Really Straight Graph Drawings*

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Abstract. We study straight-line drawings of graphs with few segments and few slopes. Optimal results are obtained for all trees. Tight bounds are obtained for outerplanar graphs, 2-trees, and planar 3-trees. We prove that every 3-connected plane graph on n vertices has a plane drawing with at most $5n/2$ segments and at most $2n$ slopes, and that every cubic 3-connected plane graph has a plane drawing with three slopes (and three bends on the outerface). Drawings of non-planar graphs with few slopes are also considered. For example, it is proved that graphs of bounded degree and bounded treewidth have drawings with $\mathcal{O}(\log n)$ slopes.

1 Introduction

A common requirement for an aesthetically pleasing drawing of graph is that the edges are straight. This paper studies the following additional requirements of straight-line graph drawings:

1. minimise the number of segments in the drawing
2. minimise the number of distinct edge slopes in the drawing

First we formalise these notions. Consider a mapping of the vertices of a graph to distinct points in the plane. Now represent each edge by the closed line segment between its endpoints. Such a mapping is a (*straight-line*) *drawing* if each edge does not intersect any vertex, except for its own endpoints. By a *segment* in a drawing, we mean a maximal set of edges that form a line segment. The *slope* of a line L is the angle swept from the X-axis in an anticlockwise direction to L (and is thus in $[0, \pi)$). The *slope* of an edge or segment is the slope of the line that extends it. A *crossing* in a drawing is a pair of edges that intersect at some point other than a common endpoint. A drawing is *plane* if it has no crossings. A *plane graph* is a planar graph with a fixed combinatorial embedding and a specified outerface. We emphasise that a plane drawing of a plane graph must preserve the embedding and outerface. That every plane graph has a plane drawing is a classical result independently due to Wagner and Fáry.

* Research initiated at the International Workshop on Fixed Parameter Tractability in Geometry and Games, organised by Sue Whitesides; Bellairs Research Institute of McGill University, Holetown, Barbados, Feb. 7-13, 2004. Research supported by NSERC and COMBSTRU.

J. Pach (Ed.): GD 2004, LNCS 3383, pp. 122–132, 2004.

It is easily seen that a graph has a (plane) drawing on two slopes if and only if it has a (plane) drawing on any two slopes [3]. Garg and Tamassia [8] proved that it is $\mathcal{NP}$-complete to decide whether a graph has a rectilinear planar drawing (that is, with vertical and horizontal edges). Thus it is $\mathcal{NP}$-complete to decide whether a graph has a plane drawing with two slopes.

Our results include lower and upper bounds on the minimum number of segments and slopes in plane drawings of graphs, as summarised in Table 1. Due to space limitations, a number of auxiliary results and most proofs are omitted from this paper; see [3] for all the details. We refer the reader to the survey of Bodlaender [1] for the definition of treewidth, pathwidth, and k-tree.

First observe that the minimum number of slopes in a drawing of (plane) graph G is at most the minimum number of segments in a drawing of G. Upper bounds for plane graphs are stronger than for planar graphs, since for planar graphs one has the freedom to choose the embedding and outerface. On the other hand, lower bounds for planar graphs are stronger than for plane graphs. For example, consider the n-vertex planar triangulation illustrated in Figure 1. It has at least $n + 2$ slopes in every plane drawing. Now fix the outerface to that illustrated in (a). Then there are at least $2n - 2$ slopes in every plane drawing. However, using the embedding shown in (b), there is a plane drawing with only $\lceil 3n/2 \rceil$ slopes.

Section 2 studies plane drawings of 3-connected plane and planar graphs. In the case of slope-minimisation for plane graphs we obtain a bound that is tight in the worst case. However, our lower bound examples have linear maximum degree. In Section 3 we (drastically) improve this result in the case of cubic graphs, by proving that every 3-connected plane cubic graph has a plane drawing with three slopes, except for three edges on the outerface that have their own slope. As a corollary we prove that every 3-connected plane cubic graph has a plane 'drawing' with three slopes and three bends on the outerface. Section 4 considers non-plane drawings of arbitrary graphs with few slopes. For example, we prove that every graph with bounded degree and bounded treewidth has a drawing with $\mathcal{O}(\log n)$ slopes.

