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Computational Geometry 38 (2007) 181–193
Computational Geometry
Theory and Applications
www.elsevier.com/locate/comgeo

Graph drawings with few slopes ☆

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Received 28 July 2005; received in revised form 19 June 2006; accepted 9 August 2006

Available online 14 September 2006

Communicated by D. Wagner

Dedicated to Godfried Toussaint on his 60th birthday

Abstract

The *slope-number* of a graph G is the minimum number of distinct edge slopes in a straight-line drawing of G in the plane. We prove that for $\Delta \geqslant 5$ and all large n, there is a Δ-regular n-vertex graph with slope-number at least $n^{1-\frac{8+\varepsilon}{\Delta+4}}$. This is the best known lower bound on the slope-number of a graph with bounded degree. We prove upper and lower bounds on the slope-number of complete bipartite graphs. We prove a general upper bound on the slope-number of an arbitrary graph in terms of its bandwidth. It follows that the slope-number of interval graphs, cocomparability graphs, and AT-free graphs is at most a function of the maximum degree. We prove that graphs of bounded degree and bounded treewidth have slope-number at most $\mathcal{O}(\log n)$. Finally we prove that every graph has a drawing with one bend per edge, in which the number of slopes is at most one more than the maximum degree. In a companion paper, planar drawings of graphs with few slopes are also considered.

Keywords: Graph drawing; Slope; Slope-number

☆ A preliminary version of this paper was published as: Really straight graph drawings, in: Proceedings of the 12th International Symposium on Graph Drawing (GD '04), Lecture Notes in Computer Science, vol. 3383, Springer, 2004, pp. 122–132.

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[1] Supported by NSERC.

[2] Supported by a Marie Curie Fellowship of the European Community under contract 023865, and by the projects MCYT-FEDER BFM2003-00368 and Gen. Cat 2001SGR00224.

0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2006.08.002

1. Introduction

This paper studies straight-line drawings of graphs[3] in the plane with few distinct edge slopes.[4] Wade and Chu [40] introduced this topic, and defined the *slope-number* of a graph G to be the minimum number of distinct edge slopes in a drawing of G. Let the *convex slope-number* of G be the minimum number of distinct edge slopes in a convex drawing[5] of G. Let $\mathsf{sn}(G)$ and $\mathsf{csn}(G)$ respectively denote the slope-number and convex slope-number of G. By definition $\mathsf{sn}(G) \leqslant \mathsf{csn}(G)$ for every graph G. In this paper we prove lower and upper bounds on $\mathsf{sn}(G)$ and $\mathsf{csn}(G)$ for various (families of) graphs G. In a companion paper [13], planar drawings of graphs with few slopes are also considered.

We start by considering some elementary lower bounds on the number of slopes. In a drawing of a graph, at most two edges incident to a vertex v can have the same slope. Thus the edges incident to v use at least $\frac{1}{2}\deg(v)$ slopes. Hence the number of slopes is at least half the maximum degree. For some vertex v on the convex hull of the drawing, every edge incident to v has a distinct slope. Thus the number of slopes is at least the minimum degree. In a convex drawing, every edge incident to each vertex v has a distinct slope. Thus the number of slopes is at least the maximum degree. Summarising:

$$\text{(a) } \mathsf{sn}(G) \geqslant \frac{1}{2}\Delta(G), \quad \text{(b) } \mathsf{sn}(G) \geqslant \delta(G), \quad \text{and} \quad \text{(c) } \mathsf{csn}(G) \geqslant \Delta(G). \tag{1}$$

Given these three lower bounds, it is natural ask whether there is a function f such that $\mathsf{sn}(G) \leqslant f(\Delta(G))$ for every graph G. (A result by Malitz [29] implies that there is no such function f for convex slope-number.[6]) This question was first posed in the conference version of this paper [14]. It was subsequently solved in the negative for $\Delta \geqslant 5$ independently by Pach and Pálvölgyi [32] and Barát et al. [2].[7] The best bound, due to Pach and Pálvölgyi [32], states that for all $\Delta \geqslant 5$ and for all sufficiently large n, there exists an n-vertex graph G with maximum degree Δ and slope-number

$$\mathsf{sn}(G) > n^{\frac{1}{2}-\frac{1}{\Delta-2}-o(1)}.$$

The first contribution of this paper is to prove an analogous lower bound of

$$\mathsf{sn}(G) > n^{1-\frac{8+\varepsilon}{\Delta+4}}$$

for all $\Delta \geqslant 5$ (Section 2). This is the best known bound for all $\Delta \geqslant 9$. More importantly, our bound tends to n for large Δ, whereas the previous bounds by Pach and Pálvölgyi [32] and Barát et al. [2] both tend to $\sqrt{n}$.

The other main contributions of this paper establish graph families for which the slope-number is at most a function of the maximum degree. First, we consider the slope-number of complete k-partite graphs (Section 3).

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[3] We consider undirected, finite, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are respectively denoted by $n = |V(G)|$ and $m = |E(G)|$. The minimum and maximum degrees of G are respectively denoted by $\delta(G)$ and $\Delta(G)$.

[4] Consider a mapping of the vertices of a graph to distinct points in the plane. Now represent each edge by the closed line segment between its endpoints. Such a mapping is a (*straight-line*) *drawing* if the only vertices that each edge intersects is its own endpoints. The *slope* of a line L is the angle swept from the X-axis in an anticlockwise direction to L (and is thus in $[0, \pi)$). The *slope* of an edge or segment is the slope of the line that contains it. Of course two edges have the same *slope* if and only if they are parallel. A *crossing* in a drawing is a pair of edges that intersect at some point other than a common endpoint. A drawing is *plane* if it has no crossings.

[5] A drawing is *convex* if all the vertices are on the convex hull, and no three vertices are collinear.

[6] The *book thickness* of a graph G is the minimum integer k such that G has a convex drawing in which each edge receives one of k colours, and edges with the same colour do not cross; see [15]. Since parallel edges do not cross, the book thickness of G is a lower bound on $\mathsf{csn}(G)$. Malitz [29] proved that there are Δ-regular n-vertex graphs G with book thickness $\Omega(\sqrt{\Delta}n^{1/2-1/\Delta})$. Barát et al. [2] proved the same result for all $\Delta \geqslant 3$. Thus $\mathsf{csn}(G) \geqslant \Omega(\sqrt{\Delta}n^{1/2-1/\Delta})$.

