Planar Graphs have Bounded Queue-Number*

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Abstract

We show that planar graphs have bounded queue-number, thus proving a conjecture of Heath, Leighton and Rosenberg from 1992. The key to the proof is a new structural tool called layered partitions, and the result that every planar graph has a vertex-partition and a layering, such that each part has a bounded number of vertices in each layer, and the quotient graph has bounded treewidth. This result generalises for graphs of bounded Euler genus. Moreover, we prove that every graph in a minor-closed class has such a layered partition if and only if the class excludes some apex graph. Building on this work and using the graph minor structure theorem, we prove that every proper minor-closed class of graphs has bounded queue-number.

Layered partitions have strong connections to other topics, including the following two examples. First, they can be interpreted in terms of strong products. We show that every planar graph is a subgraph of the strong product of a path with some graph of bounded treewidth. Similar statements hold for all proper minor-closed classes. Second, we give a simple proof of the result by DeVos et al. (2004) that graphs in a proper minor-closed class have low treewidth colourings.

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Contents

1 Introduction

Stacks and queues are fundamental data structures in computer science. But what is more powerful, a stack or a queue? In 1992, Heath, Leighton, and Rosenberg [63] developed a graph-theoretic formulation of this question, where they defined the graph parameters stack-number and queue-number which respectively measure the power of stacks and queues to represent a given graph. Intuitively speaking, if some class of graphs has bounded stack-number and unbounded queue-number, then we would consider stacks to be more powerful than queues for that class (and vice versa). It is known that the stack-number of a graph may be much larger than the queue-number. For example, Heath et al. [63] proved that the n-vertex ternary Hamming graph has queue-number at most $O(\log n)$ and stack-number at least $\Omega(n^{1/9-\epsilon})$. Nevertheless, it is open whether every graph has stack-number bounded by a function of its queue-number, or whether every graph has queue-number bounded by a function of its stack-number [51, 63].

Planar graphs are the simplest class of graphs where it is unknown whether both stack and queue-number are bounded. In particular, Buss and Shor [19] first proved that planar graphs have bounded stack-number; the best known upper bound is 4 due to Yannakakis [106]. However, for the last 27 years of research on this topic, the most important open question in this field has been whether planar graphs have bounded queue-number. This question was first proposed by Heath et al. [63] who conjectured that planar graphs have bounded queue-number.[1] This paper proves this conjecture. Moreover, we generalise this result for graphs of bounded Euler genus, and for every proper minor-closed class of graphs.[2]

First we define the stack-number and queue-number of a graph G. Let $V(G)$ and $E(G)$ respectively denote the vertex and edge set of G. Consider disjoint edges $vw, xy \in E(G)$ in a linear ordering $\preccurlyeq$ of $V(G)$. Without loss of generality, $v \prec w$ and $x \prec y$ and $v \prec x$. Then vw and xy are said to *cross* if $v \prec x \prec w \prec y$ and are said to *nest* if $v \prec x \prec y \prec w$. A *stack* (with respect to $\preccurlyeq$) is a set of pairwise non-crossing edges, and a *queue* (with respect to $\preccurlyeq$) is a set of pairwise non-nested edges. Stacks resemble the stack data structure in the following sense. In a stack, traverse the vertex ordering left-to-right. When visiting vertex v, because of the non-crossing property, if $x_1, \dots, x_d$ are the neighbours of v to the left of v in left-to-right order, then the edges $x_d v, x_{d-1} v, \dots, x_1 v$ will be on top of the stack in this order. Pop these edges off the stack. Then if $y_1, \dots, y_{d'}$ are the neighbours of v to the right of v in left-to-right order, then push $vy_{d'}, vy_{d'-1}, \dots, vy_1$ onto the stack in this order. In this way, a stack of edges with respect to a linear ordering resembles a stack data structure. Analogously, the non-nesting condition in the definition of a queue implies that a queue of edges with respect to a linear ordering resembles a queue data structure.

For an integer $k \geqslant 0$, a *k-stack layout* of a graph G consists of a linear ordering $\preccurlyeq$ of $V(G)$ and a partition $E_1, E_2, \dots, E_k$ of $E(G)$ into stacks with respect to $\preccurlyeq$. Similarly, a *k-queue layout* of G

[1] Curiously, in a later paper, Heath and Rosenberg [66] conjectured that planar graphs have unbounded queue-number.

[2] The *Euler genus* of the orientable surface with h handles is $2h$. The *Euler genus* of the non-orientable surface with c cross-caps is c. The *Euler genus* of a graph G is the minimum integer k such that G embeds in a surface of Euler genus k. Of course, a graph is planar if and only if it has Euler genus 0; see [80] for more about graph embeddings in surfaces. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. A class $\mathcal{G}$ of graphs is *minor-closed* if for every graph $G \in \mathcal{G}$, every minor of G is in $\mathcal{G}$. A minor-closed class is *proper* if it is not the class of all graphs. For example, for fixed $g \geqslant 0$, the class of graphs with Euler genus at most g is a proper minor-closed class.

consists of a linear ordering $\preccurlyeq$ of $V(G)$ and a partition $E_1, E_2, \ldots, E_k$ of $E(G)$ into queues with respect to $\preccurlyeq$. The *stack-number* of G, denoted by $\mathrm{sn}(G)$, is the minimum integer k such that G has a k-stack layout. The *queue-number* of a graph G, denoted by $\mathrm{qn}(G)$, is the minimum integer k such that G has a k-queue layout. Note that k-stack layouts are equivalent to k-page book embeddings, first introduced by Ollmann [81], and stack-number is also called page-number, book thickness, or fixed outer-thickness.

Stack and queue layouts are inherently related to depth-first search and breadth-first search respectively. For example, a DFS ordering of the vertices of a tree has no two crossing edges, and thus defines a 1-stack layout. Similarly, a BFS ordering of the vertices of a tree has no two nested edges, and thus defines a 1-queue layout. Hence every tree has stack-number 1 and queue-number 1.

As mentioned above, Heath et al. [63] conjectured that planar graphs have bounded queue-number. This conjecture has remained open despite much research on queue layouts [2, 11, 32, 33, 45, 46, 48, 49, 51, 62, 63, 65, 85, 89, 98]. We now review progress on this conjecture.

Pemmaraju [85] studied queue layouts and wrote that he "suspects" that a particular planar graph with n vertices has queue-number $\Theta(\log n)$. The example he proposed had treewidth 3; see Section 2.2 for the definition of treewidth. Dujmović et al. [45] proved that graphs of bounded treewidth have bounded queue-number. So Pemmaraju's example in fact has bounded queue-number.

The first $o(n)$ bound on the queue-number of planar graphs with n vertices was proved by Heath et al. [63], who observed that every graph with m edges has a $O(\sqrt{m})$-queue layout using a random vertex ordering. Thus every planar graph with n vertices has queue-number $O(\sqrt{n})$, which can also be proved using the Lipton-Tarjan separator theorem. Di Battista et al. [32] proved the first breakthrough on this topic, by showing that every planar graph with n vertices has queue-number $O(\log^2 n)$. Dujmović [40] improved this bound to $O(\log n)$ with a simpler proof. Building on this work, Dujmović et al. [46] established (poly-)logarithmic bounds for more general classes of graphs. For example, they proved that every graph with n vertices and Euler genus g has queue-number $O(g + \log n)$, and that every graph with n vertices excluding a fixed minor has queue-number $\log^{O(1)} n$.

Recently, Bekos et al. [11] proved a second breakthrough result, by showing that planar graphs with bounded maximum degree have bounded queue-number. In particular, every planar graph with maximum degree Δ has queue-number at most $O(\Delta^6)$. Subsequently, Dujmović, Morin, and Wood [47] proved that the algorithm of Bekos et al. [11] in fact produces a $O(\Delta^2)$-queue layout. This was the state of the art prior to the current work.[3]

1.1 Main Results

The fundamental contribution of this paper is to prove the conjecture of Heath et al. [63] that planar graphs have bounded queue-number.

Theorem 1. *The queue-number of planar graphs is bounded.*

[3]Wang [97] claimed to prove that planar graphs have bounded queue-number, but despite several attempts, we have not been able to understand the claimed proof.

The best upper bound that we obtain for the queue-number of planar graphs is 49.

We extend Theorem 1 by showing that graphs with bounded Euler genus have bounded queue-number.

Theorem 2. *Every graph with Euler genus g has queue-number at most $O(g)$.*

The best upper bound that we obtain for the queue-number of graphs with Euler genus g is $4g+49$.

We generalise further to show the following:

Theorem 3. *Every proper minor-closed class of graphs has bounded queue-number.*

These results are obtained through the introduction of a new tool, *layered partitions*, that have applications well beyond queue layouts. Loosely speaking, a layered partition of a graph G consists of a partition $\mathcal{P}$ of $V(G)$ along with a layering of G, such that each part in $\mathcal{P}$ has a bounded number of vertices in each layer (called the *layered width*), and the quotient graph $G/\mathcal{P}$ has certain desirable properties, typically bounded treewidth. Layered partitions are the key tool for proving the above theorems. Subsequent to the initial release of this paper, layered partitions and the results in this paper have been used to solve other problems [15, 26, 43]. For example, our results for layered partitions were used by Dujmović et al. [43] to prove that planar graphs have bounded nonrepetitive chromatic number, thus solving a well-known open problem of Alon, Grytczuk, Hałuszczak, and Riordan [5]. As above, this result generalises for any proper minor-closed class.

1.2 Outline

The remainder of the paper is organized as follows. In Section 2 we review relevant background including treewidth, layerings, and partitions, and we introduce layered partitions.

Section 3 proves a fundamental lemma which shows that every graph that has a partition of bounded layered width has queue-number bounded by a function of the queue-number of the quotient graph.

In Section 4, we prove that every planar graph has a partition of layered width 1 such that the quotient graph has treewidth at most 8. Since graphs of bounded treewidth are known to have bounded queue-number [45], this implies Theorem 1 with an upper bound of 766. We then prove a variant of this result with layered width 3, where the quotient graph is planar with treewidth 3. This variant coupled with a better bound on the queue-number of treewidth-3 planar graphs [2] implies Theorem 1 with an upper bound of 49.

In Section 5, we prove that graphs of Euler genus g have partitions of layered width $O(g)$ such that the quotient graph has treewidth $O(1)$. This immediately implies that such graphs have queue-number $O(g)$. These partitions are also required for the proof of Theorem 3 in Section 6. A more direct argument that appeals to Theorem 1 proves the bound $4g + 49$ in Theorem 2.

In Section 6, we extend our results for layered partitions to the setting of almost embeddable graphs with no apex vertices. Coupled with other techniques, this allows us to prove Theorem 3. We also characterise those minor-closed graph classes with the property that every graph in the class has a partition of bounded layered width such that the quotient has bounded treewidth.

In Section 7, we provide an alternative and helpful perspective on layered partitions in terms of strong products of graphs. With this viewpoint, we derive results about universal graphs that contain all planar graphs. Similar results are obtained for more general classes.

In Section 8, we prove that some well-known non-minor-closed classes of graphs, such as k-planar graphs, also have bounded queue-number.

Section 9 explores further applications and connections. We start off by giving an example where layered partitions lead to a simple proof of a known and difficult result about low treewidth colourings in proper minor-closed classes. Then we point out some of the many connections that layered partitions have with other graph parameters. We also present other implications of our results such as resolving open problems on 3-dimensional graph drawing.

Finally Section 10 summarizes and concludes with open problems and directions for future work.

2 Tools

Undefined terms and notation can be found in Diestel's text [34]. Throughout the paper, we use the notation $\overrightarrow{X}$ to refer to a particular linear ordering of a set X.

2.1 Layerings

The following well-known definitions are key concepts in our proofs, and that of several other papers on queue layouts [11, 45–47, 49]. A *layering* of a graph G is an ordered partition $(V_0, V_1, \dots)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. If $i = j$ then vw is an *intra-level* edge. If $|i - j| = 1$ then vw is an *inter-level* edge.

If r is a vertex in a connected graph G and $V_i := \{v \in V(G) : \mathrm{dist}_G(r, v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \dots)$ is called a *BFS layering* of G. Associated with a BFS layering is a *BFS spanning tree* T obtained by choosing, for each non-root vertex $v \in V_i$ with $i \geqslant 1$, a neighbour w in V_{i-1}, and adding the edge vw to T. Thus $\mathrm{dist}_T(r, v) = \mathrm{dist}_G(r, v)$ for each vertex v of G.

These notions extend to disconnected graphs. If $G_1, \dots, G_c$ are the components of G, and r_j is a vertex in G_j for $j \in \{1, \dots, c\}$, and $V_i := \bigcup_{j=1}^{c} \{v \in V(G_j) : \mathrm{dist}_{G_j}(r_j, v) = i\}$ for all $i \geqslant 0$, then $(V_0, V_1, \dots)$ is called a *BFS layering* of G.

2.2 Treewidth and Layered Treewidth

First we introduce the notion of H-decomposition and tree-decomposition. For graphs H and G, an *H-decomposition* of G consists of a collection $(B_x \subseteq V(G) : x \in V(H))$ of subsets of $V(G)$, called *bags*, indexed by the vertices of H, and with the following properties:

- for every vertex v of G, the set $\{x \in V(H) : v \in B_x\}$ induces a non-empty connected subgraph of H, and
- for every edge vw of G, there is a vertex $x \in V(H)$ for which $v, w \in B_x$.

The *width* of such an H-decomposition is $\max\{|B_x| : x \in V(H)\} - 1$. The elements of $V(H)$ are called *nodes*, while the elements of $V(G)$ are called *vertices*.

A *tree-decomposition* is a T-decomposition for some tree T. The *treewidth* of a graph G is the minimum width of a tree-decomposition of G. Treewidth measures how similar a given graph is to a tree. It is particularly important in structural and algorithmic graph theory; see [13, 61, 87] for surveys. Tree decompositions were introduced by Robertson and Seymour [90]; the more general notion of H-decomposition was introduced by Diestel and Kühn [35].

As mentioned in Section 1, Dujmović et al. [45] first proved that graphs of bounded treewidth have bounded queue-number. Their bound on the queue-number was doubly exponential in the treewidth. Wiechert [98] improved this bound to singly exponential.

Lemma 4 ([98]). *Every graph with treewidth k has queue-number at most $2^k - 1$.*

Alam, Bekos, Gronemann, Kaufmann, and Pupyrev [2] also improved the bound in the case of planar 3-trees. The following lemma that will be useful later is implied by this result and the fact that every planar graph of treewidth at most 3 is a subgraph of a planar 3-tree [76].

Lemma 5 ([2, 76]). *Every planar graph with treewidth at most 3 has queue-number at most 5.*

Graphs with bounded treewidth provide important examples of minor-closed classes. However, planar graphs have unbounded treewidth. For example, the $n \times n$ planar grid graph has treewidth n. So the above results do not resolve the question of whether planar graphs have bounded queue-number.

Dujmović et al. [46] and Shahrokhi [96] independently introduced the following concept. The *layered treewidth* of a graph G is the minimum integer k such that G has a tree-decomposition $(B_x : x \in V(T))$ and a layering $(V_0, V_1, \dots)$ such that $|B_x \cap V_i| \leqslant k$ for every bag B_x and layer V_i. Applications of layered treewidth include graph colouring [46, 70, 77], graph drawing [10, 46], book embeddings [44], and intersection graph theory [96]. The related notion of layered pathwidth has also been studied [10, 41]. Most relevant to this paper, Dujmović et al. [46] proved that every graph with n vertices and layered treewidth k has queue-number at most $O(k \log n)$. They then proved that planar graphs have layered treewidth at most 3, that graphs of Euler genus g have layered treewidth at most $2g + 3$, and more generally that a minor-closed class has bounded layered treewidth if and only if it excludes some apex graph.[4] This implies $O(\log n)$ bounds on the queue-number for all these graphs, and was the basis for the $\log^{O(1)} n$ bound for proper minor-closed classes mentioned in Section 1.

2.3 Partitions and Layered Partitions

The following definitions are central notions in this paper. A *vertex-partition*, or simply *partition*, of a graph G is a set $\mathcal{P}$ of non-empty sets of vertices in G such that each vertex of G is in exactly one element of $\mathcal{P}$. Each element of $\mathcal{P}$ is called a *part*. The *quotient* (sometimes called the *touching pattern*) of $\mathcal{P}$ is the graph, denoted by $G/\mathcal{P}$, with vertex set $\mathcal{P}$ where distinct parts $A, B \in \mathcal{P}$ are adjacent in $G/\mathcal{P}$ if and only if some vertex in A is adjacent in G to some vertex in B.

A partition of G is *connected* if the subgraph induced by each part is connected. In this case, the quotient is the minor of G obtained by contracting each part into a single vertex. Our results

[4] A graph G is *apex* if $G - v$ is planar for some vertex v.

for queue layouts do not depend on the connectivity of partitions. But we consider it to be of independent interest that many of the partitions constructed in this paper are connected. Then the quotient is a minor of the original graph.

