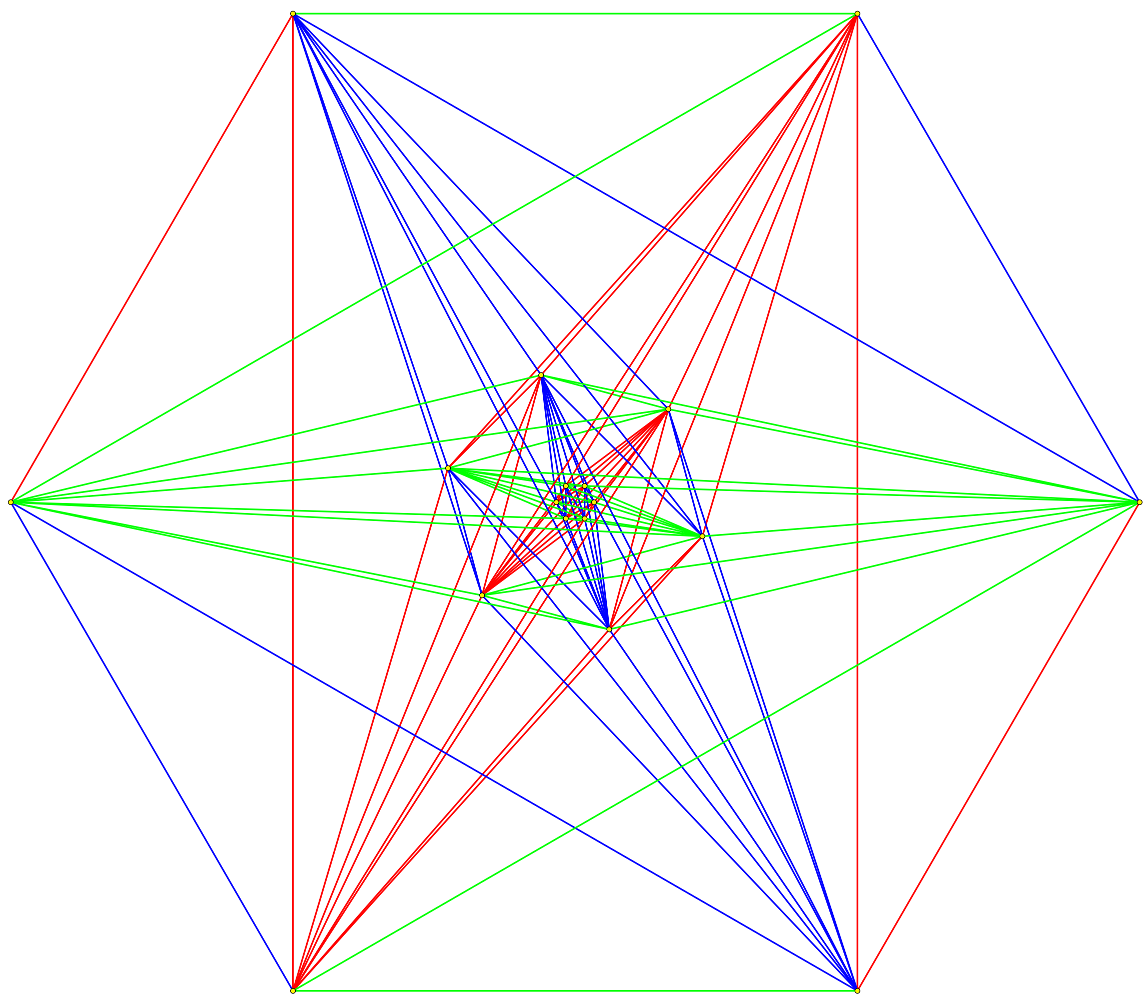

Figure 11: The construction in Theorem 20 with $k = 3$ and $s = 3$.

Now consider the antithickness of K_n.

Proposition 21. *The antithickness of K_n is at least $\frac{n}{3}$ and at most $\lceil\frac{n-1}{2}\rceil$.*

Proof. The lower bound follows from the fact that every graph with antithickness at most k has at most $\frac{3}{2}k(n-1)$ edges; see Section 6. For the upper bound, first consider the case of odd n. Walecki proved K_n has a edge-partition into $\frac{n-1}{2}$ Hamiltonian cycles [74]. Each such cycle is a thrackle. By Corollary 2, the antithickness of K_n is at most $\frac{n-1}{2}$. For even n, (applying the odd case) there is an edge-partition into $\frac{n-2}{2}$ odd cycles of length $n-1$, plus one $(n-1)$-edge star. Each such cycle and the star is a thrackle. By Corollary 2, the antithickness of K_n is at most $\frac{n-2}{2} + 1 = \lceil\frac{n-1}{2}\rceil$. □

We conjecture that the antithickness of K_n equals $\lceil\frac{n-1}{2}\rceil$ (which is implied by Conway's thrackle conjecture). Determining the geometric antithickness of K_n is an open problem. The best known upper bound is $n - \left\lfloor \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rfloor$, which follows from the convex case. The best lower bound is only $\frac{n-1}{2}$, which follows from the fact that every n-vertex graph with geometric antithickness k has at most kn edges.

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