Before continuing, we outline some related research from the literature.

- Eppstein [6] characterised those planar graphs that have plane drawings with a segment between every pair of vertices. In some sense, these are the plane drawings with the least number of slopes.
- The *geometric thickness* of a graph G is the minimum k such that G has a drawing in which every edge receives one of k colours, and monochromatic

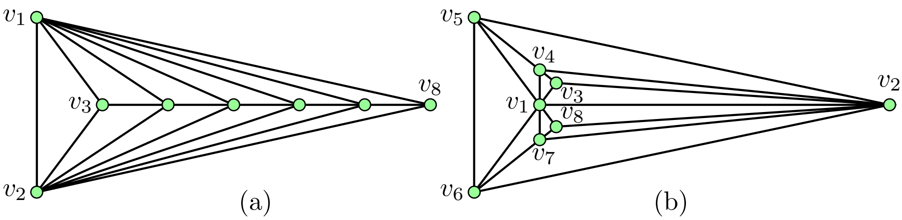

Fig. 1.

Table 1. Summary of results (ignoring additive constants). Here n is the number of vertices, η is the number of vertices of odd degree, and Δ is the maximum degree. The lower bounds are existential, except for trees, for which the lower bounds are universal.

graph family	# segments		# slopes	
	$\geq$	$\leq$	$\geq$	$\leq$
trees	$\eta/2$	$\eta/2$	$\lceil\Delta/2\rceil$	$\lceil\Delta/2\rceil$
maximal outerplanar	n	n	-	n
plane 2-trees	$2n$	$2n$	$2n$	$2n$
plane 3-trees	$2n$	$2n$	$2n$	$2n$
plane 2-connected	$5n/2$	-	$2n$	-
planar 2-connected	$2n$	-	n	-
plane 3-connected	$2n$	$5n/2$	$2n$	$2n$
planar 3-connected	$2n$	$5n/2$	n	$2n$
plane 3-connected cubic	-	$n+2$	3	3

edges do not cross (see [5, 7]). In any drawing, edges with the same slope do not cross. Thus the geometric thickness of G is a lower bound on the minimum number of slopes in a drawing of G.

- A drawing is *convex* if all the vertices are on the convex hull, and no three vertices are collinear. The *book thickness* of a graph (also called *pagenumber* and *stacknumber*) is the same as geometric thickness except that the drawing must be convex (see [4] for numerous references). Since edges with the same slope do not cross, the book thickness of G is a lower bound on the minimum number of slopes in a convex drawing of G.
- Plane orthogonal drawings with two slopes (and few bends) have been extensively studied (see [12]). For example, Ungar [14] proved that every cyclically 4-edge-connected plane cubic graph has a plane drawing with two slopes and four bends on the outerface. Thus our above-mentioned result for 3-connected plane cubic graphs nicely complements this theorem of Ungar.
- A drawing of the complete graph K_n is defined by a set of n points with no three collinear. Jamison [9] proved that the minimum number of slopes in a drawing of K_n is n. The upper bound is obtained by positioning the vertices of K_n on the vertices of a regular n-gon, as illustrated in Figure 2(a) and (b). In fact, Jamison [9] proved that every drawing of K_n with exactly n slopes is affinely equivalent to a regular n-gon. In [3] we study drawings of complete multi-partite graphs. For example, we prove that the minimum number of slopes in a convex drawing of $K_{n,n}$ is n, as illustrated in Figure 2(c).
- Wade and Chu [15] recognised that drawing arbitrary graphs with few slopes is an interesting problem. They defined the *slope-number* of a graph G to be the minimum number of slopes in a drawing of G. However, the results of Wade and Chu only pertain to K_n. Seemingly unaware of the earlier work of Scott and Jamison, they rediscovered that the minimum number of slopes in a drawing of K_n is n. In addition, they presented an algorithm to test if K_n can be drawn using a given set of slopes.