A partition $\mathcal{P}$ of a graph G is called an *H-partition* if H is a graph that contains a spanning subgraph isomorphic to the quotient $G/\mathcal{P}$. Alternatively, an *H-partition* of a graph G is a partition $(A_x : x \in V(H))$ of $V(G)$ indexed by the vertices of H, such that for every edge $vw \in E(G)$, if $v \in A_x$ and $w \in A_y$ then $x = y$ (and vw is called an *intra-bag* edge) or $xy \in E(H)$ (and vw is called an *inter-bag* edge). The *width* of such an H-partition is $\max\{|A_x| : x \in V(H)\}$. Note that a layering is equivalent to a path-partition.

A *tree-partition* is a T-partition for some tree T. Tree-partitions are well studied with several applications [14, 36, 37, 95, 102]. For example, every graph with treewidth k and maximum degree Δ has a tree-partition of width $O(k\Delta)$; see [36, 102]. This easily leads to a $O(k\Delta)$ upper bound on the queue-number [45]. However, dependence on Δ seems unavoidable when studying tree-partitions [102], so we instead consider H-partitions where H has bounded treewidth greater than 1. This idea has been used by many authors in a variety of applications, including cops and robbers [7], fractional colouring [88, 94], generalised colouring numbers [68], and defective and clustered colouring [70]. See [38, 39] for more on partitions of graphs in a proper minor-closed class.

A key innovation of this paper is to consider a layered variant of partitions (analogous to layered treewidth being a layered variant of treewidth). The *layered width* of a partition $\mathcal{P}$ of a graph G is the minimum integer ℓ such that for some layering $(V_0, V_1, \dots)$ of G, each part in $\mathcal{P}$ has at most ℓ vertices in each layer V_i.

Throughout this paper we consider partitions with bounded layered width such that the quotient has bounded treewidth. We therefore introduce the following definition. A class $\mathcal{G}$ of graphs is said to *admit bounded layered partitions* if there exist $k, \ell \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has a partition $\mathcal{P}$ with layered width at most ℓ such that $G/\mathcal{P}$ has treewidth at most k. We first show that this property immediately implies bounded layered treewidth.

Lemma 6. *If a graph G has an H-partition with layered width at most ℓ such that H has treewidth at most k, then G has layered treewidth at most $(k+1)\ell$.*

Proof. Let $(B_x : x \in V(T))$ be a tree-decomposition of H with bags of size at most $k+1$. Replace each instance of a vertex v of H in a bag B_x by the part corresponding to v in the H-partition. Keep the same layering of G. Since $|B_x| \leqslant k+1$, we obtain a tree-decomposition of G with layered width at most $(k+1)\ell$. □

Lemma 6 means that any property that holds for graph classes with bounded layered treewidth also holds for graph classes that admit bounded layered partitions. For example, Norin proved that every n-vertex graph with layered treewidth at most k has treewidth less than $2\sqrt{kn}$ (see [46]). With Lemma 6, this implies that if an n-vertex graph G has a partition with layered width ℓ such that the quotient graph has treewidth at most k, then G has treewidth at most $2\sqrt{(k+1)\ell n}$. This in turn leads to $O(\sqrt{n})$ balanced separator theorems for such graphs.

Lemma 6 suggests that having a partition of bounded layered width, whose quotient has bounded treewidth, seems to be a more stringent requirement than having bounded layered treewidth.

Indeed the former structure leads to $O(1)$ bounds on the queue-number, instead of $O(\log n)$ bounds obtained via layered treewidth. That said, it is open whether graphs of bounded layered treewidth have bounded queue-number. It is even possible that graphs of bounded layered treewidth admit bounded layered partitions.

Before continuing, we show that if one does not care about the exact treewidth bound, then it suffices to consider partitions with layered width 1.

Lemma 7. *If a graph G has an H-partition of layered width ℓ with respect to a layering $(V_0, V_1, \dots)$, for some graph H of treewidth at most k, then G has an H'-partition of layered width 1 with respect to the same layering, for some graph H' of treewidth at most $(k+1)\ell - 1$.*

Proof. Let $(A_v : v \in V(H))$ be an H-partition of G of layered width ℓ with respect to $(V_0, V_1, \dots)$, for some graph H of treewidth at most k. Let $(B_x : x \in V(T))$ be a tree-decomposition of H with width at most k. Let H' be the graph obtained from H by replacing each vertex v of H by an ℓ-clique X_v and replacing each edge vw of H by a complete bipartite graph $K_{\ell,\ell}$ between X_v and X_w. For each $x \in V(T)$, let $B'_x := \cup\{X_v : v \in B_x\}$. Observe that $(B'_x : x \in V(T))$ is a tree-decomposition of H' of width at most $(k+1)\ell - 1$. For each vertex v of H, and layer V_i, there are at most ℓ vertices in $A_v \cap V_i$. Assign each vertex in $A_v \cap V_i$ to a distinct element of X_v. We obtain an H'-partition of G with layered width 1, and the treewidth of H is at most $(k+1)\ell - 1$. □

3 Queue Layouts via Layered Partitions

The next lemma is at the heart of all our results about queue layouts.

Lemma 8. *For all graphs H and G, if H has a k-queue layout and G has an H-partition of layered width ℓ with respect to some layering $(V_0, V_1, \dots)$ of G, then G has a $(3\ell k + \lfloor \frac{3}{2}\ell \rfloor)$-queue layout using vertex ordering $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$, where $\overrightarrow{V_i}$ is some ordering of V_i. In particular,*

$$\operatorname{qn}(G) \leqslant 3\ell \operatorname{qn}(H) + \left\lfloor \tfrac{3}{2}\ell \right\rfloor.$$

The next lemma is useful in the proof of Lemma 8.

Lemma 9. *Let $v_1, \dots, v_n$ be the vertex ordering in a 1-queue layout of a graph H. Let G be the graph obtained from H by replacing each vertex v_i by a 'block' B_i of at most ℓ consecutive vertices in the ordering, and by replacing each edge $v_iv_j \in E(H)$ by a complete bipartite graph between B_i and B_j. Then this ordering admits an ℓ-queue layout of G.*

Proof. A *rainbow* in a vertex ordering of a graph G is a set of pairwise nested edges (and thus a matching). Say R is a rainbow in the ordering of $V(G)$. Heath and Rosenberg [65] proved that a vertex ordering of any graph admits a k-queue layout if and only if every rainbow has size at most k. Thus it suffices to prove that $|R| \leqslant \ell$. If the right endpoints of R belong to at least two different blocks, and the left endpoints of R belong to at least two different blocks, then no endpoint of the innermost edge in R and no endpoint of the outermost edge in R are in a common block, implying that the corresponding edges in H have no endpoint in common, and therefore are nested. Since no two edges in H are nested, without loss of generality, the left endpoints of R belong to one block. Hence there are at most ℓ left endpoints of R, implying $|R| \leqslant \ell$, as desired. □

In what follows, the graph G in Lemma 9 is called an *ℓ-blowup* of H.

Proof of Lemma 8. Let $(A_x : x \in V(H))$ be an H-partition of G such that $|A_x \cap V_i| \leqslant \ell$ for all $x \in V(H)$ and $i \geqslant 0$. Let $(x_1, \dots, x_h)$ be the vertex ordering and $E_1, \dots, E_k$ be the queue assignment in a k-queue layout of H.

We now construct a $(3\ell k + \lfloor \frac{3}{2}\ell \rfloor)$-queue layout of G. Order each layer V_i by

$$\overrightarrow{V_i} := A_{x_1} \cap V_i, A_{x_2} \cap V_i, \dots, A_{x_h} \cap V_i,$$

where each set $A_{x_j} \cap V_i$ is ordered arbitrarily. We use the ordering $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$ of $V(G)$ in our queue layout of G. It remains to assign the edges of G to queues. We consider four types of edges, and use distinct queues for edges of each type.

Intra-level intra-bag edges: Let $G^{(1)}$ be the subgraph formed by the edges $vw \in E(G)$, where $v, w \in A_x \cap V_i$ for some $x \in V(H)$ and $i \geqslant 0$. Heath and Rosenberg [65] noted that the complete graph on ℓ vertices has queue-number $\lfloor \frac{\ell}{2} \rfloor$. Since $|A_x \cap V_i| \leqslant \ell$, at most $\lfloor \frac{\ell}{2} \rfloor$ queues suffice for edges in the subgraph of G induced by $A_x \cap V_i$. These subgraphs are separated in $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$. Thus $\lfloor \frac{\ell}{2} \rfloor$ queues suffice for all intra-level intra-bag edges.

Intra-level inter-bag edges: For $\alpha \in \{1, \dots, k\}$ and $i \geqslant 0$, let $G^{(2)}_{\alpha,i}$ be the subgraph of G formed by those edges $vw \in E(G)$ such that $v \in A_x \cap V_i$ and $w \in A_y \cap V_i$ for some edge $xy \in E_\alpha$. Let $Z^{(2)}_\alpha$ be the 1-queue layout of the subgraph $(V(H), E_\alpha)$ of H on all edges in queue α. Observe that $G^{(2)}_{\alpha,i}$ is a subgraph of the graph isomorphic to the ℓ-blowup of $Z^{(2)}_\alpha$. By Lemma 9, $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$ admits an ℓ-queue layout of $G^{(2)}_{\alpha,i}$. As the subgraphs $G^{(2)}_{\alpha,i}$ for fixed α but different i are separated in $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$, ℓ queues suffice for edges in $\bigcup_{i \geqslant 0} G^{(2)}_{\alpha,i}$ for each $\alpha \in \{1, \dots, k\}$. Hence $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$ admits an ℓk-queue layout of the intra-level inter-bag edges.

Inter-level intra-bag edges: Let $G^{(3)}$ be the subgraph of G formed by those edges $vw \in E(G)$ such that $v \in A_x \cap V_i$ and $w \in A_x \cap V_{i+1}$ for some $x \in V(H)$ and $i \geqslant 0$. Consider the graph $Z^{(3)}$ with ordered vertex set

$$z_{0,x_1}, \dots, z_{0,x_h};\; z_{1,x_1}, \dots, z_{1,x_h};\; \dots$$

and edge set $\{z_{i,x} z_{i+1,x} : i \geqslant 0, x \in V(H)\}$. Then no two edges in $Z^{(3)}$ are nested. Observe that $G^{(3)}$ is isomorphic to a subgraph of the ℓ-blowup of $Z^{(3)}$. By Lemma 9, $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$ admits an ℓ-queue layout of the intra-level inter-bag edges.

Inter-level inter-bag edges: We partition these edges into $2k$ sets. For $\alpha \in \{1, \dots, k\}$, let $G^{(4a)}_\alpha$ be the spanning subgraph of G formed by those edges $vw \in E(G)$ where $v \in A_x \cap V_i$ and $w \in A_y \cap V_{i+1}$ for some $i \geqslant 0$ and for some edge xy of H in E_α, with $x \prec y$ in the ordering of H. Similarly, for $\alpha \in \{1, \dots, k\}$, let $G^{(4b)}_\alpha$ be the spanning subgraph of G formed by those edges $vw \in E(G)$ where $v \in A_x \cap V_i$ and $w \in A_y \cap V_{i+1}$ for some $i \geqslant 0$ and for some edge xy of H in E_α, with $y \prec x$ in the ordering of H.

For $\alpha \in \{1, \dots, k\}$, let $Z^{(4a)}_\alpha$ be the graph with ordered vertex set

$$z_{0,x_1}, \dots, z_{0,x_h};\; z_{1,x_1}, \dots, z_{1,x_h};\; \dots$$

and edge set $\{z_{i,x} z_{i+1,y} : i \geqslant 0, x, y \in V(H), xy \in E_\alpha, x \prec y\}$. Suppose that two edges in $Z^{(4a)}$ nest. This is only possible for edges $z_{i,x} z_{i+1,y}$ and $z_{i,p} z_{i+1,q}$, where $z_{i,x} \prec z_{i,p} \prec z_{i+1,q} \prec z_{i+1,y}$.

Thus, in H, we have $x \prec p$ and $q \prec y$. By the definition of $Z^{(4a)}$, we have $x \prec y$ and $p \prec q$. Hence $x \prec p \prec q \prec y$, which contradicts that $xy, pq \in E_\alpha$. Therefore no two edges are nested in $Z^{(4a)}$.

Observe that $G_\alpha^{(4a)}$ is isomorphic to a subgraph of the ℓ-blowup of $Z_\alpha^{(4)}$. By Lemma 9, $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$ admits an ℓ-queue layout of $G_\alpha^{(4a)}$. An analogous argument shows that $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$ admits an ℓ-queue layout of $G_\alpha^{(4b)}$. Hence $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots$ admits a $2k\ell$-queue layout of all the inter-level inter-bag edges.

In total, we use $\lfloor \frac{\ell}{2} \rfloor + k\ell + \ell + 2k\ell$ queues. □

The upper bound of $3\ell \operatorname{qn}(H) + \lfloor \frac{3}{2}\ell \rfloor$ in Lemma 8 is tight, in the sense that the vertex ordering allows for a set of this many pairwise nested edges, and thus at least that many queues are needed.

Lemmas 4 and 8 imply that a graph class that admits bounded layered partitions has bounded queue-number. In particular:

Corollary 10. *If a graph G has a partition $\mathcal{P}$ of layered width ℓ such that $G/\mathcal{P}$ has treewidth at most k, then G has queue-number at most $3\ell(2^k - 1) + \lfloor \frac{3}{2}\ell \rfloor$.*

4 Proof of Theorem 1: Planar Graphs

Our proof that planar graphs have bounded queue-number employs Corollary 10. Thus our goal is to show that planar graphs admit bounded layered partitions, which is achieved in the following key contribution of the paper.

Theorem 11. *Every planar graph G has a connected partition $\mathcal{P}$ with layered width 1 such that $G/\mathcal{P}$ has treewidth at most 8. Moreover, there is such a partition for every BFS layering of G.*

This theorem and Corollary 10 imply that planar graphs have bounded queue-number (Theorem 1) with an upper bound of $3(2^8 - 1) + \lfloor \frac{3}{2}3 \rfloor = 766$.

We now set out to prove Theorem 11. The proof is inspired by the following elegant result of Pilipczuk and Siebertz [86]: Every planar graph G has a partition $\mathcal{P}$ into geodesics such that $G/\mathcal{P}$ has treewidth at most 8. Here, a *geodesic* is a path of minimum length between its endpoints. We consider the following particular type of geodesic. If T is a tree rooted at a vertex r, then a non-empty path $(x_1, \dots, x_p)$ in T is *vertical* if for some $d \geqslant 0$ for all $i \in \{0, \dots, p\}$ we have $\operatorname{dist}_T(x_i, r) = d + i$. The vertex x_1 is called the *upper endpoint* of the path and x_p is its *lower endpoint*. Note that every vertical path in a BFS spanning tree is a geodesic. Thus the next theorem strengthens the result of Pilipczuk and Siebertz [86].

Theorem 12. *Let T be a rooted spanning tree in a connected planar graph G. Then G has a partition $\mathcal{P}$ into vertical paths in T such that $G/\mathcal{P}$ has treewidth at most 8.*

Proof of Theorem 11 assuming Theorem 12. We may assume that G is connected (since if each component of G has the desired partition, then so does G). Let T be a BFS spanning tree of G. By Theorem 12, G has a partition $\mathcal{P}$ into vertical paths in T such that $G/\mathcal{P}$ has treewidth at most 8. Each path in $\mathcal{P}$ is connected and has at most one vertex in each BFS layer corresponding to T. Hence $\mathcal{P}$ is connected and has layered width 1. □

The proof of Theorem 12 is an inductive proof of a stronger statement given in Lemma 13 below. A *plane graph* is a graph embedded in the plane with no crossings. A *near-triangulation* is a plane graph, where the outer-face is a simple cycle, and every internal face is a triangle. For a cycle C, we write $C = [P_1, \dots, P_k]$ if $P_1, \dots, P_k$ are pairwise disjoint non-empty paths in C, and the endpoints of each path P_i can be labelled x_i and y_i so that $y_i x_{i+1} \in E(C)$ for $i \in \{1, \dots, k\}$, where x_{k+1} means x_1. This implies that $V(C) = \bigcup_{i=1}^{k} V(P_i)$.

Lemma 13. *Let G^+ be a plane triangulation, let T be a spanning tree of G^+ rooted at some vertex r on the outer-face of G^+, and let $P_1, \dots, P_k$ for some $k \in \{1, 2, \dots, 6\}$, be pairwise disjoint vertical paths in T such that $F = [P_1, \dots, P_k]$ is a cycle in G^+. Let G be the near-triangulation consisting of all the edges and vertices of G^+ contained in F and the interior of F.*

Then G has a partition $\mathcal{P}$ into paths in G that are vertical in T, such that $P_1, \dots, P_k \in \mathcal{P}$ and the quotient $H := G/\mathcal{P}$ has a tree-decomposition in which every bag has size at most 9 and some bag contains all the vertices of H corresponding to $P_1, \dots, P_k$.