[7] The *geometric thickness* of a graph G is the minimum integer k such that G has a drawing in which each edge receives one of k colours, and edges with the same colour do not cross; see [11,16,17,19]. Since parallel edges do not cross, the geometric thickness of G is a lower bound on $\mathsf{sn}(G)$. Barát et al. [2] proved that for all $\Delta \geqslant 9$ and $\varepsilon > 0$, for all sufficiently large $n > n(\Delta, \varepsilon)$, there exists a Δ-regular n-vertex graph with geometric thickness at least $c\sqrt{\Delta}\,n^{1/2-4/\Delta-\varepsilon}$.

We then show that the slope-number is at most a function of the maximum degree for interval graphs,[8] cocomparability graphs,[9] and AT-free graphs.[10] These results are established by first proving a general upper bound on the slope-number in terms of the bandwidth (Section 4.1).

For graphs with bounded degree and bounded treewidth,[11] we prove a $\mathcal{O}(\log n)$ upper bound on the slope-number (Section 4.2). The proof is based on a result of independent interest: every tree T has a drawing with $\Delta(T) - 1$ slopes and $2k - 1$ distinct edge lengths, where k is the pathwidth of T.

Our final contribution is to show that every graph G has a drawing with $\Delta(G) + 1$ slopes, if we allow one bend in each edge (Section 5).

1.1. Related research

We now outline some related research from the literature. Drawings of lattices and posets with few slopes have been considered by Ferber and Jürgensen [20], Czyzowicz et al. [8–10] and Freese [22].

Ambrus et al. [1] introduced the following slope parameter of graphs. Let $P \subset \mathbb{R}^2$ be a finite set of points in the plane. Let $S \subset \mathbb{R} \cup \{\infty\}$ be a set of slopes. Let $G(P, S)$ be the graph with vertex set P where two points $v, w \in P$ are adjacent if and only if the slope of the line $\overline{vw}$ is in S. The *slope parameter* of a graph G is the minimum integer k such that $G \cong G(P, S)$ for some point set P and slope set S with $|S|k$. This idea differs from our definition in that a clique can be represented by a set of collinear points. Amongst other results, [1] characterised the graphs with slope parameter 2, and proved that the slope parameter of a tree T equals $\Delta(T)$.

A famous result by Ungar [39], settling an open problem of Scott [37], states that n non-collinear points determine at least $n - 1$ distinct slopes. The configurations of n points that determine exactly $n - 1$ distinct slopes have been investigated by Jamison [24,25]. Jamison [27] generalised the result of Ungar by proving that any set of non-collinear points has a spanning tree whose edges have distinct slopes. Jamison [27] conjectured that any set of points in general position has a spanning path whose edges have distinct slopes. In this direction, Kleitman and Pinchasi [28] proved that every n-vertex caterpillar has a drawing on any n prespecified points in general position such that no two edges have the same slope.

Multi-dimensional graph drawings with few slopes are also of interest. Since an orthogonal projection preserves parallel lines, and since there always is a 'nice' orthogonal projection from $d \geqslant 3$ dimensions into the plane, the best bounds on the number of slopes are obtained in two dimensions. Here a projection is 'nice', if no vertex-vertex or vertex-edge occlusions occur; see [7,18,23]. Thus multi-dimensional drawings with few slopes are only interesting if the vertices are restricted to not all lie in a single plane. Under this assumption, Pach et al. [34] proved that the minimum number of slopes determined by n points in $\mathbb{R}^3$ is (exactly) $2n - 5$ if n is odd, and at least $2n - 7$ if n is even. Earlier, Pach et al. [33] proved that under the additional assumption that no three points are collinear (which is needed for a drawing of K_n), the minimum number of slopes is (exactly) $2n - 2$ if n is odd and $2n - 3$ if n is even. These proofs are based on generalisations of the above-mentioned result of Ungar [39]. In related work, Onn and Pinchasi [31] studied the minimum number of edge slopes in a d-dimensional convex polytope.

2. Graphs of bounded degree

Here we prove the following theorem, which was introduced in Section 1.

[8] A graph G is an *interval graph* if one can assign to each vertex $v \in V(G)$ a closed interval $[L_v, R_v] \subset \mathbb{R}$ such that $vw \in E(G)$ if and only if $[L_v, R_v] \cap [L_w, R_w] \neq \emptyset$. The *pathwidth* of a graph G is the minimum k such that G is a spanning subgraph of an interval graph with no clique on $k + 2$ vertices.

[9] Let $\preceq$ be a partial order on a ground set P. The *cocomparability* graph of $\preceq$ has vertex set P, where two vertices are adjacent if they are incomparable under $\preceq$. For example, every permutation graph is a cocomparability graph.

[10] An *asteroidal triple* in a graph is an independent set of three vertices such that each pair is joined by a path that avoids the neighborhood of the third. A graph is *asteroidal triple-free* (or *AT-free*) if it contains no asteroidal triple. AT-free graphs include interval, trapezoid, and cocomparability graphs.

[11] A graph is *chordal* if every induced cycle is a triangle. The *treewidth* of a graph G is the minimum integer k such that G is a subgraph of a chordal graph with no clique on $k + 2$ vertices. This parameter is particularly important in algorithmic and structural graph theory; see [5,35] for surveys. The treewidth of a graph is at most its pathwidth.

Theorem 1. *For all $\Delta \geqslant 5$ and $\varepsilon > 0$, for all sufficiently large $n > n(\Delta, \varepsilon)$, there exists a Δ-regular n-vertex graph G with slope-number*

$$\mathsf{sn}(G) > n^{1-\frac{8+\varepsilon}{\Delta+4}}.$$

Proof. In this proof, c is an positive (absolute) constant that might change from one line to the next. We proceed as in the proof by Barát et al. [2]. The idea is to show that there are more Δ-regular graphs than Δ-regular graphs with slope-number k, for an appropriately chosen k. For ease of counting we work with labelled graphs.