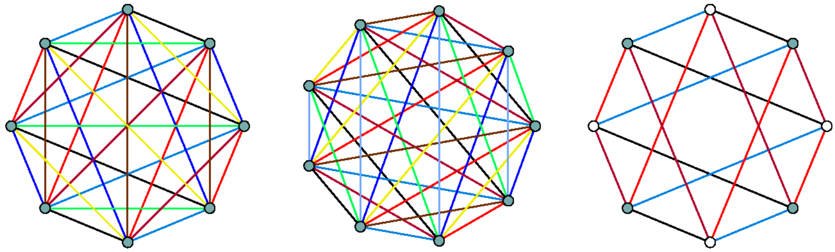

Fig. 2. Drawings of K_n and $K_{n,n}$ with n slopes.

2 3-Connected Plane Graphs

Theorem 1. *Every* 3*-connected plane graph with* n *vertices has a plane drawing with at most* $5n/2 - 3$ *segments and at most* $2n - 10$ *slopes.*

The proof of Theorem 1 is based on the canonical ordering of Kant [10]. Let G be a 3-connected plane graph. Kant [10] proved that G has a canonical ordering defined as follows. Let $\sigma = (V_1, V_2, \dots, V_K)$ be an ordered partition of $V(G)$. That is, $V_1 \cup V_2 \cup \dots \cup V_K = V(G)$ and $V_i \cap V_j = \emptyset$ for all $i \neq j$. Define G_i to be the plane subgraph of G induced by $V_1 \cup V_2 \cup \dots \cup V_i$. Let C_i be the subgraph of G induced by the edges on the boundary of the outerface of G_i. Then σ is a *canonical ordering* of G if:

- $V_1 = \{v_1, v_2\}$, where v_1 and v_2 lie on the outerface and $v_1v_2 \in E(G)$.
- $V_K = \{v_n\}$, where v_n lies on the outerface, $v_1v_n \in E(G)$, and $v_n \neq v_2$.
- Each C_i ($i > 1$) is a cycle containing v_1v_2.
- Each G_i is biconnected and internally 3-connected; that is, removing any two interior vertices of G_i does not disconnect it.
- For each $i \in \{2, 3, \dots, K-1\}$, one of the following condition holds:
  1. $V_i = \{v_i\}$ where v_i is a vertex of C_i with at least three neighbours in C_{i-1}, and v_i has at least one neighbour in $G \setminus G_i$.
  2. $V_i = (s_1, s_2, \dots, s_\ell, v_i)$, $\ell \geq 0$, is a path in C_i, where each vertex in V_i has at least one neighbour in $G \setminus G_i$. Furthermore, the first and the last vertex in V_i have one neighbour in C_{i-1}, and these are the only two edges between V_i and G_{i-1}.

The vertex v_i is called the *representative* vertex of V_i, $2 \leq i \leq K$. The vertices $\{s_1, s_2, \dots, s_\ell\} \subseteq V_i$ are called *division* vertices. Let $S \subset V(G)$ be the set of all division vertices. A vertex u is a *successor* of a vertex $w \in V_i$ if uw is an edge and $u \in G \setminus G_i$, and u is a *predecessor* of $w \in V_i$ if uw is an edge and $u \in V_j$ for some $j < i$. We also say that u is a predecessor of V_i. Let $P(V_i) = (p_1, p_2, \dots, p_q)$ be the set of predecessors of V_i ordered by the path from v_1 to v_2 in $C_{i-1} \setminus v_1v_2$. Vertex p_1 and p_q are the *left* and *right predecessors* of V_i respectively, and vertices $p_2, p_3, \dots p_{q-1}$ are called *middle predecessors* of V_i.

Theorem 2. *Let σ be a canonical ordering of an n-vertex m-edge plane 3-connected graph G. Define S as above. Then G has a plane drawing D with at most $m - \max\{\lceil n/2 \rceil - |S| - 3,\ |S|\}$ segments, and at most $m - \max\{n - |S| - 4, |S|\}$ slopes.*

Proof Construction. For every vertex v, let $X(v)$ and $Y(v)$ denote the x and y coordinates of v, respectively. If a vertex v has a neighbour w, such that $X(w) < X(v)$ and $Y(w) < Y(v)$, then we say vw is a *left edge* of v. Similarly, if v has a neighbour w, such that $X(w) > X(v)$ and $Y(w) < Y(v)$, then we say vw is a *right edge* of v. If vw is an edge such that $X(v) = X(w)$ and $Y(v) < Y(w)$, than we say vw is a *vertical edge above* v and *below* w.