Proof of Theorem 12 assuming Lemma 13. The result is trivial if $|V(G)| < 3$. Now assume $|V(G)| \geqslant 3$. Let r be the root of T. Let G^+ be a plane triangulation containing G as a spanning subgraph with r on the outer-face of G. The three vertices on the outer-face of G are vertical (singleton) paths in T. Thus G^+ satisfies the assumptions of Lemma 13, which implies that G^+ has a partition $\mathcal{P}$ into vertical paths in T such that $G^+/\mathcal{P}$ has treewidth at most 8. Note that $G/\mathcal{P}$ is a subgraph of $G^+/\mathcal{P}$. Hence $G/\mathcal{P}$ has treewidth at most 8. □

Our proof of Lemma 13 employs the following well-known variation of Sperner's Lemma (see [1]):

Lemma 14 (Sperner's Lemma)**.** *Let G be a near-triangulation whose vertices are coloured* $1, 2, 3$*, with the outer-face $F = [P_1, P_2, P_3]$ where each vertex in P_i is coloured i. Then G contains an internal face whose vertices are coloured* $1, 2, 3$.

Proof of Lemma 13. The proof is by induction on $n = |V(G)|$. If $n = 3$, then G is a 3-cycle and $k \leqslant 3$. The partition into vertical paths is $\mathcal{P} = \{P_1, \dots, P_k\}$. The tree-decomposition of H consists of a single bag that contains the $k \leqslant 3$ vertices corresponding to $P_1, \dots, P_k$.

For $n > 3$ we wish to make use of Sperner's Lemma on some (not necessarily proper) 3-colouring of the vertices of G. We begin by colouring the vertices of F, as illustrated in Figure 1. There are three cases to consider:

1. If $k = 1$ then, since F is a cycle, P_1 has at least three vertices, so $P_1 = [v, P_1', w]$ for two distinct vertices v and w. We set $R_1 := v$, $R_2 := P_1'$ and $R_3 := w$.
2. If $k = 2$ then we may assume without loss of generality that P_1 has at least two vertices so $P_1 = [v, P_1']$. We set $R_1 := v$, $R_2 := P_1'$ and $R_3 := P_2$.
3. If $k \in \{3, 4, 5, 6\}$ then we group consecutive paths by taking $R_1 := [P_1, \dots, P_{\lfloor k/3 \rfloor}]$, $R_2 := [P_{\lfloor k/3 \rfloor + 1}, \dots, P_{\lfloor 2k/3 \rfloor}]$ and $R_3 := [P_{\lfloor 2k/3 \rfloor + 1}, \dots, P_k]$. Note that in this case each R_i consists of one or two of $P_1, \dots, P_k$.

For $i \in \{1, 2, 3\}$, colour each vertex in R_i by i. Now, for each remaining vertex v in G, consider the path P_v from v to the root of T. Since r is on the outer-face of G^+, P_v contains at least one vertex of F. If the first vertex of P_v that belongs to F is in R_i then assign the colour i to v. In this way we obtain a 3-colouring of the vertices of G that satisfies the conditions of Sperner's Lemma.

Therefore, by Sperner's Lemma there exists a triangular face $\tau = v_1v_2v_3$ of G whose vertices are coloured $1, 2, 3$ respectively.

For each $i \in \{1, 2, 3\}$, let Q_i be the path in T from v_i to the first ancestor v'_i of v_i in T that is contained in F. Observe that Q_1, Q_2, and Q_3 are disjoint since Q_i consists only of vertices coloured i. Note that Q_i may consist of the single vertex $v_i = v'_i$. Let Q'_i be Q_i minus its final vertex v'_i.

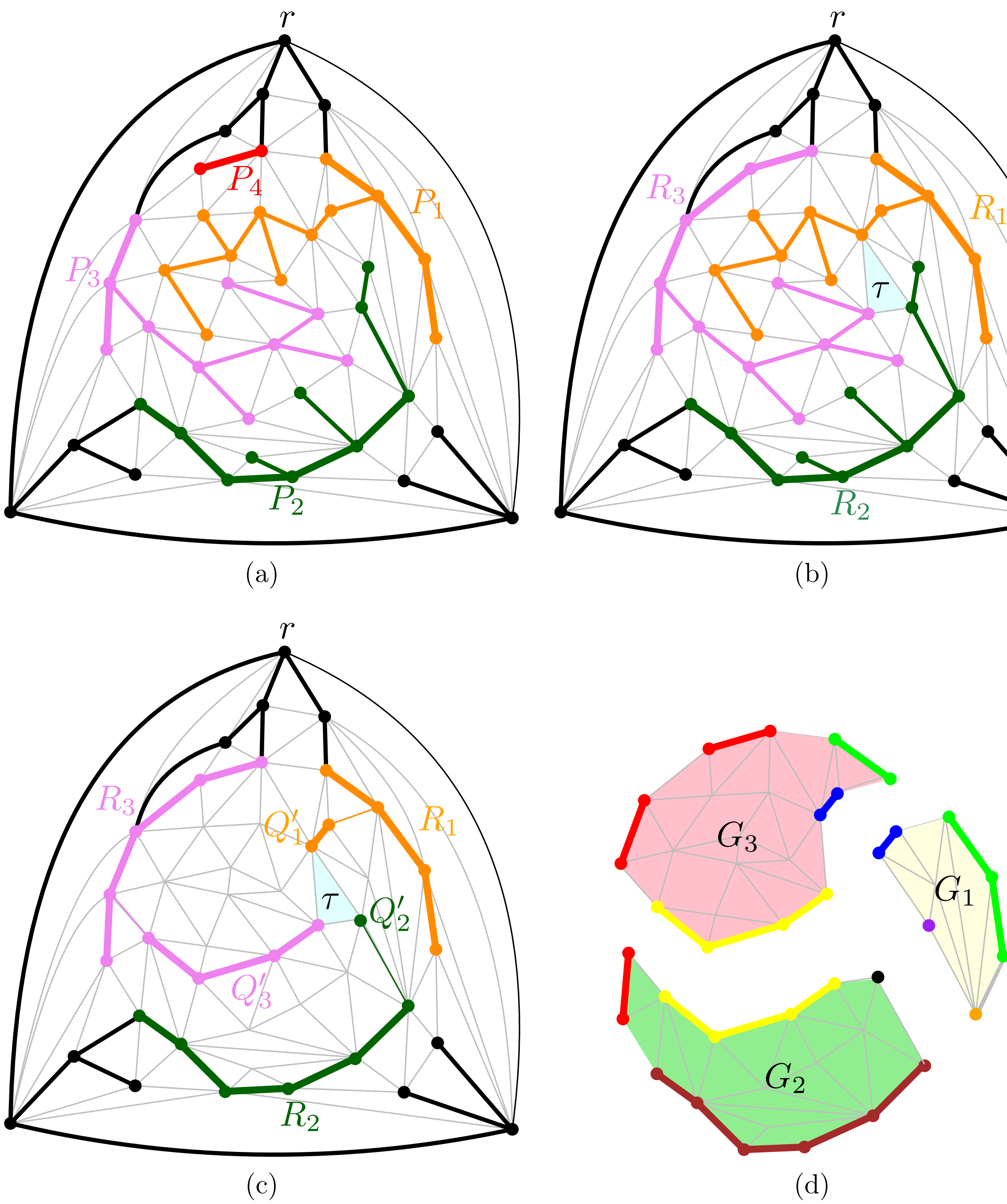

Figure 1: The inductive proof of Lemma 13: (a) the spanning tree T and the paths $P_1, \ldots, P_4$; (b) the paths R_1, R_2, R_3, and the Sperner triangle τ; (c) the paths Q'_1, Q'_2 and Q'_3; (d) the near-triangulations G_1, G_2, and G_3, with the vertical paths of T on F_1, F_2, and F_3.

Imagine for a moment that the cycle F is oriented clockwise, which defines an orientation of R_1, R_2 and R_3. Let R_i^- be the subpath of R_i that contains v_i' and all vertices that precede it, and let R_i^+ be the subpath of R_i that contains v_i' and all vertices that succeed it.

Consider the subgraph of G that consists of the edges and vertices of F, the edges and vertices of τ, and the edges and vertices of $Q_1 \cup Q_2 \cup Q_3$. This graph has an outer-face, an inner face τ, and up to three more inner faces F_1, F_2, F_3 where $F_i = [Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$, where we use the convention that $Q_4 = Q_1$ and $R_4 = R_1$. Note that F_i may be *degenerate* in the sense that $[Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$ may consist only of a single edge $v_i v_{i+1}$.

Consider any non-degenerate $F_i = [Q_i', R_i^+, R_{i+1}^-, Q_{i+1}']$. Note that these four paths are pairwise disjoint, and thus F_i is a cycle. If Q_i' and Q_{i+1}' are non-empty, then each is a vertical path in T. Furthermore, each of R_i^- and R_{i+1}^+ consists of at most two vertical paths in T. Thus, F_i is the concatenation of at most six vertical paths in T. Let G_i be the near-triangulation consisting of all the edges and vertices of G^+ contained in F_i and the interior of F_i. Observe that G_i contains v_i and v_{i+1} but not the third vertex of τ. Therefore F_i satisfies the conditions of the lemma and has fewer than n vertices. So we may apply induction on F_i to obtain a partition $\mathcal{P}_i$ of G_i into vertical paths in T, such that $H_i := G_i/\mathcal{P}_i$ has a tree-decomposition $(B_x^i : x \in V(J_i))$ in which every bag has size at most 9, and some bag $B_{u_i}^i$ contains the vertices of H_i corresponding to the at most six vertical paths that form F_i. We do this for each non-degenerate F_i.

We now construct the desired partition $\mathcal{P}$ of G. Initialise $\mathcal{P} := \{P_1, \dots, P_k\}$. Then add each non-empty Q_i' to $\mathcal{P}$. Now for each non-degenerate F_i, each path in $\mathcal{P}_i$ is either an *external* path (that is, fully contained in F_i) or is an *internal path* with none of its vertices in F_i. Add all the internal paths of $\mathcal{P}_i$ to $\mathcal{P}$. By construction, $\mathcal{P}$ partitions $V(G)$ into vertical paths in T and $\mathcal{P}$ contains $P_1, \dots, P_k$.

Let $H := G/\mathcal{P}$. Next we exhibit the desired tree-decomposition $(B_x : x \in V(J))$ of H. Let J be the tree obtained from the disjoint union of J_i, taken over the $i \in \{1, 2, 3\}$ such that F_i is non-degenarate, by adding one new node u adjacent to each u_i. (Recall that u_i is the node of J_i for which the bag $B_{u_i}^i$ contains the vertices of H_i corresponding to the paths that form F_i.) Let the bag B_u contain all the vertices of H corresponding to $P_1, \dots, P_k, Q_1', Q_2', Q_3'$. For each non-degenerate F_i, and for each node $x \in V(J_i)$, initialise $B_x := B_x^i$. Recall that vertices of H_i correspond to contracted paths in $\mathcal{P}_i$. Each internal path in $\mathcal{P}_i$ also lies in $\mathcal{P}$. Each external path P in $\mathcal{P}_i$ is a subpath of P_j for some $j \in \{1, \dots, k\}$ or is one of the paths among Q_1', Q_2', Q_3'. For each such path P, for every $x \in V(J)$, in bag B_x, replace each instance of the vertex of H_i corresponding to P by the vertex of H corresponding to the path among $P_1, \dots, P_k, Q_1', \dots, Q_3'$ that contains P. This completes the description of $(B_x : x \in V(J))$. By construction, $|B_x| \leqslant 9$ for every $x \in V(J)$.

First we show that for each vertex a in H, the set $X := \{x \in V(J) : a \in B_x\}$ forms a subtree of J. If a corresponds to a path distinct from $P_1, \dots, P_k, Q_1', Q_2', Q_3'$ then X is fully contained in J_i for some $i \in \{1, 2, 3\}$. Thus, by induction X is non-empty and connected in J_i, so it is in J. If a corresponds to P which is one of the paths among $P_1, \dots, P_k, Q_1', Q_2', Q_3'$ then $u \in X$ and whenever X contains a vertex of J_i it is because some external path of $\mathcal{P}_i$ was replaced by P. In particular, we would have $u_i \in X$ in that case. Again by induction each $X \cap J_i$ is connected and since $uu_i \in E(T)$, we conclude that X induces a (connected) subtree of J.

Finally we show that, for every edge ab of H, there is a bag B_x that contains a and b. If a and b

are both obtained by contracting any of $P_1, \dots, P_k, Q'_1, Q'_2, Q'_3$, then a and b both appear in B_u. If a and b are both in H_i for some $i \in \{1,2,3\}$, then some bag B^i_x contains both a and b. Finally, when a is obtained by contracting a path P_a in $G_i - V(F_i)$ and b is obtained by contracting a path P_b not in G_i, then the cycle F_i separates P_a from P_b so the edge ab is not present in H. This concludes the proof that $(B_x : x \in V(J))$ is the desired tree-decomposition of H. □

4.1 Reducing the Bound

We now set out to reduce the constant in Theorem 1 from 766 to 49. This is achieved by proving the following variant of Theorem 11.

Theorem 15. *Every planar graph G has a partition $\mathcal{P}$ with layered width 3 such that $G/\mathcal{P}$ is planar and has treewidth at most 3. Moreover, there is such a partition for every BFS layering of G.*

This theorem with Lemmas 5 and 8 imply that planar graphs have bounded queue-number (Theorem 1) with an upper bound of $3 \cdot 3 \cdot 5 + \lfloor \frac{3}{2} \cdot 3 \rfloor = 49$.

Note that Theorem 15 is stronger than Theorem 11 in that the treewidth bound is smaller, whereas Theorem 11 is stronger than Theorem 15 in that the partition is connected and the layered width is smaller. Also note that Theorem 15 is tight in terms of the treewidth of H: For every ℓ, there exists a planar graph G such that, if G has a partition $\mathcal{P}$ of layered width ℓ, then $G/\mathcal{P}$ has treewidth at least 3. We give this construction at the end of this section, and prove Theorem 15 first. Theorem 11 was proved via an inductive proof of a stronger statement given in Lemma 13. Similarly, the proof of Theorem 15 is via an inductive proof of a stronger statement given in Lemma 17, below.

While Theorem 12 partitions the vertices of a planar graph into vertical paths, to prove Theorem 15 we instead partition the vertices of a triangulation G^+ into parts each of which is a union of up to three vertical paths. Formally, in a rooted spanning tree T of a graph G, a *tripod* consists of up to three pairwise disjoint vertical paths in T whose lower endpoints form a clique in G. Theorem 15 quickly follows from the next result.

Theorem 16. *Let T be a rooted spanning tree in a triangulation G. Then G has a partition $\mathcal{P}$ into tripods in T such that $G/\mathcal{P}$ has treewidth at most 3.*

Proof of Theorem 15 assuming Theorem 16. We may assume that G is connected (since if each component of G has the desired partition, then so does G). Let T be a BFS spanning tree of G. Let $(V_0, V_1, \dots)$ be the BFS layering corresponding to T. Let G' be a plane triangulation containing G as a spanning subgraph. By Theorem 16, G' has a partition $\mathcal{P}$ into tripods in T such that $G'/\mathcal{P}$ is planar with treewidth at most 3. Then $\mathcal{P}$ is a partition of G such that $G/\mathcal{P}$ is planar with treewidth at most 3. Each part in $\mathcal{P}$ corresponds to a tripod, which has at most three vertices in each layer V_i. Hence $\mathcal{P}$ has layered width at most 3. □

Theorem 16 is proved via the following lemma.

Lemma 17. *Let G^+ be a plane triangulation, let T be a spanning tree of G^+ rooted at some vertex r on the boundary of the outer-face of G^+, and let $P_1, \dots, P_k$, for some $k \in \{1,2,3\}$, be pairwise* *disjoint bipods such that $F = [P_1, \ldots, P_k]$ is a cycle in G^+ with r in its exterior. Let G be the near triangulation consisting of all the edges and vertices of G^+ contained in F and the interior of F.*

Then G has a partition $\mathcal{P}$ into tripods such that $P_1, \ldots, P_k \in \mathcal{P}$, and the graph $H := G/\mathcal{P}$ is planar and has a tree-decomposition in which every bag has size at most 4 and some bag contains all the vertices of H corresponding to $P_1, \ldots, P_k$.

Proof of Theorem 16 assuming Lemma 17. Let T be a spanning tree in a triangulation G rooted at vertex v. We may assume that v is on the boundary of the outer-face of G. Let G^+ be the plane triangulation obtained from G by adding one new vertex r into the outer-face of G and adjacent to each vertex on the boundary of the outer-face of G. Let T^+ be the spanning tree of G^+ obtained from T by adding r and the edge rv. Consider T^+ to be rooted at r. Let P_1, P_2, P_3 be the singleton paths consisting of the three vertices on the boundary of the outer-face of G. Then P_1, P_2, P_3 are disjoint bipods such that $F = [P_1, P_2, P_3]$ is a cycle in G^+ with r in its exterior. Moreover, the near triangulation consisting of all the edges and vertices of G^+ contained in F and the interior of F is G itself. Thus G and G^+ satisfy the assumptions of Lemma 17, which implies that G has a partition $\mathcal{P}$ into tripods in T such $G/\mathcal{P}$ has treewidth at most 3. □

The remainder of this section is devoted to proving Lemma 17.