Let $\mathcal{G}$ be the set of labelled Δ-regular n-vertex graphs. The first asymptotic bounds on $|\mathcal{G}|$ were independently obtained by Bender and Canfield [3] and Wormald [43]. Based on a further refinement by McKay [30], Barát et al. [2] proved that

$$|\mathcal{G}| \geqslant \left(\frac{n}{3\Delta}\right)^{\Delta n/2} \quad \text{for all } n \geqslant c\Delta. \tag{2}$$

The key contribution of Barát et al. [2] was to show that the number of labelled n-vertex m-edge graphs with slope-number at most k is at most

$$\left(\frac{50n^2(k+1)}{2n+k}\right)^{2n+k}\binom{k(n-1)}{m}. \tag{3}$$

Suppose, on the contrary, that for some $\Delta \geqslant 5$, for some $\varepsilon > 0$, and for some n, every Δ-regular n-vertex graph has slope-number at most

$$k := n^{1-\frac{8+\varepsilon}{\Delta+4}}.$$

We now derive a contradiction for all sufficiently large $n > n(\Delta, \varepsilon)$. By (2) and (3),

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leqslant |\mathcal{G}| \leqslant \left(\frac{50n^2(k+1)}{2n+k}\right)^{2n+k}\binom{k(n-1)}{\Delta n/2} < (ckn)^{2n+k}\binom{kn}{\Delta n/2}.$$

Since $\binom{a}{b} \leqslant (\frac{\mathrm{e}a}{b})^b$,

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} < (ckn)^{2n+k}\left(\frac{ck}{\Delta}\right)^{\Delta n/2}.$$

Hence

$$n^{4\Delta n} < (ckn)^{16n+8k}(ck)^{4\Delta n}.$$

Observe that $8k < \varepsilon n$ for all large $n > n(\Delta, \varepsilon)$. Thus

$$n^{4\Delta} < (ckn)^{16+\varepsilon}(ck)^{4\Delta}.$$

That is,

$$n^{4\Delta-16-\varepsilon} < c^{4\Delta+16+\varepsilon}k^{4\Delta+16+\varepsilon}.$$

Since $c^{4\Delta+16+\varepsilon} < n^{2\varepsilon}$ for all large $n > n(\Delta, \varepsilon)$,

$$n^{4\Delta-16-3\varepsilon} < k^{4\Delta+16+\varepsilon}.$$

That is,

$$k > n^{\frac{4\Delta-16-3\varepsilon}{4\Delta+16+\varepsilon}} = n^{1-\frac{32+4\varepsilon}{4\Delta+16+\varepsilon}} > n^{1-\frac{8+\varepsilon}{\Delta+4}},$$

which is the desired contradiction. Therefore for all sufficiently large $n > n(\Delta, \varepsilon)$, there exists a Δ-regular n-vertex graph G with $\mathsf{sn}(G) > k$. □

The following open problem remains unsolved.

Open Problem 2. Does every graph with maximum degree at most 4 have bounded slope-number? Note that Duncan et al. [17] proved that such graphs have geometric thickness at most 2.

Another interesting problem is to determine the best possible bounds on the slope-number of graphs with bounded degree.

Open Problem 3. Does every n-vertex graph with bounded degree have $o(n)$ slope-number?

3. Complete multipartite graphs

We start this section by considering the slope-number of the complete graph K_n on n vertices. Consider a drawing of a graph G on a regular n-gon with vertex ordering $(v_1, v_2, \dots, v_n)$. Scott [37] observed that the number of slopes is

$$\left|\left\{(i+j) \bmod n\colon\ v_i v_j \in E(G)\right\}\right|. \tag{4}$$

Thus for K_n, drawn on a regular n-gon, the number of slopes is n, as illustrated in Fig. 1. Thus $\mathsf{sn}(K_n) \leqslant \mathsf{csn}(K_n) \leqslant n$. To see that this construction is optimal, let u, v, w be three consecutive vertices on the convex hull of an arbitrary drawing of K_n. Jamison [26] observed that the $n-1$ edges incident to v and the edge uw have distinct slopes.[12] Thus:

Proposition 4. [26] $\mathsf{csn}(K_n) = \mathsf{sn}(K_n) = n$.

For $k \geqslant 2$, the *complete k-partite graph* $K_{n_1,n_2,\dots,n_k}$ has vertex set $V(G) := \{v_{i,j}\colon\ 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n_i\}$ and edge set $E(G) = \{v_{i,p}v_{j,q}\colon\ 1 \leqslant i < j \leqslant k,\ 1 \leqslant p \leqslant n_i,\ 1 \leqslant q \leqslant n_j\}$.

The slope-number of the balanced complete bipartite graph is easily determined.

Proposition 5. $\mathsf{sn}(K_{n,n}) = \mathsf{csn}(K_{n,n}) = n$.

Proof. Since $K_{n,n}$ is n-regular, $\mathsf{sn}(K_{n,n}) \geqslant n$ by Eq. (1b). For the upper bound, position the vertices of $K_{n,n}$ on a regular $2n$-gon $(v_1, v_2, \dots, v_{2n})$, alternating between the colour classes, as illustrated in Fig. 2. Thus $v_i v_j$ is an edge if and only if $i+j$ is odd. By (4), the number of slopes is $|\{(i+j) \bmod 2n\colon\ 1 \leqslant i < j \leqslant 2n,\ i+j \text{ is odd}\}| = n$. □

Proposition 5 implies that $\mathsf{sn}(K_{a,b}) \leqslant \mathsf{csn}(K_{a,b}) \leqslant \max\{a,b\}$. In fact, by Eq. (1c), $\mathsf{csn}(K_{a,b}) \geqslant \Delta(K_{a,b}) = \max\{a,b\}$. Thus $\mathsf{csn}(K_{a,b}) = \max\{a,b\}$. Determining $\mathsf{sn}(K_{a,b})$ is more challenging. We have the following bounds.

Theorem 6. *For all $a \leqslant b$, $\frac{1}{2}(a+b-1) \leqslant \mathsf{sn}(K_{a,b}) \leqslant \min\{b, \lceil \frac{b}{2} \rceil + a - 1\}$.*

Proof. That $\mathsf{sn}(K_{a,b}) \leqslant b$ follows from Proposition 5.

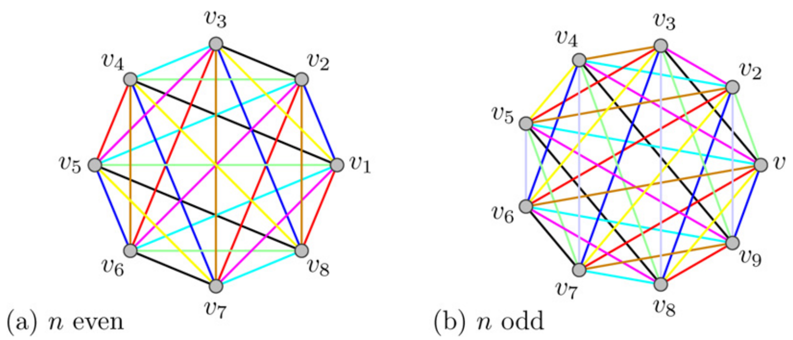

Fig. 1. Drawings of K_n with n slopes.