We define D inductively on $\sigma = (V_1, V_2, \ldots, V_K)$ as follows. Let D_i denote a drawing of G_i. A vertex v is a *peak in* D_i, if each neighbour w of v has $Y(w) \leq Y(v)$ in D_i. We say that a point p in the plane is *visible in* D_i from vertex $v \in D_i$, if the segment $\overline{pv}$ does not intersect D_i except at v. At the i^{th} induction step, $2 \leq i \leq K$, D_i will satisfy the following invariants:

Invariant 1: $C_i \setminus v_1v_2$ is *strictly X-monotone*; that is, the path from v_1 to v_2 in $C_i \setminus v_1v_2$ has (strictly) increasing X-coordinates.
Invariant 2: Every peak in D_i, $i < K$, has a successor.
Invariant 3: Every representative vertex $v_j \in V_j$, $2 \leq j \leq i$ has a left and a right edge. Moreover, if $|P(V_j)| \geq 3$ then there is a vertical edge below v_j.
Invariant 4: D_i has no edge crossings.

For the base case $i = 2$, position the vertices v_1, v_2 and v_3 at the corners of an equilateral triangle so that $X(v_1) < X(v_3) < X(v_2)$ and $Y(v_1) < Y(v_2) < Y(v_3)$. Draw the division vertices of V_2 on the segment v_1v_3. This drawing of D_2 satisfies all four invariants. Now suppose that we have a drawing of D_{i-1} that satisfies the invariants. There are two cases to consider in the construction of D_i, corresponding to the two cases in the definition of the canonical ordering.

Case 1. $|P(V_i)| \geq 3$: If v_i has a middle predecessor v_j with $|P(V_j)| \geq 3$, let $w = v_j$. Otherwise let w be any middle predecessor of v_i. Let L be the open ray $\{(X(w), y) : y > Y(w)\}$. By invariant 1 for D_{i-1}, there is a point in L that is visible in D_{i-1} from every predecessor of v_i. Represent v_i by such a point, and draw segments between v_i and each of its predecessors. That the resulting drawing D_i satisfies the four invariants can be immediately verified.

Case 2. $|P(V_i)| = 2$: Suppose that $P(V_i) = \{w, u\}$, where w and u are the left and the right predecessors of V_i, respectively. Suppose $Y(w) \geq Y(u)$. (The other case is symmetric.) Let P be the path between w and u on $C_{i-1} \setminus v_1v_2$. As illustrated in Figure 3, let A_i be the region $\{(x, y) : y > Y(w)$ and $X(w) \leq x \leq X(u)\}$. Assume on the contrary that $D_{i-1} \cap A_i \neq \emptyset$. By the monotonicity of D_{i-1}, $P \cap A_i \neq \emptyset$. Let $p \in P \cap A_i$. Since $Y(p) > Y(w) \geq Y(u)$, P is X-monotone and thus has a vertex between w and u that is a peak. By the definition of the canonical ordering σ, the addition of V_i creates a face of G, since V_i is added in the outerface of G_{i-1}. Therefore, each vertex between w and u on P has no successor, and is thus not a peak in D_{i-1} by invariant 2, which is the desired contradiction. Therefore $D_{i-1} \cap A_i = \emptyset$.

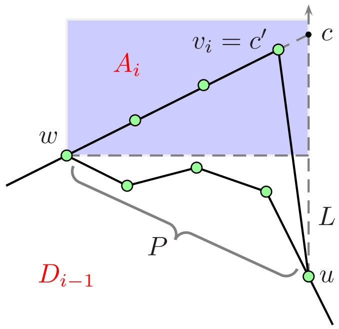

Fig. 3. Illustration for Case 2.