Proof of Lemma 17. This proof follows the same approach as the proof of Lemma 13, by induction on $n = |V(G)|$. We focus mainly on the differences here. The base case $n = 3$ is trivial.

As before we partition the vertices of F into paths R_1, R_2, and R_3. If $k = 3$, then $R_i := P_i$ for $i \in \{1, 2, 3\}$. Otherwise, as before, we split P_1 into two (when $k = 2$) or three (when $k = 1$) paths.

We apply the same colouring as in the proof of Lemma 13. Then Sperner's Lemma gives a face $\tau = v_1v_2v_3$ of G whose vertices are coloured 1, 2, 3 respectively. As in the proof of Lemma 13, we obtain vertical paths Q_1, Q_2, and Q_3 where each Q_i is a path in T from v_i to R_i. Remove the last vertex from each Q_i to obtain (possibly empty) paths Q'_1, Q'_2, and Q'_3. Let Y be the tripod consisting of $Q'_1 \cup Q'_2 \cup Q'_3$ plus the edges of τ between non-empty Q'_1, Q'_2, Q'_3.

As before we consider the graph consisting of the edges and vertices of τ, the edges and vertices of F and the edges and vertices of Q_1, Q_2, Q_3. This graph has up to three internal faces F_1, F_2, F_3 where each $F_i = [Q'_i, R^+_i, R^-_{i+1}, Q'_{i+1}]$ and R^+_i and R^-_i are the same portions of R_i as defined in Lemma 13. Observe that $F_i = [R^+_i, R^-_{i+1}, I_i]$, where R^+_i and R^-_{i+1} are bipods, and I_i is the bipod formed by $Q'_i \cup Q'_{i+1}$. As before, let G_i be the subgraph of G whose vertices and edges are in F_i or its interior.

For $i \in \{1, 2, 3\}$, if F_i is non-empty, then G_i and $F_i = [R^+_i, R^-_{i+1}, I_i]$ satisfy the conditions of the lemma, and G_i has fewer vertices than G. Thus we may apply induction to G_i. (Note that one or two of R^+_i, R^-_{i+1} and I_i may be empty, in which case we apply the inductive hypothesis with $k = 2$ or $k = 1$, respectively.) This gives a partition $\mathcal{P}_i$ of G_i such that $H_i := G_i/\mathcal{P}_i$ satisfies the conclusions of the lemma. Let $(B^i_x : x \in V(J_i))$ be a tree-decomposition of H_i, in which every bag has size at most 4, and some bag $B^i_{u_i}$ contains the vertices of H_i corresponding to R^+_i, R^-_{i+1} and I_i (if they are non-empty).

We construct $\mathcal{P}$ as before. Initialise $\mathcal{P} := \{P_1, \ldots, P_k, Y\}$. Then, for $i \in \{1, 2, 3\}$, each tripod in $\mathcal{P}_i$ is either fully contained in F_i or it is *internal* with none of its vertices in F_i. Add all these internal tripods in $\mathcal{P}_i$ to $\mathcal{P}$. By construction, $\mathcal{P}$ partitions $V(G)$ into tripods. The graph $H := G/\mathcal{P}$ is planar since G is planar and each tripod in $\mathcal{P}$ induces a connected subgraph of G.

Next we produce the tree-decomposition $(B_x : x \in V(J))$ of H that satisfies the requirements of the lemma. Let J be the tree obtained from the disjoint union of J_1, J_2 and J_3 by adding one new node u adjacent to u_1, u_2 and u_3. Let B_u be the set of at most four vertices of H corresponding to $Y, P_1, \dots, P_k$. For $i \in \{1, 2, 3\}$ and for each node $x \in V(J_i)$, initialise $B_x := B_x^i$.

As in the proof of Lemma 13, the resulting structure, $(B_x : x \in V(J))$, is not yet a tree-decomposition of H since some bags may contain vertices of H_i that are not necessarily vertices of H. Note that unlike in Lemma 13 this does not only include elements of $\mathcal{P}_i$ that are contained in F. In particular, I_i is also not an element of $\mathcal{P}$ and thus does not correspond to a vertex of H. We remedy this as follows. For $x \in V(J)$, in bag B_x, replace each instance of the vertex of H_i corresponding to I_i by the vertex of H corresponding to Y. Similarly, by construction, R_i^+ is a subgraph of P_{α_i} for some $\alpha_i \in \{1, \dots, k\}$. For $x \in V(J)$, in bag B_x, replace each instance of the vertex of H_i corresponding to R_i^+ by the vertex of H corresponding to P_{α_i}. Finally, R_{i+1}^- is a subgraph of P_{β_i} for some $\beta_i \in \{1, \dots, k\}$. For $x \in V(J)$, in bag B_x, replace each instance of the vertex of H_i corresponding to R_{i+1}^- by the vertex of H corresponding to P_{β_i}.

This completes the description of $(B_x : x \in V(J))$. Clearly, every bag B_x has size at most 4. The proof that $(B_x : x \in V(J))$ is indeed a tree-decomposition of H is completely analogous to the proof in Lemma 13. $\square$

The following lemma, which is implied by Theorem 15 and Lemmas 4 and 8, will be helpful for generalising our results to bounded genus graphs.

Lemma 18. *For every BFS layering $(V_0, V_1, \dots)$ of a planar graph G, there is a 49-queue layout of G using vertex ordering $\overrightarrow{V_0}, \overrightarrow{V_1}, \dots,$, where $\overrightarrow{V_i}$ is some ordering of V_i, $i \geqslant 0$.*

As promised above, we now show that Theorem 15 is tight in terms of the treewidth of H.

Theorem 19. *For all integers $k \geqslant 2$ and $\ell \geqslant 1$ there is a graph G with treewidth k such that if G has a partition $\mathcal{P}$ with layered width at most ℓ, then $G/\mathcal{P}$ contains K_{k+1} and thus has treewidth at least k. Moreover, if $k = 2$ then G is outer-planar, and if $k = 3$ then G is planar.*

Proof. We proceed by induction on k. Consider the base case with $k = 2$. Let G be the graph obtained from the path on $9\ell^2 + 3\ell$ vertices by adding one dominant vertex v (the so-called *fan graph*). Consider an H-partition $(A_x : x \in V(H))$ of G with layered width at most ℓ. Since v is dominant in G, there are at most three layers, and each part A_x has at most 3ℓ vertices. Say v is in part A_x. Consider deleting A_x from G. This deletes at most $3\ell - 1$ vertices from the path $G - v$. Thus $G - A_x$ is the union of at most 3ℓ paths, with at least $9\ell^2 + 1$ vertices in total. Thus, one such path P in $G - A_x$ has at least $3\ell + 1$ vertices. Thus there is an edge yz in $H - x$, such that $P \cap A_y \neq \emptyset$ and $P \cap A_z \neq \emptyset$. Since v is dominant, x is dominant in H. Hence $\{x, y, z\}$ induces K_3 in H.

Now assume the result for $k - 1$. Thus there is a graph Q with treewidth $k - 1$ such that if Q has an H-partition with width at most ℓ, then H contains K_k. Let G be obtained by taking 3ℓ copies of Q and adding one dominant vertex v. Thus G has treewidth k. Consider an H-partition $(A_x : x \in V(H))$ of G with layered width at most ℓ. Since v is dominant there are at most three layers, and each part has at most 3ℓ vertices. Say v is in part A_x. Since $|A_x| \leqslant 3\ell$, some copy of Q avoids A_x. Thus this copy of Q has an $(H-x)$-partition of layered width at most ℓ. By assumption, $H-x$ contains K_k. Since v is dominant, x is dominant in H. Thus H contains K_{k+1}, as desired.

In the $k=2$ case, G is outer-planar. Thus, in the $k=3$ case, G is planar. □

5 Proof of Theorem 2: Bounded-Genus Graphs

As was the case for planar graphs, our proof that bounded genus graphs have bounded queue-number employs Corollary 10. Thus the goal of this section is to show that our construction of bounded layered partitions for planar graphs can be generalised for graphs of bounded Euler genus. In particular, we show the following theorem of independent interest.

Theorem 20. *Every graph G of Euler genus g has a connected partition $\mathcal{P}$ with layered width at most* $\max\{2g, 1\}$ *such that $G/\mathcal{P}$ has treewidth at most* 9*. Moreover, there is such a partition for every BFS layering of G.*

This theorem and Corollary 10 imply that graphs of Euler genus g have bounded queue-number (Theorem 2) with an upper bound of $3 \cdot 2g \cdot (2^9 - 1) + \left\lfloor \frac{3}{2}\, 2g \right\rfloor = O(g)$.

Note that Theorem 20 is best possible in the following sense. Suppose that every graph G of Euler genus g has a partition $\mathcal{P}$ with layered width at most ℓ such that $G/\mathcal{P}$ has treewidth at most k. By Lemma 6, G has layered treewidth $O(k\ell)$. Dujmović et al. [46] showed that the maximum layered treewidth of graphs with Euler genus g is $\Theta(g)$. Thus $k\ell \geqslant \Omega(g)$.

The rest of this section is devoted to proving Theorem 20. The next lemma is the key to the proof. Many similar results are known in the literature (for example, [20, Lemma 8] or [80, Section 4.2.4]), but none prove exactly what we need.

Lemma 21. *Let G be a connected graph with Euler genus g. For every BFS spanning tree T of G rooted at some vertex r with corresponding BFS layering $(V_0, V_1, \dots)$, there is a subgraph $Z \subseteq G$ with at most $2g$ vertices in each layer V_i, such that Z is connected and $G - V(Z)$ is planar. Moreover, there is a connected planar graph G^+ containing $G - V(Z)$ as a subgraph, and there is a BFS spanning tree T^+ of G^+ rooted at some vertex r^+ with corresponding BFS layering $(W_0, W_1, \dots)$ of G^+, such that $W_i \cap (V(G) \setminus V(Z)) = V_i \setminus V(Z)$ for all $i \geqslant 0$, and $P \cap (V(G) \setminus V(Z))$ is a vertical path in T for every vertical path P in T^+.*

Proof. Fix an embedding of G in a surface of Euler genus g. Say G has n vertices, m edges, and f faces. By Euler's formula, $n-m+f = 2-g$. Let D be the graph with $V(D) = F(G)$, where for each edge e of $G - E(T)$, if f_1 and f_2 are the faces of G with e on their boundary, then there is an edge $f_1 f_2$ in D. (Think of D as the spanning subgraph of G^* consisting of those edges that do not cross edges in T.) Note that $|V(D)| = f = 2-g-n+m$ and $|E(D)| = m-(n-1) = |V(D)|-1+g$. Since T is a tree, D is connected; see [46, Lemma 11] for a proof. Let T^* be a spanning tree of D. Let $Q := E(D) \setminus E(T^*)$. Thus $|Q| = g$. Say $Q = \{v_1w_1, v_2w_2, \dots, v_gw_g\}$. For $i \in \{1, 2, \dots, g\}$, let Z_i be the union of the v_ir-path and the w_ir-path in T, plus the edge v_iw_i. Let $Z := Z_1 \cup Z_2 \cup \dots \cup Z_g$, considered to be a subgraph of G. By construction, Z is connected. Say Z has p vertices and q edges. Since Z consists of a subtree of T plus the g edges in Q, we have $q = p - 1 + g$.

We now describe how to 'cut' along the edges of Z to obtain a new graph G'; see Figure 2. First, each edge e of Z is replaced by two edges e' and e'' in G'. Each vertex of G that is incident with no edges in Z is untouched. Consider a vertex v of G incident with edges $e_1, e_2, \dots, e_d$ in Z in clockwise order. In G' replace v by new vertices $v_1, v_2, \dots, v_d$, where v_i is incident with e'_i, e''_{i+1} and all the edges incident with v clockwise from e_i to e_{i+1} (exclusive). Here e_{d+1} means e_1 and e''_{d+1} means e''_1. This operation defines a cyclic ordering of the edges in G' incident with each vertex (where e''_{i+1} is followed by e'_i in the cyclic order at v_i). This in turn defines an embedding of G' in some orientable surface. (Note that if G is embedded in a non-orientable surface, then the edge signatures for G are ignored in the embedding of G'.) Let Z' be the set of vertices introduced in G' by cutting through vertices in Z.

Say G' has n' vertices and m' edges, and the embedding of G' has f' faces and Euler genus g'. Each vertex v in G with degree d in Z is replaced by d vertices in G'. Each edge in Z is replaced

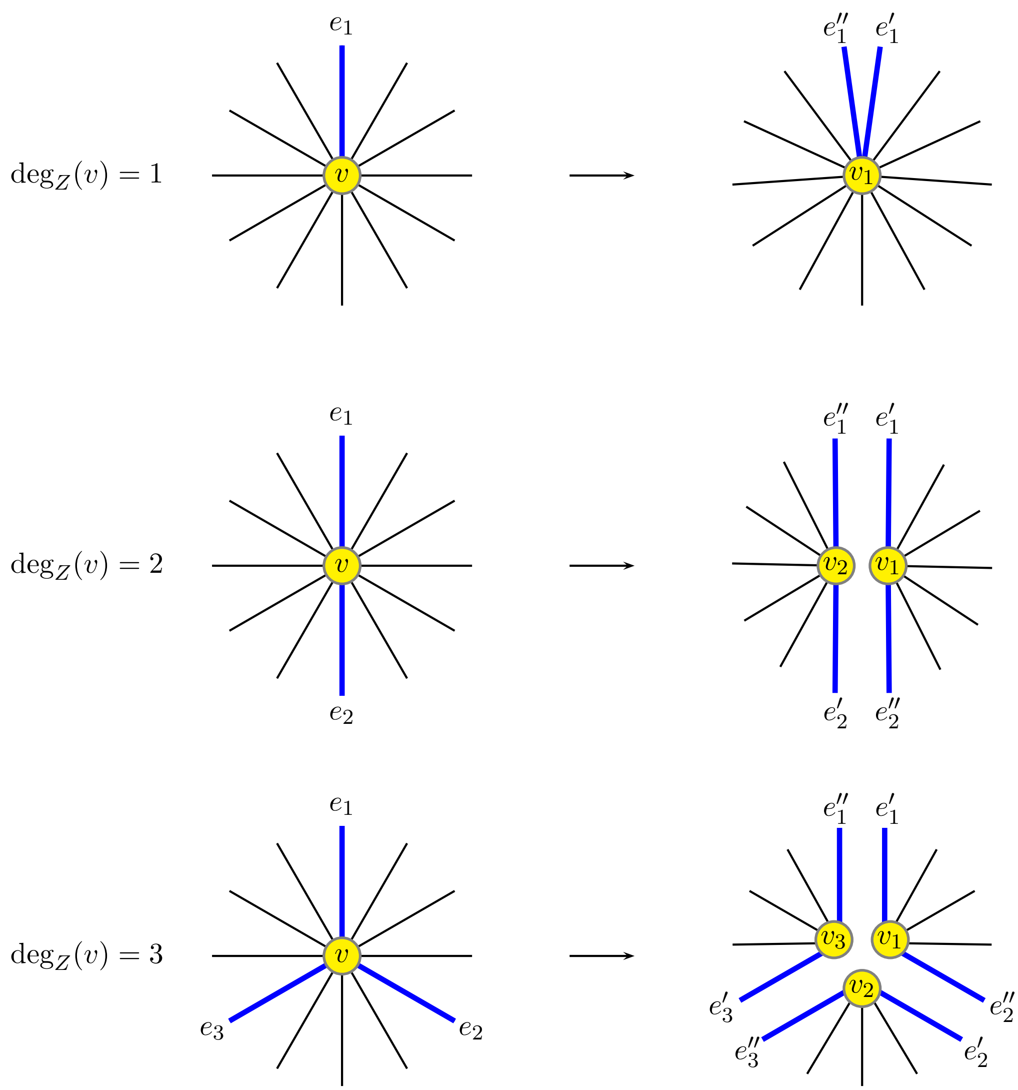

Figure 2: Cutting the blue edges in Z at each vertex.

by two edges in G', while each edge of $G - E(Z)$ is maintained in G'. Thus

$$n' = n - p + \sum_{v \in V(G)} \deg_Z(v) = n + 2q - p = n + 2(p - 1 + g) - p = n + p - 2 + 2g$$

and $m' = m + q = m + p - 1 + g$. Each face of G is preserved in G'. Say s new faces are created by the cutting. Thus $f' = f + s$. Since D is connected, it follows that G' is connected. By Euler's formula, $n' - m' + f' = 2 - g'$. Thus $(n + p - 2 + 2g) - (m + p - 1 + g) + (f + s) = 2 - g'$, implying $(n - m + f) - 1 + g + s = 2 - g'$. Hence $(2 - g) - 1 + g + s = 2 - g'$, implying $g' = 1 - s$. Since $s \geqslant 1$ and $g' \geqslant 0$, we have $g' = 0$ and $s = 1$. Therefore G' is planar, and all the vertices in Z' are on the boundary of a single face, f, of G'.

Note that $G - V(Z)$ is a subgraph of G', and thus $G - V(Z)$ is planar. By construction, each path Z_i has at most two vertices in each layer V_j. Thus Z has at most $2g$ vertices in each V_j.