[12] More generally, Jamison [26] proved that if a drawing of K_n has k vertices on the convex hull then the number of slopes is at least $k(n-2)/(k-2)$, and that every drawing of K_n with exactly n slopes is affinely equivalent to a regular n-gon. Note that Wade and Chu [40] independently proved that $\mathsf{sn}(K_n) = n$, and also presented an algorithm to test if K_n can be drawn using a given set of slopes.

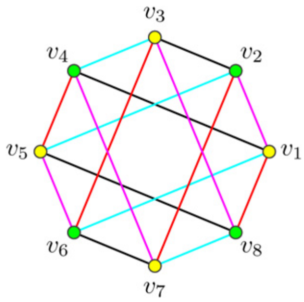

Fig. 2. Drawing of $K_{4,4}$ with 4 slopes.

Now we prove the upper bound, $\mathsf{sn}(K_{a,b}) \leqslant \lceil \frac{b}{2} \rceil + a - 1$. Without loss of generality b is even. Suppose $V(K_{a,b}) = \{v_1, v_2, \dots, v_a\} \cup \{u_1, u_2, \dots, u_{\frac{b}{2}}\} \cup \{w_1, w_2, \dots, w_{\frac{b}{2}}\}$, and $E(K_{a,b}) = \{v_i u_j, v_i w_j\colon 1 \leqslant i \leqslant a,\ 1 \leqslant j \leqslant \frac{b}{2}\}$. Position each vertex u_j at $(j, 1)$; position each vertex v_i at $(\frac{b}{2} + i, 0)$; and position each vertex w_j at $(\frac{b}{2} + a + j, -1)$. Then every edge is parallel with one of the $\frac{b}{2} + a - 1$ edges $\{v_1 u_j\colon 1 \leqslant j \leqslant \frac{b}{2}\} \cup \{u_1 v_i\colon 2 \leqslant i \leqslant a\}$, as illustrated in Fig. 3.

Now we prove the lower bound (which is due to an anonymous referee). Let A and B be the two colour classes of $K_{a,b}$ where $|A| = a$ and $|B| = b$. Given a drawing of $K_{a,b}$, rotate it so that no two vertices are horizontal. Let L be a horizontal line that intersects no vertex, and has at least $\lfloor \frac{1}{2}(a+b) \rfloor$ vertices above and below L. Let a_1 and a_2 be the number of vertices in A respectively above and below L. Let b_1 and b_2 be the number of vertices in B respectively above and below L. Thus $a_1 + b_1 \geqslant \lfloor \frac{1}{2}(a+b) \rfloor$ and $a_2 + b_2 \geqslant \lfloor \frac{1}{2}(a+b) \rfloor$. Since $(a_1 + b_2) + (a_2 + b_1) = a + b$, without loss of generality, $a_1 + b_2 \geqslant \lceil \frac{1}{2}(a+b) \rceil$.

We claim that $a_1 > 0$ and $b_2 > 0$. Suppose on the contrary that $a_1 = 0$. Thus $b = b_1 + b_2 \geqslant \lfloor \frac{1}{2}(a+b) \rfloor + \lceil \frac{1}{2}(a+b) \rceil = a + b$, implying $a = 0$, which is a contradiction. Thus $a_1 > 0$, and similarly, $b_2 > 0$. Consider the drawing of K_{a_1,b_2} induced by the a_1 vertices in A above L and the b_2 vertices in B below L. Every edge of K_{a_1,b_2} crosses L and there is some edge in K_{a_1,b_2}. Let vw be the leftmost edge of K_{a_1,b_2} crossing L. Then the $a_1 + b_2 - 1$ edges of K_{a_1,b_2} incident to v or w all have distinct slopes, as illustrated in Fig. 4. □

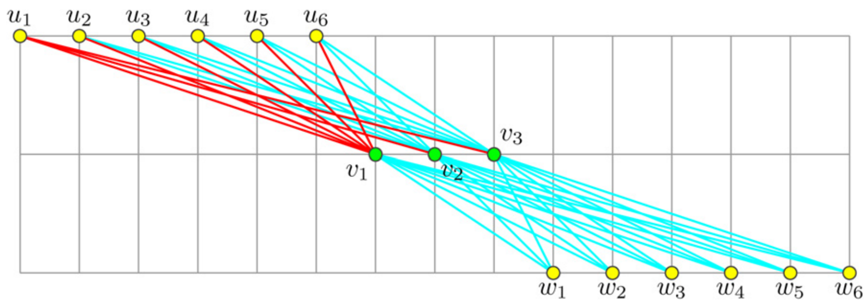

Fig. 3. Drawing of $K_{3,12}$ with 8 slopes (highlighted).

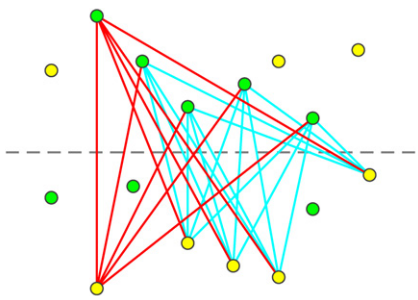

Fig. 4. Finding a large separated subgraph in $K_{a,b}$.

Closing the gap in the bounds in Theorem 6 remains an interesting open problem.

Open Problem 7. What is the slope-number $\mathsf{sn}(K_{a,b})$ of the complete bipartite graph $K_{a,b}$?

Now consider the general case of a complete k-partite graph G. Say G has n vertices. Since $\mathsf{csn}(G) \leqslant n$ and $\Delta(G) \geqslant \frac{k-1}{k}n$, we have $\mathsf{sn}(G) \leqslant \mathsf{csn}(G) \leqslant \frac{k}{k-1}\Delta(G)$.

Open Problem 8. Does every complete multipartite graph G with maximum degree Δ have a (convex) drawing with at most $\Delta + \mathrm{o}(\Delta)$ slopes?

We have the following partial solution to Open Problem 8.