Let L be the open ray $\{(X(u), y) : y > Y(u)\}$. If $w \notin S$, then by invariant 3, w has a left and a right edge in D_{i-1}. Let c be the point of intersection between L and the line extending the left edge at w. If $w \in S$, then let c be any point in A_i on L. By invariant 1, there is a point $c' \notin \{c, w\}$ on $\overline{wc}$ such that c' is visible in D_{i-1} from u. Represent v_i by c', and draw two segments $\overline{v_i u}$ and $\overline{v_i w}$. These two segments do not intersect any part of D_{i-1} (and neither is horizontal). Represent any division vertices in V_i by arbitrary points on the open segment $\overline{wv_i \cap A_i}$. Therefore, in the resulting drawing D_i, there are no crossings and the remaining three invariants are maintained. This completes the construction of D. The analysis for the number of segments and slopes is in [3]. □

Proof (of Theorem 1). Whenever a set V_i is added to G_{i-1}, at least $|V_i| - 1$ edges that are not in G can be added so that the resulting graph is planar. Thus $|S| = \sum_i (|V_i| - 1) \leq 3n - 6 - m$. Hence Theorem 2 implies that G has a plane drawing with at most $m - n/2 + |S| + 3 \leq 5n/2 - 3$ segments, and at most $m - n + |S| - 4 \leq 2n - 10$ slopes. □

Since deleting an edge from a drawing cannot increase the number of slopes, and every plane graph can be triangulated to a 3-connected plane graph, Theorem 1 implies that every n-vertex plane graph has a plane drawing with at most $2n - 10$ slopes. Note that we cannot draw the same conclusion for segments, since deleting an edge in a drawing may increase the number of segments. The famous 'nested-triangles' planar graph leads to the following lower bound.

Lemma 1. *For all $n \equiv 0 \pmod 3$, there is an n-vertex planar triangulation with maximum degree six that has at least $2n-6$ segments in every plane drawing, regardless of the choice of outerface.*

3 Cubic 3-Connected Plane Graphs

A graph in which every vertex has degree three is *cubic*.

Theorem 3. *Every cubic 3-connected plane graph has a plane drawing in which every edge has slope in $\{\pi/4, \pi/2, 3\pi/4\}$, except for three edges on the outerface.*

Proof. Let $\sigma = (V_1, V_2, \ldots, V_K)$ be a canonical ordering of G. We re-use the notation from Theorem 2, except that a representative vertex of V_i may be the first or last vertex in V_i. Since G is cubic, $|P(V_i)| = 2$ for all $1 < i < K$, and every vertex not in $\{v_1, v_2, v_n\}$ has exactly one successor. We proceed by induction on i with the hypothesis that G_i has a plane drawing D_i that satisfies:

Invariant 1: $C_i \backslash v_1v_2$ is X*-monotone*; that is, the path from v_1 to v_2 in $C_i \backslash v_1v_2$ has non-decreasing X-coordinates.
Invariant 2: Every peak in D_i, $i < K$, has a successor.
Invariant 3: If there is a vertical edge above v in D_i, then all the edges of G that are incident to v are in G_i.
Invariant 4: D_i has no edge crossings.

Let D_2 be the drawing of G_2 constructed as follows. Draw v_1v_2 horizontally with $X(v_1) < X(v_2)$. This accounts for one edge whose slope is not in $\{\pi/4, \pi/2, 3\pi/4\}$. Now draw v_1v_3 with slope $\pi/4$, and draw v_2v_3 with slope $3\pi/4$. Add any division vertices on the segment v_1v_3. Now v_3 is the only peak in D_2, and it has a successor by the definition of the canonical ordering. Thus all the invariants are satisfied for the base case D_2.

Now suppose that $2 < i < K$ and we have a drawing of D_{i-1} that satisfies the invariants. Suppose that $P(V_i) = \{u, w\}$, where u and w are the left and the right predecessors of V_i, respectively. Without loss of generality, $Y(w) \leq Y(u)$. Let the representative vertex v_i be last vertex in V_i. Position v_i at the intersection of a vertical segment above w, and a segment of slope $\pi/4$ from u, and add any division vertices on $\overline{uv_i}$, as illustrated in Figure 4(a). Note that there is no vertical edge above w by invariant 3 for D_{i-1}. (For the case in which $Y(u) < Y(w)$, we take the representative vertex v_i to be the first vertex in V_i, and the edge wv_i has slope $3\pi/4$, as illustrated in Figure 4(b).)