Now construct a supergraph G'' of G' by adding a vertex r_0 in f and some paths from r_0 to vertices in Z'. Specifically, for each vertex $v_i \in Z'$ corresponding to some vertex $v \in V(Z)$, add to G'' a path Q_{v_i} from r_0 to v_i of length $1 + \mathrm{dist}_G(r, v)$. Note that G'' is planar.

Claim 1. $\mathrm{dist}_{G''}(r_0, v') = 1 + \mathrm{dist}_G(r, v)$ for every vertex v' in G' corresponding to $v \in V(Z)$.

Proof. By construction, $\mathrm{dist}_{G''}(r_0, v') \leqslant 1 + \mathrm{dist}_G(r, v)$, so it is sufficient to show that $\mathrm{dist}_{G''}(r_0, v') \geqslant 1 + \mathrm{dist}_G(r, v)$, which we now do. Let P be a shortest path from r_0 to v' in G''. By construction $P = P_1 P_2$, where P_1 is a path from r_0 to w' of length $1 + \mathrm{dist}_G(r, w)$ for some vertex w' in G' corresponding to $w \in V(Z)$, and P_2 is a path in G' from w' to v' of length $\mathrm{dist}_{G''}(r_0, v') - 1 - \mathrm{dist}_G(r, w)$. By construction, $\mathrm{dist}_G(v, w) \leqslant \mathrm{dist}_{G'}(v', w') \leqslant \mathrm{dist}_{G''}(r_0, v') - 1 - \mathrm{dist}_G(r, w)$. Thus $\mathrm{dist}_G(v, r) \leqslant \mathrm{dist}_G(v, w) + \mathrm{dist}_G(w, r) \leqslant \mathrm{dist}_{G''}(r_0, v') - 1$, as desired. □

Claim 2. $\mathrm{dist}_{G''}(r_0, x) = 1 + \mathrm{dist}_G(r, x)$ for each vertex $x \in V(G) \setminus V(Z)$.

Proof. We first prove that $\mathrm{dist}_{G''}(r_0, x) \leqslant 1 + \mathrm{dist}_G(r, x)$. Let P be a shortest path from x to r in G. Let v be the first vertex in Z on P (which is well defined since r is in Z). So $\mathrm{dist}_G(x, r) = \mathrm{dist}_G(x, v) + \mathrm{dist}_G(v, r)$. Let z be the vertex prior to v on the xv-subpath of P. Then z is adjacent to some copy v' of v in G'. In G'', there is a path from r_0 to v' of length $1 + \mathrm{dist}_G(r, v)$. Thus $\mathrm{dist}_{G''}(r_0, x) \leqslant 1 + \mathrm{dist}_G(r, v) + \mathrm{dist}_G(v, x) = 1 + \mathrm{dist}_G(r, x)$.

We now prove that $\mathrm{dist}_{G''}(r_0, x) \geqslant 1 + \mathrm{dist}_G(r, x)$. Let P be a shortest path from x to r_0 in G''. Let v' be the first vertex not in G on P. Then v' corresponds to some vertex v in Z. Since P is shortest, $\mathrm{dist}_{G''}(r_0, x) = \mathrm{dist}_{G''}(r_0, v') + \mathrm{dist}_{G''}(v', x)$. By Claim 1, $\mathrm{dist}_{G''}(r_0, v') = 1 + \mathrm{dist}_G(r, v)$. By the choice of v, the subpath of P from x to v' corresponds to a shortest path in G from x to v. Thus $\mathrm{dist}_{G''}(v', x) = \mathrm{dist}_G(v, x)$. Combining these equalities, $\mathrm{dist}_{G''}(r_0, x) = 1 + \mathrm{dist}_G(r, v) + \mathrm{dist}_G(v, x) \geqslant 1 + \mathrm{dist}_G(r, x)$, as desired. □

Let T'' be the following spanning tree of G'' rooted at r_0. Initialise T'' to be the union of the above-defined paths Q_{v_i} taken over all vertices $v_i \in Z'$. Consider each edge $vw \in E(T)$ where $v \in Z$ and $w \in V(G) \setminus V(Z)$. Then w is adjacent to exactly one vertex v_i introduced when cutting through v. Add the edge wv_i to T''. Finally, add the induced forest $T[V(G) \setminus V(Z)]$ to T''. Observe that T'' is a spanning tree of G''.

Construct the desired graph G^+ by contracting r_0 and all its neighbours in G'' into a single vertex r^+. Let T^+ be the spanning tree of G^+ obtained from T'' by the same contraction. Then G^+ is planar because G'' is planar. By Claim 2, the BFS layering of G^+ from r^+ satisfies the conditions of the lemma.

Every maximal vertical path in T'' consists of some path Q_{v_i} (where $v_i \in Z'$), followed by some edge $v_i w$ (where $w \in V(G) \setminus V(Z)$, followed by a path in $T[V(G) \setminus V(Z)]$ from w to a leaf in T. Since every vertical path P in T^+ is contained in some maximal vertical path in T'', it follows that $P \cap V(G) \setminus V(Z)$ is a vertical path in T. $\square$

We are now ready to complete the proof of Theorem 20.

Proof of Theorem 20. We may assume that G is connected (since if each component of G has the desired partition, then so does G). Let T be a BFS spanning tree of G rooted at some vertex r with corresponding BFS layering $(V_0, V_1, \dots)$. By Lemma 21, there is a subgraph $Z \subseteq G$ with at most $2g$ vertices in each layer V_i, a connected planar graph G^+ containing $G - V(Z)$ as a subgraph, and a BFS spanning tree T^+ of G^+ rooted at some vertex r^+ with corresponding BFS layering $(W_0, W_1, \dots)$, such that $W_i \cap V(G) \setminus V(Z) = V_i \setminus V(Z)$ for all $i \geqslant 0$, and $P \cap V(G) \setminus V(Z)$ is a vertical path in T for every vertical path P in T^+.

By Theorem 12, G^+ has a partition $\mathcal{P}^+$ into vertical paths in T^+ such that $G^+/\mathcal{P}^+$ has treewidth at most 8. Let $\mathcal{P} := \{P \cap V(G) \setminus V(Z) : P \in \mathcal{P}^+\} \cup \{V(Z)\}$. Thus $\mathcal{P}$ is a partition of G. Since $P \cap V(G) \setminus V(Z)$ is a vertical path in T and Z is a connected subgraph of G, $\mathcal{P}$ is a connected partition. Note that the quotient $G/\mathcal{P}$ is obtained from a subgraph of $G^+/\mathcal{P}^+$ by adding one vertex corresponding to Z. Thus $G/\mathcal{P}$ has treewidth at most 9. Since $P \cap V(G) \setminus V(Z)$ is a vertical path in T, it has at most one vertex in each layer V_i. Thus each part of $\mathcal{P}$ has at most $\max\{2g, 1\}$ vertices in each layer V_i. Hence $\mathcal{P}$ has layered width at most $\max\{2g, 1\}$. $\square$

The same proof in conjunction with Theorem 16 instead of Theorem 12 shows the following.

Theorem 22. *Every graph of Euler genus g has a partition $\mathcal{P}$ with layered width at most* $\max\{2g, 3\}$ *such that $G/\mathcal{P}$ has treewidth at most* 4.

Note that Theorem 22 is stronger than Theorem 20 in that the treewidth bound is smaller, whereas Theorem 20 is stronger than Theorem 22 in that the partition is connected (and the layered width is smaller for $g \in \{0, 1\}$). Both Theorems 20 and 22 (with Lemma 8) imply that graphs with Euler genus g have $O(g)$ queue-number, but better constants are obtained by the following more direct argument that uses Lemma 21 and Theorem 1 to circumvent the use of Theorem 20 and obtain a proof of Theorem 2 with the best known bound.

Proof of Theorem 2 with a $4g + 49$ upper bound. Let G be a graph G with Euler genus g. We may assume that G is connected. Let $(V_0, V_1, \dots, V_t)$ be a BFS layering of G. By Lemma 21, there is a subgraph $Z \subseteq G$ with at most $2g$ vertices in each layer V_i, such that $G - V(Z)$ is planar, and there is a connected planar graph G^+ containing $G - V(Z)$ as a subgraph, such that there is a BFS layering $(W_0, \dots, W_t)$ of G^+ such that $W_i \cap V(G) \setminus V(Z) = V_i \setminus V(Z)$ for all $i \in \{0, 1, \dots, t\}$.

By Lemma 18, there is a 49-queue layout of G^+ with vertex ordering $\overrightarrow{W_0}, \dots, \overrightarrow{W_t}$, where $\overrightarrow{W_i}$ is some ordering of W_i. Delete the vertices of G^+ not in $G - V(Z)$ from this queue layout. We obtain a 49-queue layout of $G - V(Z)$ with vertex ordering $\overrightarrow{V_0 \setminus V(Z)}, \dots, \overrightarrow{V_t \setminus V(Z)}$, where $\overrightarrow{V_i - V(Z)}$ is some ordering $V_i - V(Z)$. Recall that $|V_j \cap V(Z)| \leqslant 2g$ for all $j \in \{0, 1, \dots, t\}$. Let $\overrightarrow{V_j \cap V(Z)}$ be an arbitrary ordering of $V_j \cap V(Z)$. Let $\preccurlyeq$ be the ordering

$$\overrightarrow{V_0 \cap V(Z)}, \overrightarrow{V_0 \setminus V(Z)},\ \overrightarrow{V_1 \cap V(Z)}, \overrightarrow{V_1 \setminus V(Z)},\ \dots, \overrightarrow{V_t \cap V(Z)}, \overrightarrow{V_t \setminus V(Z)}$$

of $V(G)$. Edges of $G - V(Z)$ inherit their queue assignment. We now assign edges incident with vertices in $V(Z)$ to queues. For $i \in \{1, \dots, 2g\}$ and odd $j \geqslant 1$, put each edge incident with the i-th vertex in $\overrightarrow{V_j \cap V(Z)}$ in a new queue S_i. For $i \in \{1, \dots, 2g\}$ and even $j \geqslant 0$, put each edge incident with the i-th vertex in $\overrightarrow{V_j \cap V(Z)}$ (not already assigned to a queue) in a new queue T_i. Suppose that two edges vw and pq in S_i are nested, where $v \prec p \prec q \prec w$. Say $v \in V_a$ and $p \in V_b$ and $q \in V_c$ and $w \in V_d$. By construction, $a \leqslant b \leqslant c \leqslant d$. Since vw is an edge, $d \leqslant a + 1$. At least one endpoint of vw is in $V_j \cap V(Z)$ for some odd j, and one endpoint of pq is in $V_\ell \cap V(Z)$ for some odd ℓ. Since v, w, p, q are distinct, $j \neq \ell$. Thus $|i - j| \geqslant 2$. This is a contradiction since $a \leqslant b \leqslant c \leqslant d \leqslant a + 1$. Thus S_i is a queue. Similarly T_i is a queue. Hence this step introduces $4g$ new queues, and in total we have $4g + 49$ queues. □

6 Proof of Theorem 3: Excluded Minors

This section first introduces the graph minor structure theorem of Robertson and Seymour, which shows that every graph in a proper minor-closed class can be constructed using four ingredients: graphs on surfaces, vortices, apex vertices, and clique-sums. We then use this theorem to prove that every proper minor-closed class has bounded queue-number (Theorem 3).

Let G_0 be a graph embedded in a surface Σ. Let F be a facial cycle of G_0 (thought of as a subgraph of G_0). An *F-vortex* is an F-decomposition $(B_x \subseteq V(H) : x \in V(F))$ of a graph H such that $V(G_0 \cap H) = V(F)$ and $x \in B_x$ for each $x \in V(F)$. For $g, p, a, k \geqslant 0$, a graph G is (g, p, k, a)-*almost-embeddable* if for some set $A \subseteq V(G)$ with $|A| \leqslant a$, there are graphs $G_0, G_1, \dots, G_s$ for some $s \in \{0, \dots, p\}$ such that:

- $G - A = G_0 \cup G_1 \cup \cdots \cup G_s$,
- $G_1, \dots, G_s$ are pairwise vertex-disjoint;
- G_0 is embedded in a surface of Euler genus at most g,
- there are s pairwise vertex-disjoint facial cycles $F_1, \dots, F_s$ of G_0, and
- for $i \in \{1, \dots, s\}$, there is an F_i-vortex $(B_x \subseteq V(G_i) : x \in V(F_i))$ of G_i of width at most k.

The vertices in A are called *apex* vertices. They can be adjacent to any vertex in G.

A graph is *k-almost-embeddable* if it is (k, k, k, k)-almost-embeddable.

Let $C_1 = \{v_1, \dots, v_k\}$ be a k-clique in a graph G_1. Let $C_2 = \{w_1, \dots, w_k\}$ be a k-clique in a graph G_2. Let G be the graph obtained from the disjoint union of G_1 and G_2 by identifying v_i and w_i for $i \in \{1, \dots, k\}$, and possibly deleting some edges in C_1 $(= C_2)$. Then G is a *clique-sum* of G_1 and G_2.

The following graph minor structure theorem by Robertson and Seymour [91] is at the heart of graph minor theory.

Theorem 23 ([91])**.** *For every proper minor-closed class $\mathcal{G}$, there is a constant k such that every graph in $\mathcal{G}$ is obtained by clique-sums of k-almost-embeddable graphs.*

We now set out to show that graphs that satisfy the ingredients of the graph minor structure theorem have bounded queue-number. First consider the case of no apex vertices.

Lemma 24. *Every $(g, p, k, 0)$-almost embeddable graph G has a connected partition $\mathcal{P}$ with layered width at most* $\max\{2g + 4p - 4, 1\}$ *such that $G/\mathcal{P}$ has treewidth at most* $11k + 10$.

Proof. By definition, $G = G_0 \cup G_1 \cup \cdots \cup G_s$ for some $s \leqslant p$, where G_0 has an embedding in a surface of Euler genus g with pairwise disjoint facial cycles $F_1, \dots, F_s$, and there is an F_i-vortex $(B^i_x \subseteq V(G_i) : x \in V(F_i))$ of G_i of width at most k. If $s = 0$ then Theorem 20 implies the result. Now assume that $s \geqslant 1$.

We may assume that G_0 is connected. Fix an arbitrary vertex r in F_1. Let G_0^+ be the graph obtained from G_0 by adding an edge between r and every other vertex in $F_1 \cup \cdots \cup F_s$. Note that we may add $s - 1$ handles, and embed G_0^+ on the resulting surface. Thus G_0^+ has Euler genus at most $g + 2(s - 1) \leqslant g + 2p - 2$.

Let $(V_0, V_1, \dots)$ be a BFS layering of G_0^+ rooted at r. So $V_0 = \{r\}$ and $V(F_1) \cup \cdots \cup V(F_s) \subseteq V_0 \cup V_1$. By Theorem 20, there is a graph H_0 with treewidth at most 9, and there is a connected H_0-partition $(A_x : x \in V(H_0))$ of G_0^+ of layered width at most $\max\{2g + 4p - 4, 1\}$ with respect to $(V_0, V_1, \dots)$. Let $(C_y : y \in V(T))$ be a tree-decomposition of H_0 with width at most 9.

Let $X := \bigcup_{i=1}^{s} V(G_i) \setminus V(G_0)$. Note that $(V_0 \cup X, V_1, V_2, \dots)$ is a layering of G (since all the neighbours of vertices in X are in $V_0 \cup V_1 \cup X$). We now add the vertices in X to the partition of G_0^+ to obtain the desired partition of G. We add each such vertex as a singleton part. Formally, let H be the graph with $V(H) := V(H_0) \cup X$. For each vertex $v \in X$, let $A_v := \{v\}$. Initialise $E(H) := E(H_0)$. For each edge vw in some vortex G_i, if x and y are the vertices of H for which $v \in A_x$ and $w \in A_y$, then add the edge xy to H. Now $(A_x : x \in V(H))$ is a connected H-partition of G with width $\max\{2g + 4p - 4, 1\}$ with respect to $(V_0 \cup X, V_1, V_2, V_3, \dots)$ (since each new part is a singleton).