Proposition 9. *Given integers $p \geqslant 0$ and $k \geqslant 2$, where $k-1$ is a power of two, let G be the complete k-partite graph $K_{2^p,2^p,2^{p+1},\dots,2^{p+1}}$. Then* $\mathsf{sn}(G) \leqslant \mathsf{csn}(G) = \Delta(G)$.

Proof. Eq. (1c) implies that $\mathsf{csn}(G) \geqslant \Delta(G)$. We now prove the upper bound.

Let $n := (k-1)2^{p+1}$ be the number of vertices in G. Note that n is a power of two, and $\Delta(G) = n - 2^p$. In what follows $a \equiv b$ means that $a \equiv b \pmod{n/2^p}$, and $a \equiv \pm b$ means that $a \equiv b$ or $a \equiv -b$. For all $0 \leqslant i \leqslant k-1$, let $P_i = \{j \in V(G)\colon\ i \equiv \pm j\}$. Let $V(G) := \{0, 1, \dots, n-1\}$. Below we prove that $\{P_0, P_1, \dots, P_{k-1}\}$ is a partition of $V(G)$ with $|P_0| = |P_{k-1}| = 2^p$, and $|P_i| = 2^{p+1}$ for all $1 \leqslant i \leqslant k-2$. Thus $\{P_0, P_1, \dots, P_{k-1}\}$ defines a valid assignment of the vertices to the colour classes. To obtain the drawing of G, place the vertices in numerical order on the vertices of a regular n-gon.

For each vertex $j \in V(G)$, let $j' := j \bmod n/2^p$. If $0 \leqslant j' \leqslant n/2^{p+1}$, then $j \in P_{j'}$. Otherwise, $n/2^{p+1} < j' < n/2^p$, and $j \in P_{n/2^p - j'}$. Thus, each vertex belongs to at least one P_i. Suppose that $j \in P_i \cap P_h$. Thus $i \equiv \pm j$ and $h \equiv \pm j$, implying $i \equiv \pm h$. Since $0 \leqslant i \leqslant n/2^{p+1}$, we have $h = i$. Thus, each vertex belongs to exactly one P_i, and $\{P_0, P_1, \dots, P_{k-1}\}$ is a partition of $V(G)$. The set P_0 has size 2^p because it is the set of all multiples of $n/2^p$ in $\{0, 1, \dots, n-1\}$. Similarly, P_{k-1} has size 2^p because it is the set of all odd multiples of $n/2^{p+1}$ in $\{0, 1, \dots, n-1\}$. The remainder of the P_i's have the same size, 2^{p+1}, by symmetry.

To prove that the number of slopes $|\{(i+j) \bmod n\colon\ ij \in E(G)\}| = n - 2^p$, by (4), it suffices to prove that $i + j \equiv 0$ implies $ij \notin E(G)$. Suppose that $i \in P_h$. Thus $h + i \equiv 0$ or $h - i \equiv 0$. In the first case, we have $h + i \equiv i + j$, implying $h - j \equiv 0$. In the second case, we have $h - i + (i + j) \equiv 0$, implying $h + j \equiv 0$. In both cases $j \in P_h$, implying $ij \notin E(G)$. □

Corollary 10. *Given integers $p \geqslant 0$, $q \leqslant 2^p$, and $k \geqslant 2$, where $k-1$ is a power of two, let G be the complete k-partite graph $K_{q,2^p,2^{p+1},\dots,2^{p+1}}$. Then* $\mathsf{csn}(G) = \Delta(G)$.

Proof. Let G' be the complete k-partite graph $K_{2^p,2^p,2^{p+1},\dots,2^{p+1}}$. Then G is a subgraph of G', and $\Delta(G) = \Delta(G') = (k-2)2^{p+1} + 2^p$. The result follows from Proposition 9. □

4. General graphs

While Theorem 1 proves that there exist graphs of bounded degree with unbounded slope-number, in this section, we prove that the slope-number of various classes of graphs is bounded by a function of the maximum degree. For graphs of bounded degree and bounded treewidth we prove a $\mathcal{O}(\log n)$ bound on the slope-number.

Our results are based on the following structure. Let H be a (*host*) graph. The vertices of H are called *nodes*. An *H-partition* of a graph G is a function $f : V(G) \to V(H)$ such that for every edge $vw \in E(G)$ we have $f(v) = f(w)$ or $f(v)f(w) \in E(H)$. In the latter case, we say vw is *mapped* to the edge $f(v)f(w)$. The *width* of f is the maximum of $|f^{-1}(x)|$, taken over all nodes $x \in V(H)$, where $f^{-1}(x) := \{v \in V(G)\colon\ f(v) = x\}$. The following general result describes how to produce a drawing of a graph G given an H-partition of G and a drawing of H.

Theorem 11. *Let D be a drawing of a graph H with s distinct slopes and ℓ distinct edge lengths. Let $t := |\{(\mathrm{slope}(e), \mathrm{length}(e))\colon\ e \in E(D)\}|$ (which is at most $s\ell$). Let G be a graph with an H-partition of width k. Then G*

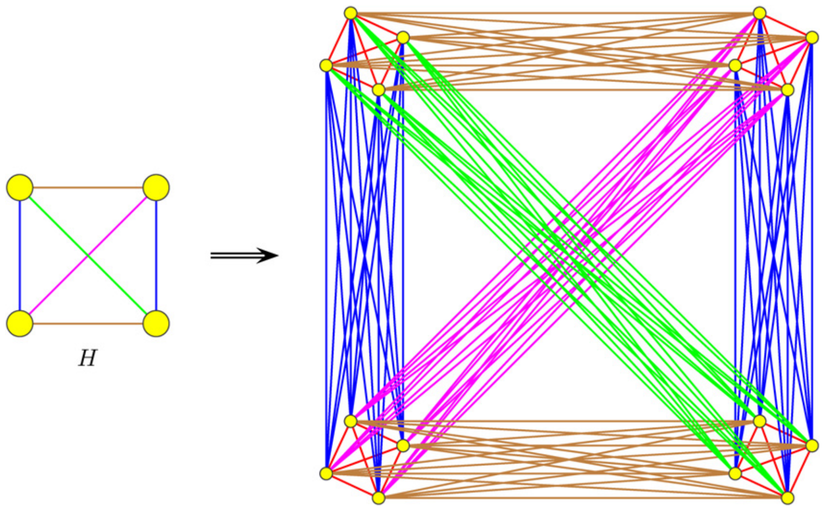

Fig. 5. Illustration of the construction in Theorem 11 with $H = K_4$, $s = 4$, $\ell = 2$, and $k = 4$.

has a drawing with at most $k + s + t(k^2 - k) \leqslant k + s + s\ell(k^2 - k)$ *distinct slopes* (*and at most* $\lfloor \frac{k}{2} \rfloor + \ell + t(k^2 - k) \leqslant \lfloor \frac{k}{2} \rfloor + \ell + s\ell(k^2 - k)$ *distinct edge lengths*).