Clearly the resulting drawing D_i is X-monotone. Thus invariant 1 is maintained. The vertex v_i is the only peak in D_i that is not a peak in D_{i-1}. Since v_i has a successor by the definition of the canonical ordering, invariant 2 is maintained. The vertical edge wv_i satisfies invariant 3, since v_i is the sole successor of w. Thus invariant 3 is maintained. No vertex between u and w (on the path from u to w in $C_{i-1} \setminus v_1v_2$) is higher than the higher of u and w. Otherwise there would be a peak, not equal to v_n, with no successor, and thus violating invariant 2 for D_{i-1}. Thus the edges in $D_i \setminus D_{i-1}$ do not cross any edges in D_i. In particular, there is no edge ux in D_{i-1} with slope $\pi/4$ and $Y(x) > Y(u)$. The vertex v_n can be easily added to the drawing to complete the construction. □

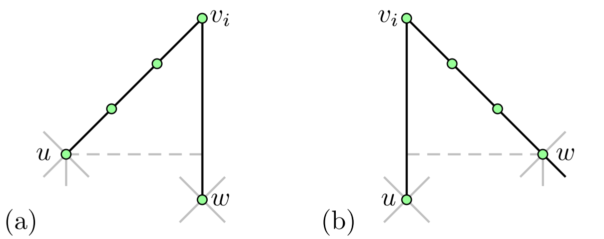

Fig. 4. Construction of a 3-slope drawing of a cubic 3-connected plane graph.

It is easily seen that the bound of six on the number of slopes in Theorem 3 is optimal for any 3-connected cubic plane graph whose outerface is a triangle. An easy variation on the algorithm in Theorem 3 gives:

Corollary 1. *Every cubic* 3*-connected plane graph has a plane 'drawing' with three slopes and three bends on the outerface.*

4 Drawings of General Graphs with Few Slopes

This section is motivated by the following fundamental open problem: Is there a function f such that every graph with maximum degree Δ has a drawing with at most $f(\Delta)$ slopes? This is open even for $\Delta = 3$. Note that:

- The best lower bound that we are aware of is $\Delta + 1$ for the complete graph.
- There is no such function f for convex drawings. Malitz [11] proved that there are Δ-regular n-vertex graphs with book thickness $\Omega(\sqrt{\Delta} n^{1/2-1/\Delta})$. Since book thickness is a lower bound on the number of slopes in a convex drawing, every convex drawing of such a graph has $\Omega(\sqrt{\Delta} n^{1/2-1/\Delta})$ slopes.
- An affirmative solution to this problem would imply that geometric thickness is bounded by maximum degree, which is an open problem due to Eppstein [7]. Duncan *et al.* [5] recently proved that graphs with maximum degree at most four have geometric thickness at most two.

Let H be a (*host*) graph. The vertices of H are called *nodes*. An H*-partition* of a graph G is a function $f : V(G) \to V(H)$ such that for every edge $vw \in E(G)$ we have $f(v) = f(w)$ or $f(v)f(w) \in E(H)$. In the latter case, we say vw is *mapped* to the edge $f(v)f(w)$. The *width* of f is the maximum of $|f^{-1}(x)|$, taken over all nodes $x \in V(H)$, where $f^{-1}(x) = \{v \in V(G) : f(v) = x\}$. In the following result, we describe how to produce a drawing of a graph G given an H-partition of G and a drawing D of H. The general approach is to scale D appropriately, and then replace each node of H by a copy of the drawing of K_k on a regular k-gon. The only difficulty is to scale D so that we obtain a valid drawing of G.