We now modify the tree-decomposition of H_0 to obtain the desired tree-decomposition of H. Let $(C'_y : y \in V(T))$ be the tree-decomposition of H obtained from $(C_y : y \in V(T))$ as follows. Initialise $C'_y := C_y$ for each $y \in V(T)$. For $i \in \{1, \dots, s\}$ and for each vertex $u \in V(F_i)$ and for each node $y \in V(T)$ with $u \in C_y$, add B^i_u to C'_y. Since $|C_y| \leqslant 10$ and $|B^i_u| \leqslant k + 1$, we have $|C'_y| \leqslant 11(k + 1)$. We now show that $(C'_y : y \in V(T))$ is a tree-decomposition of H. Consider a vertex $v \in X$. So v is in G_i for some $i \in \{1, \dots, s\}$. Let $u_1, \dots, u_t$ be the sequence of vertices in F_i for which $v \in B^i_{u_1} \cap \cdots \cap B^i_{u_t}$. Then $u_1, \dots, u_t$ is a path in G_0. Say x_j is the vertex of H for which $u_j \in A_{x_j}$. Let T_j be the subtree of T corresponding to bags that contain x_j. Since $u_j u_{j+1}$ is an edge of G_0, either $x_j = x_{j+1}$ or $x_j x_{j+1}$ is an edge of H. In each case, by the definition of tree-decomposition, T_j and T_{j+1} share a vertex in common. Thus $T_1 \cup \cdots \cup T_t$ is a (connected) subtree of T. By construction, $T_1 \cup \cdots \cup T_t$ is precisely the subtree of T corresponding to bags that contain v. This show the 'vertex-property' of $(C'_y : y \in V(T))$ holds. Since each edge of $G_1 \cup \cdots \cup G_s$ has both its endpoints in some bag B^i_u, and some bag C'_y contains B^i_u, the 'edge-property' of $(C'_y : y \in V(T))$ also holds. Hence $(C'_y : y \in V(T))$ is a tree-decomposition of H with width at most $11k + 10$. □

Lemmas 8 and 24 imply the following result, where the edges incident to each apex vertex are put in their own queue:

Lemma 25. *Every (g, p, k, a)-almost embeddable graph has queue-number at most*

$$a + 3\max\{2g + 4p - 4, 1\}\, 2^{11k+10} - \left\lceil \tfrac{3}{2} \max\{2g + 4p - 4, 1\} \right\rceil.$$

In particular, for $k \geqslant 1$, every k-almost embeddable graph has queue-number less than $9k \cdot 2^{11(k+1)}$.

We now extend Lemma 25 to allow for clique-sums using some general-purpose machinery of Dujmović et al. [46]. A tree-decomposition $(B_x \subseteq V(G) : x \in V(T))$ of a graph G is *k-rich* if $B_x \cap B_y$ is a clique in G on at most k vertices, for each edge $xy \in E(T)$. Rich tree-decomposition are implicit in the graph minor structure theorem, as demonstrated by the following lemma, which is little more than a restatement of the graph minor structure theorem.

Lemma 26 ([46]). *For every proper minor-closed class $\mathcal{G}$, there are constants $k \geqslant 1$ and $\ell \geqslant 1$, such that every graph $G_0 \in \mathcal{G}$ is a spanning subgraph of a graph G that has a k-rich tree-decomposition such that each bag induces an ℓ-almost-embeddable subgraph of G.*

Dujmović et al. [46] used so-called shadow-complete layerings to establish the following result.[5]

Lemma 27 ([46]). *Let G be a graph that has a k-rich tree-decomposition such that the subgraph induced by each bag has bounded queue-number. Then G has an $f(k)$-queue layout for some function f.*

Theorem 3, which says that every proper minor-closed class has bounded queue-number, is an immediate corollary of Lemmas 25 to 27.

6.1 Characterisation

Bounded layered partitions are the key structure in this paper. So it is natural to ask which minor-closed classes admit bounded layered partitions. The following definition leads to the answer to this question. A graph G is *strongly (g, p, k, a)-almost-embeddable* if it is (g, p, k, a)-almost-embeddable and (using the notation in the definition of (g, p, k, a)-almost-embeddable) there is no edge between an apex vertex and a vertex in $G_0 - (G_1 \cup \cdots \cup G_s)$. That is, each apex vertex is only adjacent to other apex vertices or vertices in the vortices. A graph is *strongly k-almost-embeddable* if it is strongly (k, k, k, k)-almost-embeddable.

Lemma 24 generalises as follows:

Lemma 28. *Every strongly (g, p, k, a)-almost embeddable graph G has a connected partition $\mathcal{P}$ with layered width at most $\max\{2g + 4p - 4, 1\}$ such that $G/\mathcal{P}$ has treewidth at most $11k + a + 10$.*

Proof. By definition, $G - A = G_0 \cup G_1 \cup \cdots \cup G_s$ for some $s \leqslant p$, and for some set $A \subseteq V(G)$ of size at most a, where G_0 has an embedding in a surface of Euler genus g with pairwise disjoint facial cycles $F_1, \ldots, F_s$, such that there is an F_i-vortex $(B^i_x \subseteq V(G_i) : x \in V(F_i))$ of G_i of width at most k, and $N_G(v) \subseteq A \cup \bigcup_{i=1}^{s} V(G_i)$ for each $v \in A$.

[5] In [46], Lemma 27 is expressed in terms of the track-number of a graph. However, it is known that the track-number and the queue-number of a graph are tied; see Section 9.2. So Lemma 27 also holds for queue-number.

As proved in Lemma 24, $G - A$ has a connected partition $\mathcal{P}$ with layered width at most $\max\{2g + 4p - 4, 1\}$ with respect to some layering $(V_0, V_1, V_2, \dots)$ with $\bigcup_{i=1}^{s} V(G_i) \subseteq V_0 \cup V_1$, such that $G/\mathcal{P}$ has treewidth at most $11k + 10$. Thus $(A \cup V_0, V_1, V_2, \dots)$ is a layering of G. Add each vertex in A to the partition as a singleton part. That is, let $\mathcal{P}' := \mathcal{P} \cup \{\{v\} : v \in A\}$. The treewidth of $G/\mathcal{P}'$ is at most the treewidth of $(G - A)/\mathcal{P}$ plus $|A|$. Thus $\mathcal{P}'$ is a connected partition with layered width at most $\max\{2g + 4p - 4, 1\}$ with respect to $(A \cup V_0, V_1, V_2, \dots)$, such that $G/\mathcal{P}$ has treewidth at most $11k + a + 10$. □

Let C be a clique in a graph G, and let $\{C_0, C_1\}$ and $\{P_1, \dots, P_c\}$ be partitions of C. An H-partition $(A_x : x \in V(H))$ and layering $(V_0, V_1, \dots)$ of G is $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$*-friendly* if $C_0 \subseteq V_0$ and $C_1 \subseteq V_1$ and there are vertices $x_1, \dots, x_c$ of H, such that $A_{x_i} = P_i$ for all $i \in \{1, \dots, c\}$. A graph class $\mathcal{G}$ *admits clique-friendly* (k, ℓ)*-partitions* if for every graph $G \in \mathcal{G}$, for every clique C in G, for all partitions $\{C_0, C_1\}$ and $\{P_1, \dots, P_c\}$ of C, there is a $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$-friendly H-partition of G with layered width at most ℓ, such that H has treewidth at most k.

Lemma 29. *Let $(A_x : x \in V(H))$ be an H-partition of G with layered width at most ℓ with respect to some layering $(W_0, W_1, \dots)$ of G, for some graph H with treewidth at most k. Let C be a clique in G, and let $\{C_0, C_1\}$ and $\{P_1, \dots, P_c\}$ be partitions of C such that $|C_j \cap P_i| \leqslant 2\ell$ for each $j \in \{0, 1\}$ and $i \in \{1, \dots, c\}$. Then G has a $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$-friendly $(k + c, 2\ell)$-partition.*

Proof. Since C is a clique, $C \subseteq W_i \cup W_{i+1}$ for some i. Let $V_j := (W_{i-j+1} \cup W_{i+j}) \setminus C_0$ for $j \geqslant 1$. Let $V_0 := C_0$. Thus $(V_0, V_1, \dots)$ is a layering of G and $C_1 \subseteq V_1$. Let H' be obtained from H by adding c dominant vertices $x_1, \dots, x_c$. Thus H' has treewidth at most $k + c$. Let $A'_x := A_x \setminus C$ for $x \in V(H)$. By construction, $|A'_x \cap V_j| \leqslant 2\ell$ for $x \in V(H)$ and $j \geqslant 0$. Let $A'_{x_i} := P_i$ for $i \in \{1, \dots, c\}$. Thus $(A'_x : x \in V(H'))$ is a $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$-friendly H'-partition of G with layered width at most 2ℓ with respect to $(V_0, V_1, \dots)$. □

Every clique in a strongly k-almost embeddable graph has size at most $8k$ (see [46, Lemma 21]). Thus Lemmas 28 and 29 imply:

Corollary 30. *For $k \in \mathbb{N}$, the class of strongly k-almost embeddable graphs admits clique-friendly $(20k + 10, 12k)$-partitions.*

Lemma 31. *Let $\mathcal{G}$ be a class of graphs that admit clique-friendly (k, ℓ)-partitions. Then the class of graphs obtained from clique-sums of graphs in $\mathcal{G}$ admits clique-friendly (k, ℓ)-partitions.*

Proof. Let G be obtained from summing graphs G_1 and G_2 in $\mathcal{G}$ on a clique K. Let C be a clique in G, and let $\{C_0, C_1\}$ and $\{P_1, \dots, P_c\}$ be partitions of C. Our goal is to produce a $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$-friendly (k, ℓ)-partition of G. Without loss of generality, C is in G_1. By assumption, there is a $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$-friendly H_1-partition $(A^1_x : x \in V(H_1))$ of G_1 with layered width ℓ with respect to some layering $(V_0, V_1, \dots)$ of G_1, for some graph H_1 of treewidth at most k. Thus, for some vertices $x_1, \dots, x_c$ of H, we have $A_{x_i} = P_i$ for all $i \in \{1, \dots, c\}$.

Since K is a clique, $K \subseteq V_\kappa \cup V_{\kappa+1}$ for some $\kappa \geqslant 0$. Let $K_j := K \cap V_{\kappa+j}$ for $j \in \{0, 1\}$. Thus K_0, K_1 is a partition of K. Let $y_1, \dots, y_b$ be the vertices of H_1 such that $A^1_{y_i} \cap K \neq \emptyset$. Let $Q_i := A^1_{y_i} \cap K$. Thus $Q_1, \dots, Q_b$ is a partition of K. By assumption, there is a $(K, \{K_0, K_1\}, \{Q_1, \dots, Q_b\})$-friendly H_2-partition $(A^2_x : x \in V(H_2))$ of G_2 with layered width at most ℓ with respect to some layering

$(W_0, W_1, \dots)$ of G_2, for some graph H_2 of treewidth at most k. Thus, for some vertices $z_1, \dots, z_b$ of H_2, we have $A^2_{z_i} = Q_i$ for all $i \in \{1, \dots, b\}$.

Let H be obtained from H_1 and H_2 by identifying y_i and z_i into y_i for $i \in \{1, \dots, b\}$. Since K is a clique, $y_1, \dots, y_b$ is a clique in H_1 and $z_1, \dots, z_b$ is a clique in H_2. Given tree-decompositions of H_1 and H_2 with width at most k, we obtain a tree-decomposition of H by simply adding an edge between a bag that contains $y_1, \dots, y_b$ and a bag that contains $z_1, \dots, z_b$. Thus H has treewidth at most k.

Let $X_a := V_a \cup W_{a-\kappa}$ for $a \geqslant 0$ (where $W_{a-\kappa} = \emptyset$ if $a - \kappa < 0$). Then $(X_0, X_1, \dots)$ is a layering of G, since $K_0 \subseteq V_\kappa \cap W_0$ and $K_1 \subseteq V_{\kappa+1} \cap W_1$. By construction, $C_0 \subseteq V_0 \subseteq X_0$ and $C_1 \subseteq V_1 \subseteq X_1$, as desired.

For $x \in V(H_1)$, let $A_x := A^1_x$. For $x \in V(H_2) \setminus \{z_1, \dots, z_b\}$, let $A_x := A^2_x$. For $i \in \{1, \dots, b\}$, we have $A^2_{z_i} = Q_i \subseteq A^1_{y_i}$. Thus $(A_x : x \in V(H))$ is an H-partition of G with layered width at most ℓ with respect to $(X_0, X_1, \dots)$. Moreover, since $(A^1_x : x \in V(H_1))$ is $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$-friendly with respect to $(V_0, V_1, \dots)$, and $V_i \subseteq X_i$, the partition $(A_x : x \in V(H))$ is $(C, \{C_0, C_1\}, \{P_1, \dots, P_c\})$-friendly with respect to $(X_0, X_1, \dots)$. $\square$

The following is the main result of this section. See [29, 46, 55] for the definition of (linear) local treewidth.

Theorem 32. *The following are equivalent for a minor-closed class of graphs $\mathcal{G}$:*

(1) there exists $k, \ell \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has a partition $\mathcal{P}$ with layered width at most ℓ, such that $G/\mathcal{P}$ has treewidth at most k.
(2) there exists $k \in \mathbb{N}$ such that every graph $G \in \mathcal{G}$ has a partition $\mathcal{P}$ with layered width at most 1, such that $G/\mathcal{P}$ has treewidth at most k.
(3) there exists $k \in \mathbb{N}$ such that every graph in $\mathcal{G}$ has layered treewidth at most k,
(4) $\mathcal{G}$ has linear local treewidth,
(5) $\mathcal{G}$ has bounded local treewidth,
(6) there exists an apex graph not in $\mathcal{G}$,
(7) there exists $k \in \mathbb{N}$ such that every graph in $\mathcal{G}$ is obtained from clique-sums of strongly k-almost-embeddable graphs.

Proof. Lemma 7 says that (1) implies (2). Lemma 6 says that (2) implies (3). Dujmović et al. [46] proved that (3) implies (4), which implies (5) by definition. Eppstein [55] proved that (5) and (6) are equivalent; see [28] for an alternative proof. Dvořák and Thomas [54] proved that (6) implies (7); see Theorem 33 below. Lemma 31 and Corollary 30 imply that every graph obtained from clique-sums of strongly k-almost embeddable graphs has a partition of layered width $12k$ such that the quotient has treewidth at most $20k + 10$. This says that (7) implies (1). $\square$

Note that Demaine and Hajiaghayi [29] previously proved that (3) and (4) are equivalent. Also note that the assumption of a minor-closed class in Theorem 32 is essential: Dujmović, Eppstein, and Wood [42] proved that the $n \times n \times n$ grid G_n has bounded local treewidth but has unbounded, indeed $\Omega(n)$, layered treewidth. By Lemma 6, if G_n has a partition with layered width ℓ such that the quotient has treewidth at most k, then $k\ell \geqslant \Omega(n)$. That said, it is open whether (1), (2) and (3) are equivalent in a subgraph-closed class.

The above proof that (6) implies (7) employed a structure theorem for apex-minor-free graphs by Dvořák and Thomas [54]. Dvořák and Thomas [54] actually proved the following strengthening of the graph minor structure theorem. For a graph X and a surface Σ, let $a(X, \Sigma)$ be the minimum size of a set $S \subseteq V(X)$, such that $X - S$ can be embedded in Σ. Let $a(X) := a(X, \mathbb{S}_0)$ where $\mathbb{S}_0$ is the sphere. Note that $a(X) = 1$ for every apex graph.

Theorem 33 ([54])**.** *For every graph X, there are integers p, k, a, such that every X-minor-free graph G is a clique-sum of graphs $G_1, G_2, \dots, G_n$ such that for $i \in \{1, \dots, n\}$ there exists a surface Σ_i and a set $A_i \subseteq V(G_i)$ satisfying the following:*

- *$|A_i| \leqslant a$,*
- *X cannot be embedded in Σ_i,*
- *$G_i - A_i$ can be almost embedded in Σ_i with at most p vortices of width at most k,*
- *all but at most $a(X, \Sigma_i) - 1$ vertices of A_i are only adjacent in G_i to vertices contained either in A_i or in the vortices.*

Theorem 33 leads to the following result of interest.

Theorem 34. *For every graph X there is an integer k such that every X-minor-free graph G can be obtained from clique-sums of graphs $G_1, G_2, \dots, G_n$ such that for $i \in \{1, 2, \dots, n\}$ there is a set $A_i \subseteq V(G_i)$ of size at most $\max\{a(X) - 1, 0\}$ such that $G_i - A_i$ has a partition $\mathcal{P}_i$ with layered width at most 1, such that $G_i/\mathcal{P}_i$ has treewidth at most k.*

Proof. In Theorem 33, since X cannot be embedded in Σ_i, there is an integer g depending only on X such that Σ_i has Euler genus at most g. Thus each graph G_i has a set A_i of at most $\max\{a(X, \Sigma_i) - 1, 0\} \leqslant \max\{a(X) - 1, 0\}$ vertices, such that $G_i - A_i$ is strongly (g, p, k, a)-almost embeddable. By Lemma 28, $G_i - A_i$ has a partition $\mathcal{P}$ with layered width at most $\max\{2g+4p-4, 1\}$, such that $G/\mathcal{P}$ has treewidth at most $11k + a + 10$. The result follows from Lemma 7. $\square$

7 Strong Products

This section provides an alternative and helpful perspective on layered partitions. The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A) \times V(B)$ are adjacent if:

- $v = w$ and $xy \in E(B)$, or
- $x = y$ and $vw \in E(A)$, or
- $vw \in E(A)$ and $xy \in E(B)$.

The next observation follows immediately from the definitions.