Proof. The general approach is to scale D appropriately, and then replace each node of H by a copy of the drawing of K_k on a regular k-gon (described in Section 3). The only difficulty is to scale D so that we obtain a valid drawing of G.

Observe that $|\phi_1 - \phi_2|$ is the size of the minimum angle formed by lines of slope ϕ_1 and ϕ_2. Let $\{\theta_1, \theta_2, \ldots, \theta_s\}$ be the set of slopes of the edges of D. Rotate the drawing of K_k on a regular k-gon so that, if $\{\beta_1, \beta_2, \ldots, \beta_k\}$ is the set of slopes of the edges of K_k, then $|\theta_i - \beta_j| > 0$ for all i and j. Let $\varepsilon := \min_{i,j}\{|\theta_i - \beta_j|\}$.

Replace each node x in D by a disc B_x of uniform radius r centred at x, where r is chosen small enough so that: (1) $B_x \cap B_y = \emptyset$ for all distinct nodes x and y in D; and (2) for every edge $xy \in E(H)$ with slope θ_i, every segment with endpoints in B_x and B_y and with slope ϕ intersects no other B_z, and $|\phi - \theta_i| < \varepsilon$. Position a regular k-gon on each B_x (using the orientation determined above), and position the vertices $f^{-1}(x)$ of G at its vertices. Since $|\theta_i - \beta_j| \geqslant \varepsilon$, the slope of any edge vw of G that is mapped to xy does not equal any β_j. Hence vw does not pass through any other vertex of G.

Each copy of K_k contributes the same k slopes to the drawing of G. For each edge $xy \in E(H)$, for all $1 \leqslant i \leqslant k$, the edge of G from the ith vertex on B_x to the ith vertex on B_y (if it exists) has the same slope as the edge xy in D. Thus these edges contribute s slopes to the drawing of G. Consider two edges e_1 and e_2 of H that have the same slope and the same length in D (of the t possibilities). The edges of G that are mapped to e_1 use the same set of slopes as the edges of G that are mapped to e_2. There are at most $k^2 - k$ edges of G that are mapped to a single edge of H and were not counted above. Thus in total we have at most $k + s + t(k^2 - k)$ slopes, as illustrated in Fig. 5.

Each copy of K_k contributes the same $\lceil \frac{k}{2} \rceil$ distinct edge lengths. This, along with analogous arguments to those presented above, gives an upper bound of $\lfloor \frac{k}{2} \rfloor + \ell + t(k^2 - k)$ on the number of distinct edge lengths. □

4.1. *Drawings based on paths*

Theorem 11 suggests using host graphs that have drawings with few slopes and few edge lengths. Thus a path is a natural choice for a host graph, since it has a drawing with one slope and one edge length. The *path-partition-width* of a graph G, denoted by $\mathsf{ppw}(G)$, is the minimum integer k such that G has a P-partition of width k, for some path P. Theorem 11 with $r = s = \ell = 1$ implies:

Corollary 12. *Every graph G has a drawing with* $\mathsf{ppw}(G)^2 + 1$ *slopes.*

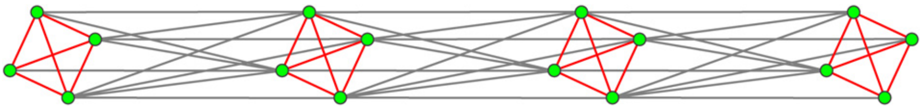

Fig. 6. Drawing of a graph with bandwidth 4 with eleven slopes.

As indicated by the following lemma, path-partition-width is closely related to the classical graph parameter bandwidth.[13] The *width* of a vertex ordering $(v_1, v_2, \dots, v_n)$ of a graph G is the maximum of $|i-j|$, taken over all edges $v_i v_j \in E(G)$. The *bandwidth* of G, denoted by $\mathsf{bw}(G)$, is the minimum width of a vertex ordering of G.

Lemma 13. *For every graph G, $\frac{1}{2}(\mathsf{bw}(G)+1) \leqslant \mathsf{ppw}(G) \leqslant \mathsf{bw}(G)$.*

Proof. Let $(v_1, v_2, \dots, v_n)$ be a vertex ordering of G with width $b = \mathsf{bw}(G)$. For all $0 \leqslant i \leqslant \lfloor n/b \rfloor$, let $B_i := \{v_{ib+1}, v_{ib+2}, \dots, v_{ib+b}\}$. Then $(B_0, B_1, \dots, B_{\lfloor n/b \rfloor})$ defines a path-partition of G with width b. Thus $\mathsf{ppw}(G) \leqslant \mathsf{bw}(G)$.

Now suppose $(B_1, B_2, \dots, B_m)$ is a path-partition of G with width $k = \mathsf{ppw}(G)$. Let $(v_1, v_2, \dots, v_n)$ be a vertex ordering of G such that $i < j$ whenever $v_i \in B_p$ and $v_j \in B_q$ and $p < q$. For every edge $v_i v_j \in E(G)$ with $v_i \in B_p$ and $v_j \in B_q$, we have $|p-q| \leqslant 1$. Thus $|i-j| \leqslant 2k-1$. Hence the width of $(v_1, v_2, \dots, v_n)$ is at most $2k-1$. Therefore $\mathsf{bw}(G) \leqslant 2\mathsf{ppw}(G) - 1$. □

Corollary 12 and Lemma 13 imply that every graph G has a drawing with $\mathsf{bw}(G)^2 + 1$ slopes. This bound can be tweaked as follows.