Lemma 2 ([3]). *Let* H *be a graph admitting a drawing* D *with* s *distinct slopes and* ℓ *distinct edge lengths. Let* G *be a graph admitting an* H*-partition of width* k*. Then* G *has a drawing with* $ks\ell(k-1) + k + s$ *slopes.*

Lemma 2 suggests looking at host graphs that admit drawings with few slopes and few edge lengths. Obviously a path has a drawing with one slope and one edge length. Based on this idea, we prove that every graph with bandwidth b has a drawing with at most $\frac{1}{2}b(b+1)+1$ slopes. Based on results from the literature that bound bandwidth in terms of maximum degree Δ, we conclude:

- Every interval graph has a drawing with at most $\frac{1}{2}\Delta(\Delta+1)+1$ slopes.
- Every co-comparability graph (which includes the permutation graphs) has a drawing with at most $\Delta(2\Delta-1)+1$ slopes.
- Every AT-free graph has a drawing with at most $\frac{3}{2}\Delta(3\Delta+1)+1$ slopes.

Lemma 2 motivates the study of drawings of trees with few slopes and few distinct edge lengths.

Lemma 3. *Every tree T with pathwidth $k \geq 1$ has a plane drawing with* $\max\{\Delta(T)-1,1\}$ *slopes and $2k-1$ distinct edge lengths.*

Lemma 4 ([13]). *Every tree T has a path P, called a "backbone", such that $T \setminus V(P)$ has smaller pathwidth than T, and the endpoints of P are leaves of T.*

Proof (of Lemma 3). We refer to T as T_0. Let n_0 be the number of vertices in T_0, and let $\Delta_0 = \Delta(T_0)$. The result holds trivially for $\Delta_0 \leq 2$. Now assume that $\Delta_0 \geq 3$. Let S be the set of slopes $S = \{\frac{\pi}{2}(1+\frac{i}{\Delta_0-2}) : 0 \leq i \leq \Delta_0 - 2\}$. We proceed by induction on n with the hypothesis: "There is a real number $\ell = \ell(n_0, \Delta_0)$, such that for every tree T with $n \leq n_0$ vertices, maximum degree at most Δ_0, and pathwidth $k \geq 1$, and for every vertex r of T with degree less than Δ_0, T has a plane drawing D in which:

- r is at the top of D (that is, no point in D has greater Y-coordinate than r),
- every edge of T has slope in S,
- every edge of T has length in $\{\ell^0, \ell^1, \dots, \ell^{2k-1}\}$, and
- if r is contained in some backbone of T, then every edge of T has length in $\{\ell^0, \ell^1, \dots, \ell^{2k-2}\}$."

The result follows from the induction hypothesis, since we can take r to be the endpoint of a backbone of T_0, in which case $\deg(r) = 1 < \Delta_0$, and thus every edge of T_0 has length in $\{\ell^0, \ell^1, \dots, \ell^{2k-2}\}$.

The base case with $n = 1$ is trivial. Now suppose that the hypothesis is true for trees on less than n vertices, and we are given a tree T with n vertices and pathwidth k, and r is a vertex of T with degree less than Δ_0.

If r is contained in some backbone B of T, then let $P = B$. Otherwise, let P be a path from r to an endpoint of a backbone B of T. Note that P has at least one edge. As illustrated in Figure 5, draw P horizontally with unit-length edges. Every vertex in P has at most $\Delta_0 - 2$ neighbours in $T \setminus V(P)$, since r has degree less than Δ_0 and the endpoints of a backbone are leaves. At each vertex $x \in P$, the children $\{y_0, y_1, \dots, y_{\Delta_0-3}\}$ of x are positioned below P and on the unit-circle centred at x, so that each edge xy_j has slope $\frac{\pi}{2}(1 + j/(\Delta_0 - 2)) \in S$.

Every connected component T' of $T \setminus V(P)$ is a tree rooted at some vertex r' adjacent to a vertex in P. Thus r' has already been positioned in the drawing of T. If T' is a single vertex, then we no longer need to consider this T'.

We consider two types of subtrees T', depending on whether the pathwidth of T' is less than k. Suppose that the pathwidth of T' is k (it cannot be more). Then $T' \cap B \neq \emptyset$ since B is a backbone of T. Thus $T' \cap B$ is a backbone of T' containing r'. Thus we can apply the stronger induction hypothesis in this case.