Observation 35. *For every graph H, a graph G has an H-partition of layered width at most ℓ if and only if G is a subgraph of $H \boxtimes P \boxtimes K_\ell$ for some path P.*

Note that a general result about the queue-number of strong products by Wood [99] implies that $\mathrm{qn}(H \boxtimes P) \leqslant 3\,\mathrm{qn}(H) + 1$. Lemma 9 and the fact that $\mathrm{qn}(K_\ell) = \lfloor \frac{\ell}{2} \rfloor$ implies that $\mathrm{qn}(Q \boxtimes K_\ell) \leqslant \ell \cdot \mathrm{qn}(Q) + \lfloor \frac{\ell}{2} \rfloor$. Together these results say that $\mathrm{qn}(H \boxtimes P \boxtimes K_\ell) \leqslant \ell(3\,\mathrm{qn}(H) + 1) + \lfloor \frac{\ell}{2} \rfloor$, which is equivalent to Lemma 8.

Several papers in the literature study minors in graph products [21, 73, 74, 103–105]. The results in this section are complimentary: they show that every graph in certain minor-closed classes is a subgraph of a particular graph product, such as a subgraph of $H \boxtimes P$ for some bounded treewidth graph H and path P. First note that Observation 35 and Theorems 11 and 15 imply the following result conjectured by Wood [101].[6]

Theorem 36. *Every planar graph is a subgraph of:*

(a) $H \boxtimes P$ *for some graph* H *with treewidth at most* 8 *and some path* P.
(b) $H \boxtimes P \boxtimes K_3$ *for some graph* H *with treewidth at most* 3 *and some path* P.

Theorem 36 generalises for graphs of bounded Euler genus as follows. Let $A + B$ be the *complete join* of graphs A and B. That is, take disjoint copies of A and B, and add an edge between each vertex in A and each vertex in B.

Theorem 37. *Every graph of Euler genus* g *is a subgraph of:*

(a) $H \boxtimes P \boxtimes K_{\max\{2g,1\}}$ *for some graph* H *of treewidth at most* 9 *and for some path* P.
(b) $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ *for some graph* H *of treewidth at most* 4 *and for some path* P.
(c) $(K_{2g} + H) \boxtimes P$ *for some graph* H *of treewidth at most* 8 *and some path* P.

Proof. Parts (a) and (b) follow from Observation 35 and Theorems 20 and 22. It remains to prove (c). We may assume that G is edge-maximal with Euler genus g, and is thus connected. Let $(V_0, V_1, \dots)$ be a BFS layering of G. By Lemma 21, there is a subgraph $Z \subseteq G$ with at most $2g$ vertices in each layer V_i, such that $G - V(Z)$ is planar, and there is a connected planar graph G^+ containing $G - V(Z)$ as a subgraph, such that there is a BFS layering $(W_0, W_1, \dots)$ of G^+ such that $W_i \cap V(G) \setminus V(Z) = V_i \setminus V(Z)$ for all $i \geqslant 0$.

By Theorem 11, there is a graph H with treewidth at most 8, such that G^+ has an H-partition $(A_x : x \in V(H))$ of layered width 1 with respect to $(W_0, \dots, W_n)$. Let $A'_x := A_x \cap V(G) \setminus V(Z)$ for each $x \in V(H)$. Thus $(A'_x : x \in V(H))$ is an H-partition of $G - V(Z)$ of layered width 1 with respect to $(V_0 \setminus V(Z), V_1 \setminus V(Z), \dots)$ (since $W_i \cap V(G) \setminus V(Z) = V_i \setminus V(Z)$).

Let $z_1, \dots, z_{2g}$ be the vertices of a complete graph K_{2g}. Say $v_{i,1}, \dots, v_{i,2g}$ are the vertices in $V(Z) \cap V_i$ for $i \geqslant 0$. (Here some $v_{i,j}$ might be undefined.) Define $A'_{z_j} := \{v_{i,j} : i \geqslant 0\}\}$. Now, $(A'_x : x \in V(H + K_{2g}))$ is an $(H + K_{2g})$-partition of G of layered width 1, which is equivalent to the claimed result by Observation 35. □

This result is generalised for (g, p, k, a)-almost embeddable graphs as follows.

Theorem 38. *Every* (g, p, k, a)*-almost embeddable graph is a subgraph of:*

(a) $(H \boxtimes P \boxtimes K_{\max\{2g+4p,1\}}) + K_a$ *for some graph* H *with treewidth at most* $11k + 10$ *and some path* P,
(b) $((H + K_{(2g+4p)(k+1)}) \boxtimes P) + K_a$ *for some graph* H *with treewidth at most* $9k + 8$ *and some path* P.

[6] To be precise, Wood [101] conjectured that for every planar graph G there are graphs X and Y, such that both X and Y have bounded treewidth, Y has bounded maximum degree, and G is a minor of $X \boxtimes Y$, such that the preimage of each vertex of G has bounded radius in $X \boxtimes Y$. Theorem 36(a) is stronger than this conjecture since it has a subgraph rather than a shallow minor, and Y is a path.

Proof. Lemma 24 and Observation 35 imply (a). It remains to prove (b). Let G be a (g, p, k, a)-almost embeddable graph. We use the notation from the definition of (g, p, k, a)-almost embeddable. In the proof of Lemma 24, since G_0^+ has Euler genus at most $g + 2p$, by Theorem 37 there is a graph H_0 with treewidth at most 8, such that G_0^+ has an $(H_0 + (K_{2g+4p})$-partition of layered width 1. That is, $G_0^+ \subseteq (H_0 + K_{2g+4p}) \boxtimes P$. Apply the proof in Lemma 24 to obtain a graph H with treewidth at most $9k + 8$, such that G has an $(H + K_{(2g+4p)(k+1)})$-partition of layered width 1. That is, $G \subseteq (H + K_{(2g+4p)(k+1)}) \boxtimes P$. Adding apex vertices, every (g, p, k, a)-almost embeddable graph is a subgraph of $((H + K_{(2g+4p)(k+1)}) \boxtimes P) + K_a$ for some graph H with treewidth at most $9k + 8$. □

Corollary 39. *For $k \geqslant 1$ every k-almost embeddable graph is a subgraph of:*

(a) $(H \boxtimes P \boxtimes K_{6k}) + K_k$ *for some graph H with treewidth at most $11k + 10$ and some path P,*
(b) $((H + K_{(6k)(k+1)}) \boxtimes P) + K_k$ *for some graph H with treewidth at most $9k + 8$ and some path P.*

Theorems 23 and 38 imply the following result for any proper minor-closed class.

Theorem 40. *For every proper minor-closed class $\mathcal{G}$ there are integers k and a such that every graph $G \in \mathcal{G}$ can be obtained by clique-sums of graphs $G_1, \dots, G_n$ such that for $i \in \{1, \dots, n\}$,*

$$G_i \subseteq (H_i \boxtimes P_i) + K_a,$$

for some graph H_i with treewidth at most k and some path P_i.

Theorem 34 and Observation 35 imply the following precise bound on a for X-minor-free graphs.

Theorem 41. *For every graph X there is an integer k such that every X-minor-free graph G can be obtained by clique-sums of graphs $G_1, \dots, G_n$ such that for $i \in \{1, \dots, n\}$,*

$$G_i \subseteq (H_i \boxtimes P_i) + K_{\max\{a(X)-1,0\}},$$

for some graph H_i with treewidth at most k and some path P_i.

Note that it is easily seen that in all of the above results, the graph H and the path P have at most $|V(G)|$ vertices.

We can interpret these results as saying that strong products and complete joins form universal graphs for the above classes. For all n and k there is a graph $H_{n,k}$ with treewidth k that contains every graph with n vertices and treewidth k as a subgraph (for example, take the disjoint union of all such graphs). The proof of Theorem 36 then shows that $H_{n,8} \boxtimes P_n$ contains every planar graph with n vertices. There is a substantial literature on universal graphs for planar graphs and other classes [3, 4, 8, 12, 17, 18]. For example, Babai, Chung, Erdős, Graham, and Spencer [8] constructed a graph on $O(n^{3/2})$ edges that contains every planar graph on n vertices as a subgraph. While $H_{n,8} \boxtimes P_n$ contains much more than $O(n^{3/2})$ edges, it has the advantage of being highly structured and with bounded average degree. Taking this argument one step further, there is an infinite graph $\mathcal{T}_k$ with treewidth k that contains every (finite) graph with treewidth k as a subgraph. Similarly, the infinite path $\mathcal{Q}$ contains every (finite) path as a subgraph. Thus our results imply that $\mathcal{T}_8 \boxtimes \mathcal{Q}$ contains every planar graph. Analogous statements can be made for the other classes above.

8 Non-Minor-Closed Classes

This section gives three examples of non-minor-closed classes of graphs that have bounded queue-number. The following lemma will be helpful.

Lemma 42. *Let G_0 be a graph with a k-queue layout. Fix integers $c \geqslant 1$ and $\Delta \geqslant 2$. Let G be the graph with $V(G) := V(G_0)$ where $vw \in E(G)$ whenever there is a vw-path P in G_0 of length at most c, such that every internal vertex on P has degree at most Δ. Then*

$$\text{qn}(G) < 2(2k(\Delta + 1))^{c+1}.$$

Proof. Consider a k-queue layout of G_0. Let $\preccurlyeq$ be the corresponding vertex ordering and let $E_1, \ldots, E_k$ be the partition of $E(G_0)$ into queues with respect to $\preccurlyeq$.

For each edge $xy \in E_i$, let $q(xy) := i$. For distinct vertices $a, b \in V(G_0)$, let $f(a, b) := 1$ if $a \prec b$ and let $f(a, b) := -1$ if $b \prec a$. For $\ell \in \{1, \ldots, c\}$, let X_ℓ be the set of edges $vw \in E(G)$ such that the corresponding vw-path P in G_0 has length exactly ℓ. We will use distinct sets of queues for the X_ℓ in our queue layout of G.

By Vizing's Theorem, there is an edge-colouring h of G with $\Delta + 1$ colours, such that any two edges incident with a vertex of degree at most Δ receive distinct colours. (Edges incident with a vertex of degree greater than Δ can be assigned the same colour.)

Consider an edge vw in X_ℓ with $v \prec w$. Say $(v = x_0, x_1, \ldots, x_\ell, x_{\ell+1} = w)$ is the corresponding path in G_0. Let

$$\begin{aligned} f(vw) &:= (f(x_0, x_1), \ldots, f(x_\ell, x_{\ell+1})) \\ q(vw) &:= (q(x_0, x_1), \ldots, q(x_\ell, x_{\ell+1})) \\ h(vw) &:= (h(x_0, x_1), \ldots, h(x_\ell, x_{\ell+1})). \end{aligned}$$

Consider edges $vw, pq \in X_\ell$ with v, w, p, q distinct and $f(vw) = f(pq)$ and $g(vw) = g(pq)$ and $h(vw) = h(pq)$. Assume $v \prec p$. Say $(v = x_0, x_1, \ldots, x_\ell, x_{\ell+1} = w)$ and $(p = y_0, y_1, \ldots, x_\ell, x_{\ell+1} = q)$ are the paths respectively corresponding to vw and pq in G_0. Thus $f(x_i, x_{i+1}) = f(y_i, y_{i+1})$ and $q(x_i x_{i+1}) = q(y_i y_{i+1})$ and $h(x_i x_{i+1}) = h(y_i y_{i+1})$ for $i \in \{0, 1, \ldots, \ell\}$. Thus $x_i x_{i+1}$ and $y_i y_{i+1}$ are not nested. Since $v = x_0 \prec y_0 = p$, we have $x_1 \preccurlyeq y_1$. Since $h(x_0 x_1) = h(y_0 y_1)$ and both x_1 and y_1 have degree at most Δ in G_0, we have $x_1 \prec y_1$. It follows by induction that $x_i \prec y_i$ for $i \in \{0, 1, \ldots, \ell + 1\}$, where in the last step we use the assumption that $w \neq q$. In particular, $w = x_{\ell+1} \prec y_{\ell+1} = q$. Thus vw and pq are not nested. There are $2^{\ell+1}$ values for f, and $k^{\ell+1}$ values for q, and $(\Delta + 1)^{\ell+1}$ values for h. Thus $(2k(\Delta + 1))^{\ell+1}$ queues suffice for X_ℓ. The total number of queues is $\sum_{\ell=1}^{c}(2k(\Delta + 1))^{\ell+1} < 2(2k(\Delta + 1))^{c+1}$. □

8.1 Allowing Crossings

Our result for graphs of bounded Euler genus generalises to allow for a bounded number of crossings per edge. A graph is (g, k)-*planar* if it has a drawing in a surface of Euler genus g with at most k crossings per edge and with no three edges crossing at the same point. A $(0, k)$-planar graph is called k-*planar*; see [72] for a survey about 1-planar graphs. Even in the simplest case, there are 1-planar graphs that contain arbitrarily large complete graph minors [42]. Nevertheless, such graphs have bounded queue-number.

Proposition 43. *Every (g, k)-planar graph G has queue-number at most* $2(40g + 490)^{k+2}$.

Proof. Let G_0 be the graph obtained from G by replacing each crossing point by a vertex. Thus G_0 has Euler genus at most g, and thus has queue-number at most $4g + 49$ by Theorem 2. Note that for every edge vw in G there is a vw-path P in G_0 of length at most $k + 1$, such that every internal vertex has degree 4. The result follows from Lemma 42 with $c = k + 1$ and $\Delta = 4$. □

Proposition 43 can also be concluded from a result of Dujmović and Wood [51] in conjunction with Theorem 2.

8.2 Map Graphs

Map graphs are defined as follows. Start with a graph G_0 embedded in a surface of Euler genus g, with each face labelled a 'nation' or a 'lake', where each vertex of G_0 is incident with at most d nations. Let G be the graph whose vertices are the nations of G_0, where two vertices are adjacent in G if the corresponding faces in G_0 share a vertex. Then G is called a *(g, d)-map graph*. A $(0, d)$-map graph is called a (plane) *d-map graph*; such graphs have been extensively studied [22–24, 27, 58]. The $(g, 3)$-map graphs are precisely the graphs of Euler genus at most g (see [42]). So (g, d)-map graphs provide a natural generalisation of graphs embedded in a surface.

Proposition 44. *Every (g, d)-map graph G has queue-number at most* $2\big((8g + 98)(d + 1)\big)^3$.

Proof. It is known that G is the half-square of a bipartite graph G_0 with Euler genus g (see [42]). This means that G_0 has a bipartition $\{A, B\}$, such that every vertex in B has degree at most k, $V(G) = A$, and for every edge $vw \in E(G)$, there is a common neighbour of v and w in B. By Theorem 2, G_0 has a $(4g + 49)$-queue layout. The result follows from Lemma 42 with $c = 2$ and $\Delta = d$. □

8.3 String Graphs

A *string graph* is the intersection graph of a set of curves in the plane with no three curves meeting at a single point [59, 60, 75, 84, 92, 93]. For an integer $k \geqslant 2$, if each curve is in at most k intersections with other curves, then the corresponding string graph is called a *k-string graph*. A *(g, k)-string* graph is defined analogously for curves on a surface of Euler genus at most g.

Proposition 45. *For all integers $g \geqslant 0$ and $k \geqslant 2$, every (g, k)-string graph has queue-number at most* $2(40g + 490)^{2k+1}$.

Proof. We may assume that in the representation of G, no curve is self-intersecting, no three curves intersect at a common point, and no two curves intersect at an endpoint of one of the curves. Let G_0 be the graph obtained by adding a vertex at the intersection point of any two distinct curves, and at the endpoints of each curve. Each section of a curve between two such vertices becomes an edge in G_0. So G_0 is embedded without crossings and has Euler genus at most g. Associate each vertex v of G with a vertex v_0 of G_0 at the endpoint of the curve representing v. For each edge vw of G, there is v_0w_0-path in G_0 of length at most $2k$, such that every internal vertex on P has degree at most 4. By Theorem 2, G_0 has a $(4g+49)$-queue layout. The result then follows from Lemma 42 with $\Delta = 4$ and $c = 2k$. $\square$

9 Applications and Connections

In this section, we show that layered partitions lead to a simple proof of a known result about low treewidth colourings, and we discuss implications of our results such as resolving open problems about 3-dimensional graph drawings.

9.1 Low Treewidth Colourings

DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan [31] proved that every graph in a proper minor-closed class can be edge 2-coloured so that each monochromatic subgraph has bounded treewidth, and more generally, that for fixed $c \geqslant 2$, every such graph can be edge c-coloured such that the union of any $c-1$ colour classes has bounded treewidth. They also showed analogous vertex-colouring results. (Of course, in both cases, by a colouring we mean a non-proper colouring). Here we show that these results can be easily proved using layered partitions.

Lemma 46. *For every k-almost embeddable graph G and integer $c \geqslant 2$, there are subgraphs $G_1, \dots, G_c$ of G, such that $G = \bigcup_{j=1}^{c} G_j$, and for $j \in \{1, \dots, c\}$ if*

$$X_j := G_1 \cup \dots \cup G_{j-1} \cup G_{j+1} \cup \dots \cup G_c,$$

then X_j has a tree-decomposition $(B_x^j : x \in V(T_j))$ of width at most $66k(k+1)(2c-1)+k-1$, such that for every clique C of G, $C \cap V(X_j)$ is a subset of some bag B_x^j.