Theorem 14. *Every graph G has slope-number*

$$\mathsf{sn}(G) \leqslant \frac{1}{2}\mathsf{bw}(G)\big(\mathsf{bw}(G)+1\big)+1.$$

Proof. Let $G[B_i, B_{i+1}]$ be the bipartite subgraph of G with vertex set $B_i \cup B_{i+1}$ and edge set $\{vw \in E(G)\colon v \in B_i, w \in B_{i+1}\}$ from the proof of Lemma 13. Observe that in the construction of the path-partition in Lemma 13, the edges of each $G[B_i, B_{i+1}]$ are a subset of $\{\{v_{ib+j}, v_{(i+1)b+\ell}\}\colon 1 \leqslant j \leqslant b,\ 1 \leqslant \ell \leqslant j\}$. If we consistently assign the vertices in each B_i to the regular b-gon in Theorem 11, then each $G[B_i, B_{i+1}]$ will use the same set of slopes, since each $G[B_i, B_{i+1}]$ is a subgraph of the same graph. The number of slopes in $G[B_i, B_{i+1}]$ is $1 + \sum_{j=1}^{b}(j-1)$, since each vertex $v_j \in B_i$ is incident to j edges with endpoints in B_{i+1}, one of which is horizontal. Thus the total number of slopes in the resulting drawing of G is $b + 1 + \frac{1}{2}(b-1)b = \frac{1}{2}b(b+1)+1$. □

The following examples of Theorem 14 are corollaries of results by Fomin and Golovach [21] and Wood [41] that bound bandwidth in terms of maximum degree.

- Every interval graph G has $\mathsf{bw}(G) \leqslant \Delta(G)$ [21,41], and thus has a drawing with at most $\frac{1}{2}\Delta(G)(\Delta(G)+1)+1$ slopes.
- Every cocomparability graph G has $\mathsf{bw}(G) \leqslant 2\Delta(G) - 1$ [41], and thus has a drawing with at most $\Delta(G)(2\Delta(G)-1)+1$ slopes.
- Every AT-free graph G has $\mathsf{bw}(G) \leqslant 3\Delta(G)$ [41], and thus has a drawing with at most $\frac{3}{2}\Delta(G)(3\Delta(G)+1)+1$ slopes.

Open Problem 15. Does every interval graph G have a drawing with $\mathcal{O}(\Delta(G))$ slopes?

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13 Bodlaender [4] found essentially the same relation in the context of emulations of networks.

4.2. Drawings based on trees

To obtain bounds on the slope-number of more general graphs, we consider T-partitions for some tree T. This structure is called a *tree-partition*, and has been extensively studied; see [6,12,42] for example. Theorem 11 motivates the study of drawings of trees with few slopes and few distinct edge lengths.

Theorem 16. *Every tree T with pathwidth $k \geqslant 1$ has a plane drawing with* $\max\{\Delta(T)-1, 1\}$ *slopes and* $2k-1$ *distinct edge lengths.*

The proof of Theorem 16 is loosely based on an algorithm of Suderman [38] for drawing trees on layers. We will need the following lemma.[14]

Lemma 17. [38] *Every tree T has a path P such that $T \setminus V(P)$ has smaller pathwidth than T, and the endpoints of P are leaves of T.*

A path P satisfying Lemma 17 is called a *backbone* of T.

Proof of Theorem 16. We refer to T as T_0. Let n_0 be the number of vertices in T_0, and let $\Delta_0 = \Delta(T_0)$. The result holds trivially for $\Delta_0 \leqslant 2$. Now assume that $\Delta_0 \geqslant 3$. Let S be the set of slopes

$$S := \left\{\frac{\pi}{2}\left(1 + \frac{i}{\Delta_0 - 2}\right) \colon 0 \leqslant i \leqslant \Delta_0 - 2\right\}.$$

We proceed by induction on n with the hypothesis: "There is a real number $\ell = \ell(n_0, \Delta_0)$, such that for every tree T with $n \leqslant n_0$ vertices, maximum degree at most Δ_0, and pathwidth $k \geqslant 1$, and for every vertex r of T with degree less than Δ_0, T has a plane drawing D in which:

- r is at the top of D; that is, no point in D has greater Y-coordinate than r,
- every edge of T has slope in S,
- every edge of T has length in $\{1, \ell, \ldots, \ell^{2k-1}\}$, and
- if r is contained in some backbone of T, then every edge of T has length in $\{1, \ell, \ldots, \ell^{2k-2}\}$.

The result follows from the induction hypothesis, since we can take r to be the endpoint of a backbone of T_0, in which case $\deg(r) = 1 < \Delta_0$, and thus every edge of T_0 has length in $\{1, \ell, \ldots, \ell^{2k-2}\}$.

The base case with $n = 1$ is trivial. Now suppose that the hypothesis is true for trees on less than n vertices, and we are given a tree T with n vertices and pathwidth k, and r is a vertex of T with degree less than Δ_0.

If r is contained in some backbone B of T, then let $P := B$. Otherwise, let P be a path from r to an endpoint of a backbone B of T. Note that P has at least one edge. As illustrated in Fig. 7, draw P horizontally with unit-length

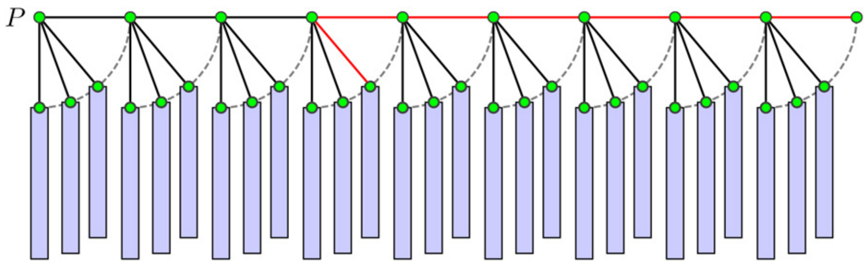

Fig. 7. Drawing of T with few slopes and few edge lengths.

[14] In fact, Lemma 17 can be viewed as the basis for an alternative definition of the pathwidth of a forest. In particular, the pathwidth of K_1 equals 0, the pathwidth of a forest F equals the maximum pathwidth of a connected component of F, and the pathwidth of a tree T equals the minimum k such that there exists a path P of T and the pathwidth of $T \setminus V(P)$ is at most $k-1$.

edges. Every vertex in P has at most $\Delta_0 - 2$ neighbours in $T \setminus V(P)$, since r has degree less than Δ_0 and the endpoints of a backbone are leaves. At each vertex $x \in P$, the children $\{y_0, y_1, \dots, y_{\Delta_0 - 3}\}$ of x are positioned below P and on the unit-circle centred at x, so that each edge xy_j has slope $\frac{\pi}{2}(1 + j/(\Delta_0 - 2)) \in S$.