Every T' has less vertices than T, and every r' has degree less than Δ_0 in T'. Thus by induction, every T' has a drawing with r' at the top, and every edge of T' has slope in S. Furthermore, if the pathwidth of T' is less than k, then every edge of T' has length in $\{\ell^0, \ell^1, \dots, \ell^{2k-3}\}$. Otherwise r' is in a backbone of T', and every edge of T' has length in $\{\ell^0, \ell^1, \dots, \ell^{2k-2}\}$.

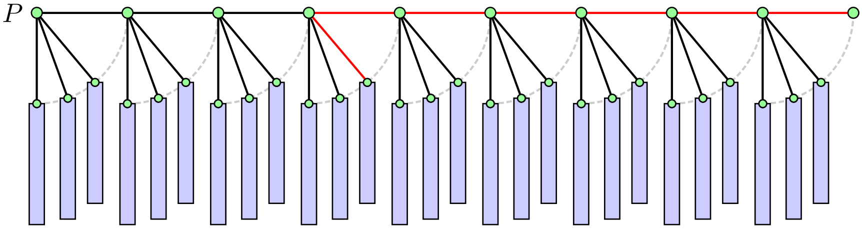

Fig. 5. Drawing of T with few slopes and few edge lengths.

There exists a scale factor $\ell < 1$, depending only on n_0 and Δ_0, so that by scaling the drawings of every T' by ℓ, the widths of the drawings are small enough so that there is no crossings when the drawings are positioned with each r' at its already chosen location. (Note that ℓ is the same value at every level of the induction.) Scaling preserves the slopes of the edges. An edge in any T' that had length ℓ^i before scaling, now has length ℓ^{i+1}.

Case 1. r is contained in some backbone B of T: By construction, $P = B$. So every T' has pathwidth at most $k-1$, and thus every edge of T' has length in $\{\ell^1, \ell^2, \ldots, \ell^{2k-2}\}$. All the other edges of T have unit-length. Thus we have a plane drawing of T with edge lengths $\{\ell^0, \ell^1, \ldots, \ell^{2k-2}\}$, as claimed.

Case 2. r is not contained in any backbone of T: Every edge in every T' has length in $\{\ell^1, \ell^2, \ldots, \ell^{2k-1}\}$. All the other edges of T have unit-length. Thus we have a plane drawing of T with edge lengths $\{\ell^0, \ell^1, \ldots, \ell^{2k-1}\}$, as claimed. □

Theorem 4. *Let G be a graph with n vertices, maximum degree Δ, and tree-width k. Then G has a drawing with $\mathcal{O}(k^3\Delta^4 \log n)$ slopes.*

Proof. Ding and Oporowski [2] proved that for some tree T, G has a T-partition of width at most $\max\{24k\Delta, 1\}$. Let $w = \max\{24k\Delta, 1\}$. For each node $x \in V(T)$, there are at most $w\Delta$ edges of G incident to vertices mapped to x. Hence we can assume that T is a forest with maximum degree at most $w\Delta$, as otherwise there is an edge of T with no edge of G mapped to it, in which case the edge of T can be deleted. Similarly, T has at most n vertices. Now, T has pathwidth at most $\log(2n+1)$ (see [1]). By Lemma 3, T has a drawing with at most $w\Delta - 1$ slopes and at most $2\log(2n+1)-1$ distinct edge lengths. By Lemma 2, G has a drawing in which the number of slopes is at most $w(w\Delta-1)(2\log(2n+1)-1)(w-1) + (w\Delta-1) + w \in \mathcal{O}(w^3\Delta \log n) \subseteq \mathcal{O}(k^3\Delta^4 \log n)$. □

Corollary 2. *Every n-vertex graph with bounded degree and bounded treewidth has a drawing with $\mathcal{O}(\log n)$ slopes.* □

Acknowledgements

Thanks to all of the participants of the Bellairs workshop for creating a stimulating working environment. Special thanks to Mike Fellows for suggesting the problem.

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