Every connected component T' of $T \setminus V(P)$ is a tree rooted at some vertex r' adjacent to a vertex in P. By the above layout procedure, r' has already been positioned in the drawing of T. If T' is a single vertex, then we no longer need to consider this T'.

We consider two types of subtrees T', depending on whether the pathwidth of T' is less than k. Suppose that the pathwidth of T' is k (it cannot be more). Then $T' \cap B \neq \emptyset$ since B is a backbone of T. Thus $T' \cap B$ is a backbone of T' containing r'. Thus we can apply the stronger induction hypothesis in this case.

Every T' has fewer vertices than T, and every r' has degree less than Δ_0 in T'. Thus by induction, every T' has a drawing with r' at the top, and every edge of T' has slope in S. Furthermore, if the pathwidth of T' is less than k, then every edge of T' has length in $\{1, \ell, \dots, \ell^{2k-3}\}$. Otherwise r' is in a backbone of T', and every edge of T' has length in $\{1, \ell, \dots, \ell^{2k-2}\}$.

There exists a scale factor $\ell < 1$, depending only on n_0 and Δ_0, so that by scaling the drawings of every T' by ℓ, the widths of the drawings are small enough so that there is no crossings when the drawings are positioned with each r' at its already chosen location. (Note that ℓ is the same value at every level of the induction.) Scaling preserves the slopes of the edges. An edge in any T' that had length ℓ^i before scaling, now has length ℓ^{i+1}.

Case 1. r is contained in some backbone B of T: By construction, $P = B$. So every T' has pathwidth at most $k - 1$, and thus every edge of T' has length in $\{\ell^1, \ell^2, \dots, \ell^{2k-2}\}$. All the other edges of T have unit-length. Thus we have a plane drawing of T with edge lengths $\{1, \ell, \dots, \ell^{2k-2}\}$, as claimed.

Case 2. r is not contained in any backbone of T: Every edge in every T' has length in $\{\ell^1, \ell^2, \dots, \ell^{2k-1}\}$. All the other edges of T have unit-length. Thus we have a plane drawing of T with edge lengths $\{1, \ell, \dots, \ell^{2k-1}\}$, as claimed. □

Theorem 18. *Let G be a graph with n vertices, maximum degree $\Delta \geqslant 1$, and treewidth $k \geqslant 1$. Then G has a drawing with $\mathcal{O}(k^3 \Delta^4 \log n)$ slopes.*

Proof. Wood [42] proved[15] that G has a T-partition of width at most $w := 2(k+1)(9\Delta - 1)$ for some forest T. For each node $x \in V(T)$, there are at most $w\Delta$ edges of G incident to vertices mapped to x. Hence we can assume that T is a forest with maximum degree at most $w\Delta$, as otherwise there is an edge of T with no edge of G mapped to it, in which case the edge of T can be deleted. Similarly, T has at most n vertices. Scheffler [36] proved that T has pathwidth at most $\log(2n+1)$; see [5]. By Theorem 16, T has a drawing with at most $w\Delta - 1$ slopes and at most $2\log(2n+1) - 1$ distinct edge lengths. By Theorem 11, G has a drawing in which the number of slopes is at most $w(w\Delta - 1)(2\log(2n+1) - 1)(w - 1) + (w\Delta - 1) + w \in \mathcal{O}(w^3 \Delta \log n) \subseteq \mathcal{O}(k^3 \Delta^4 \log n)$. □

Corollary 19. *Every n-vertex graph with bounded degree and bounded treewidth has a drawing with $\mathcal{O}(\log n)$ slopes.* □

5. 1-bend drawings

While Theorem 1 proves that some graph with bounded degree has unbounded slope-number, we now show that there is no such graph if we allow bends in the edges. For a graph G, let G' be the graph obtained from G by subdividing each edge of G; that is, for each edge $e = vw$ of G, introduce a new *subdivision* vertex x_e in G', and replace e by the path vx_ew. A 1-*bend drawing* of G is a drawing G'.

Theorem 20. *Every graph G has a 1-bend drawing with $\Delta(G) + 1$ slopes.*

Proof. Let S be a set of $\Delta(G) + 1$ distinct slopes. Suppose the vertices of G have been positioned in the plane. For each vertex v of G and each slope $\ell \in S$, consider there to be a *slope line* through v with slope ℓ. Position the vertices

[15] The proof by Wood [42] is a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski [12].

of G at distinct points in the plane so that: (1) each slope line intersects exactly one vertex, and (2) no three slope lines intersect at a single point, unless all three are the slope lines of a single vertex. This can be achieved by positioning each vertex in turn, since at each step, there are finitely many forbidden positions.

Consider each slope line to be initially *unused*. Each edge is drawn with one bend, using one slope line at each of its endpoints, in which case, we say these slope lines become *used*. Now draw each edge vw of G in turn. At most $\deg(v) - 1$ slope lines at v are used, and at most $\deg(w) - 1$ slope lines at w are used. Since $|S| \geqslant \deg(v) + 1$ and $|S| \geqslant \deg(w) + 1$, there are two unused slope lines at v, and two unused slope lines at w. Thus there is an unused slope line at v that intersects an unused slope line at w. Position the bend for vw at their intersection point.

We now prove that this defines a drawing of G'. Suppose on the contrary that there is an edge vu of G' and a vertex w of G' that intersects vu, and $v \neq w \neq u$. Without loss of generality, v is a vertex of G and u is a subdivision vertex. Since each slope line intersects exactly one vertex of G, w is a subdivision vertex of some edge w_1w_2 of G. Since edges are only drawn on unused slope lines, $w_1 \neq v$ and $w_2 \neq v$. Therefore, the three slope lines containing the edges w_1w, w_2w and vu intersect in one point, and all three do not belong to the same vertex. This is a desired contradiction. □

Acknowledgements

This research was initiated at the *International Workshop on Fixed Parameter Tractability in Geometry and Games*, organised by Sue Whitesides; Bellairs Research Institute of McGill University, Barbados, February 7–13, 2004. Thanks to all of the participants for creating a stimulating working environment. Special thanks to Mike Fellows for suggesting the topic. Thanks to the anonymous referees for many helpful suggestions, including the proof of the lower bound in Theorem 6, and for pointing out Ref. [4].

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