Proof. Note that we allow G_i and G_j to have vertices and edges in common. Let A be the set of apex vertices in G (as described in the definition of k-almost embeddable). Thus $|A| \leqslant k$. By Lemma 24, $G - A$ has an H-partition $(Z_h : h \in V(H))$ of layered width at most $6k$, for some graph H with treewidth at most $11k+10$. Let $(V_0, V_1, \dots)$ be the corresponding layering of $G - A$. Let $V_i := \emptyset$ if $i < 0$. For $j \in \{1, \dots, c\}$, let

$$G_j := G\Big[A \cup \bigcup_{i \geqslant 0} V_{2ci+2j-2} \cup V_{2ci+2j-1} \cup V_{2ci+2j}\Big].$$

Note that $G = \bigcup_{j=1}^{c} G_j$, as claimed. For $i \in \mathbb{Z}$ and $j \in \{1, \dots, c\}$, let

$$X_{i,j} := G[V_{2ci+2j} \cup V_{2ci+2j+1} \cup \dots \cup V_{2c(i+1)+2j-2}].$$

Note that $X_j = G[\bigcup_{i \in \mathbb{Z}} V(X_{i,j}) \cup A]$.

Let $(H_x : x \in V(T))$ be a tree-decomposition of H in which every bag has size at most $11(k+1)$. For $i \in \mathbb{Z}$ and $j \in \{1, \dots, c\}$, let $T_{i,j}$ be a copy of T, and let $D_x := \bigcup_{h \in H_x} Z_h \cap V(X_{i,j})$ for each node $x \in V(T_{i,j})$. Then $(D_x : x \in V(T_{i,j}))$ is a tree-decomposition of $X_{i,j}$ because: (1) each vertex v of $X_{i,j}$ is in one part Z_h of our H-partition, and thus v is in precisely those bags corresponding to nodes x of T for which $h \in H_x$, which form a subtree of T; and (2) for each edge vv' of $X_{i,j}$, v is is one part Z_h and v' is in one part $Z_{h'}$ of our H-partition, and thus $h = h'$ or $hh' \in E(H)$, implying that h and h' are in a common bag H_x, and thus v and v' are in a common bag D_x. Since $X_{i,j}$ consists of $2c-1$ layers, and our H-partition has layered width at most $6k$, we have $|Z_h \cap V(X_{i,j})| \leqslant 6k(2c-1)$. Thus $(D_x : x \in V(T_{i,j}))$ has width at most $66k(k+1)(2c-1)-1$.

For $j \in \{1,\dots,c\}$, let $(B^j_x : x \in V(T_j))$ be the tree-decomposition of X_j obtained as follows: First, let T_j be the tree obtained from the disjoint union $\bigcup_{i\in\mathbb{Z}} T_{i,j}$ by adding an edge between $T_{i,j}$ and $T_{i+1,j}$ for all $i \in \mathbb{Z}$. Then for each node x of T_j, let $B^j_x := D_x \cup A$, where D_x is the bag corresponding to x in the tree-decomposition of $X_{i,j}$ where i is such that $x \in V(T_{i,j})$. Since $X_{i,j}$ and $X_{i',j}$ are disjoint for $i \neq i'$, and A is a subset of every bag, $(B^j_x : x \in V(T_j))$ is a tree-decomposition of X_j with width at most $66k(k+1)(2c-1)+k-1$.

A key property of the above construction is that any two consecutive layers intersect X_j in at most one $X_{i,j}$. More precisely, for $\ell \geqslant 0$ and $j \in \{1,\dots,c\}$, either $(V_\ell \cup V_{\ell+1}) \cap V(X_j) = \emptyset$ or $(V_\ell \cup V_{\ell+1}) \cap V(X_j) \subseteq V(X_{i,j})$ for some $i \in \mathbb{Z}$.

Now consider a clique C of G. Our goal is to show that $C \cap V(X_j)$ is a subset of some bag B^j_x for all $j \in \{1,\dots,c\}$. If $(C \setminus A) \cap V(X_j) = \emptyset$ then $C \cap V(X_j) \subseteq A$, implying that $C \cap V(X_j)$ is a subset of every bag B^j_x. Now assume that $(C \setminus A) \cap V(X_j) \neq \emptyset$. By the definition of layering, $C \setminus A \subseteq V_\ell \cup V_{\ell+1}$ for some $\ell \geqslant 0$. By the above key property, $(C \setminus A) \cap V(X_j) \subseteq V(X_{i,j})$ for some $i \in \mathbb{Z}$ (depending on ℓ). Let W be the set of vertices h of H such that $Z_h \cap (C \setminus A) \neq \emptyset$. Since $C \setminus A$ is a clique, W is a clique of H. Hence, some bag H_x of the tree-decomposition of H contains W. By construction, D_x contains $(C \setminus A) \cap V(X_j)$ in the tree-decomposition of $X_{i,j}$, and thus B^j_x contains $C \cap V(X_j)$ in the tree-decomposition of X_j. □

Lemma 47. *Fix an integer $c \geqslant 2$. For $i \in \{1,2\}$, let G^i be a graph for which there are subgraphs $G^i_1,\dots,G^i_c$ satisfying Lemma 46. Let G be a clique-sum of G^1 and G^2. Then G has subgraphs $G_1,\dots,G_c$ satisfying Lemma 46.*

Proof. Let C^1 and C^2 be the cliques respectively in G^1 and G^2 involved in the clique-sum. For $i \in \{1,2\}$, let $X^i_j := G^i_1 \cup \dots \cup G^i_{j-1} \cup G^i_{j+1} \cup \dots \cup G^i_c$. By assumption, $G^i = \bigcup_{j=1}^c G^i_j$, and X^i_j has a tree-decomposition $(B^{i,j}_x : x \in V(T_j))$ of width at most $66k(k+1)(2c-1)+k-1$, such that for every clique C of G^i, $C \cap V(X^i_j)$ is a subset of some bag $B^{i,j}_x$.

For $j \in \{1,\dots,c\}$, let $G_j := G^1_j \cup G^2_j$. This means that for vertices $v_1 \in C^1$ and $v_2 \in C^2$, if v_1 and v_2 are identified into v in the clique-sum, and $v_1 \in V(G^1_j)$ or $v_2 \in V(G^2_j)$, then v is in $V(G_j)$. Similarly, for vertices $v_1, w_1 \in C^1$ and $v_2, w_2 \in C^2$, if v_1 and v_2 are identified into v in the clique-sum, and w_1 and w_2 are identified into w in the clique-sum, and $v_1w_1 \in E(G^1_j)$ or $v_2w_2 \in E(G^2_j)$, then vw is in $E(G_j)$. Take the disjoint union of the tree-decompositions of X^1_j and X^2_j and add an edge between the bags containing $C^1 \cap V(X^1_j)$ and $C^2 \cap V(X^2_j)$ to obtain a tree-decomposition of X_j of width $66k(k+1)(2c-1)+k-1$. Every clique C of G is a clique of G^1 or G^2, and thus $C \cap V(X_j)$ is a subset of some bag of the tree-decomposition of X_j by assumption. □

We now prove the main result of this section.

Theorem 48 ([31])**.** *For every proper minor-closed class $\mathcal{G}$ and integer $c \geqslant 2$, there is a constant k such that every graph in $\mathcal{G}$ can be edge c-coloured or vertex c-coloured so that the union of any $c-1$ colour classes has treewidth at most k.*

Proof. Theorem 23 and Lemmas 46 and 47 imply that there exists an integer k such that every graph $G \in \mathcal{G}$ has subgraphs $G_1, \dots, G_c$, such that $G = \bigcup_{j=1}^{c} G_j$, and for $j \in \{1, \dots, c\}$ the subgraph $X_j := G_1 \cup \dots \cup G_{j-1} \cup G_{j+1} \cup \dots \cup G_c$ has treewidth at most $66k(k+1)(2c-1)+k-1$.

First we prove the edge-colouring result. Colour each edge e of G by an integer j for which $e \in E(G_j)$. The subgraph of G induced by the edges not coloured j is a subgraph of X_j, and thus has treewidth at most $66k(k+1)(2c-1)+k-1$.

For the vertex-colouring result, colour each vertex v of G by an integer j for which $v \in V(G_j)$. The subgraph of G induced by the vertices not coloured j is a subgraph of X_j, and thus has treewidth at most $66k(k+1)(2c-1)+k-1$. □

9.2 Track Layouts

Track layout are a type of graph layout closely related to queue layouts. A *vertex k-colouring* of a graph G is a partition $\{V_1, \dots, V_k\}$ of $V(G)$ into independent sets; that is, for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$ then $i \neq j$. A *track* in G is an independent set equipped with a linear ordering. A partition $\{\overrightarrow{V_1}, \dots, \overrightarrow{V_k}\}$ of $V(G)$ into k tracks is a *k-track layout* if for distinct $i, j \in \{1, \dots, k\}$ no two edges of G cross between $\overrightarrow{V_i}$ and $\overrightarrow{V_j}$. That is, for all distinct edges $vw, xy \in E(G)$ with $v, x \in V_i$ and $w, y \in V_j$, if $v \prec x$ in $\overrightarrow{V_i}$ then $w \preccurlyeq y$ in $\overrightarrow{V_j}$. The minimum k such that G has a k-track layout is called the *track-number* of G, denoted by $\text{tn}(G)$. Dujmović et al. [45] proved the following connection to queue-number.

Lemma 49 ([45])**.** *For every graph G,* $\text{qn}(G) \leqslant \text{tn}(G) - 1$.

The proof of Lemma 49 simply puts the tracks one after the other to produce a queue layout. In this sense, track layouts can be thought of as a richer structure than queue layouts. This structure was the key to an inductive proof by Dujmović et al. [45] that graphs of bounded treewidth have bounded track-number (which implies bounded queue-number by Lemma 49). Nevertheless, Dujmović et al. [48] proved the following converse to Lemma 49:

Lemma 50 ([48])**.** *There is a function f such that* $\text{tn}(G) \leqslant f(\text{qn}(G))$ *for every graph G. In particular, every graph with queue-number at most k has track-number at most*

$$4k \cdot 4^{k(2k-1)(4k-1)}.$$

Lemmas 49 and 50 together say that queue-number and track-number are tied.

The following lemma often gives better bounds on the track-number than Lemma 50. A proper graph colouring is *acyclic* if every cycle gets at least three colours. The *acyclic chromatic number* of a graph G is the minimum integer c such that G has an acyclic c-colouring.

Lemma 51 ([45])**.** *Every graph G with acyclic chromatic number at most c and queue-number at most k has track-number at most $c(2k)^{c-1}$.*

Borodin [16] proved that planar graphs have acyclic chromatic number at most 5, which with Lemma 51 and Theorem 1 implies:

Theorem 52. *Every planar graph has track-number at most* $5(2 \cdot 49)^4 = 461,184,080$.

Note that the best lower bound on the track-number of planar graphs is 7, due to Dujmović et al. [48].

Heawood [67] and Alon, Mohar, and Sanders [6] respectively proved that every graph with Euler genus g has chromatic number $O(g^{1/2})$ and acyclic chromatic number $O(g^{4/7})$. Lemma 51 and Theorem 2 then imply:

Theorem 53. *Every graph with Euler genus g has track-number at most $g^{O(g^{4/7})}$.*

For proper minor-closed classes, Lemma 50 and Theorem 3 imply:

Theorem 54. *Every proper minor-closed class has bounded track-number.*

We now briefly show that (g,k)-planar graphs have bounded track-number. First note that every graph with layered treewidth k has acyclic chromatic number at most $3k$ [*Proof.* Van den Heuvel and Wood [69] proved that every graph with layered treewidth k has r-strong colouring number at most $k(2r+1)$, and it is well known that every graph has acyclic chromatic number at most its 1-strong colouring number.] Dujmović et al. [42] proved that every (g,k)-planar graph G has layered treewidth at most $(4g+6)(k+1)$. Thus G has acyclic chromatic number at most $3(4g+6)(k+1)$, and has bounded track-number by Lemma 51 and Proposition 43. Dujmović et al. [42] also proved that every (g,d)-map graph and every (g,k)-strong graph has bounded layered treewidth. By the same argument, such graphs have bounded track-number.

9.3 Three-Dimensional Graph Drawing

Further motivation for studying queue and track layouts is their connection with 3-dimensional graph drawing. A *3-dimensional grid drawing* of a graph G represents the vertices of G by distinct grid points in $\mathbb{Z}^3$ and represents each edge of G by the open segment between its endpoints so that no two edges intersect. The *volume* of a 3-dimensional grid drawing is the number of grid points in the smallest axis-aligned grid-box that encloses the drawing. For example, Cohen, Eades, Lin, and Ruskey [25] proved that the complete graph K_n has a 3-dimensional grid drawing with volume $O(n^3)$ and this bound is optimal. Pach, Thiele, and Tóth [83] proved that every graph with bounded chromatic number has a 3-dimensional grid drawing with volume $O(n^2)$, and this bound is optimal for $K_{n/2,n/2}$.

Track layouts and 3-dimensional graph drawings are connected by the following lemma.

Lemma 55 ([45, 50])**.** *If a c-colourable n-vertex graph G has a t-track layout, then G has 3-dimensional grid drawings with $O(t^2n)$ volume and with $O(c^7tn)$ volume. Conversely, if a graph G has a 3-dimensional grid drawing with $A\times B\times C$ bounding box, then G has track-number at most $2AB$.*

Lemma 55 is the foundation for all of the following results. Dujmović and Wood [50] proved that every graph with bounded maximum degree has a 3-dimensional grid drawing with volume $O(n^{3/2})$, and the same bound holds for graphs from a proper minor-closed class. In fact, every graph with bounded degeneracy has a 3-dimensional grid drawing with $O(n^{3/2})$ volume [52]. Dujmović et al. [45] proved that every graph with bounded treewidth has a 3-dimensional grid drawing with volume $O(n)$.

Prior to this work, whether planar graphs have 3-dimensional grid drawings with $O(n)$ volume was a major open problem, due to Felsner, Liotta, and Wismath [56, 57]. The previous best known bound on the volume of 3-dimensional grid drawings of planar graphs was $O(n \log n)$ by Dujmović [40]. Lemma 55 and Theorem 52 together resolve the open problem of Felsner et al. [56, 57].

Theorem 56. *Every planar graph with n vertices has a 3-dimensional grid drawing with $O(n)$ volume.*

Lemma 55 and Theorems 53 and 54 imply the following strengthenings of Theorem 56.

Theorem 57. *Every graph with Euler genus g and n vertices has a 3-dimensional grid drawing with $g^{O(g^{4/7})}n$ volume.*

Theorem 58. *For every proper minor-closed class $\mathcal{G}$, every graph in $\mathcal{G}$ with n vertices has a 3-dimensional grid drawing with $O(n)$ volume.*

As shown in Section 9.2, (g, k)-planar graphs, (g, d)-map graphs and (g, k)-string graphs have bounded track-number (for fixed g, k, d). By Lemma 55, such graphs have 3-dimensional grid drawings with $O(n)$ volume.

10 Open Problems

1. What is the maximum queue-number of planar graphs? We can tweak our proof of Theorem 1 to show that every planar graph has queue-number at most 48, but it seems new ideas are required to obtain a significant improvement. The best lower bound on the maximum queue-number of planar graphs is 4, due to Alam et al. [2].

   More generally, does every graph with Euler genus g have $o(g)$ queue-number? Complete graphs provide a $\Theta(\sqrt{g})$ lower bound. Note that every graph with Euler genus g has $O(\sqrt{g})$ stack-number [78].

2. As discussed in Section 1 it is open whether there is a function f such that $\text{sn}(G) \leqslant f(\text{qn}(G))$ for every graph G. Heath et al. [63] proved that every 1-queue graph has stack-number at most 2. Dujmović and Wood [51] showed that there is such a function f if and only if every 2-queue graph has bounded stack-number.

   Similarly, it is open whether there is a function f such that $\text{qn}(G) \leqslant f(\text{sn}(G))$ for every graph G. Heath et al. [63] proved that every 1-stack graph has queue-number at most 2. Since 2-stack graphs are planar, this paper solves the first open case of this question. Dujmović and Wood [51] showed that there is such a function f if and only if every 3-stack graph has bounded queue-number.

3. Ossona de Mendez, Oum, and Wood [82] introduced the following definition: A graph G is said to be *k-close to Euler genus g* if every subgraph H of G has a drawing in a surface of Euler genus g with at most $k\,|E(H)|$ crossings (that is, with $O(k)$ crossings per edge on average). Does every such graph have queue-number at most $f(g, k)$ for some function f?

4. Is there a proof of Theorem 3 that does not use the graph minor structure theorem and with more reasonable bounds?

5. Queue layouts naturally extend to posets. The *cover graph* G_P of a poset P is the undirected graph with vertex set P, where $vw \in E(G)$ if $v <_P w$ and $v <_P x <_P w$ for no $x \in P$ (or $w <_P v$ and $w <_P x <_P v$ for no $x \in P$). Thus the cover graph encodes relations in P that are not implied by transitivity. A k-*queue layout* of a poset P consists of a linear extension $\preccurlyeq$ of P and a partition $E_1, E_2, \ldots, E_k$ of $E(G_P)$ into queues with respect to $\preccurlyeq$. The *queue-number* of a poset P is the minimum integer k such that P has a k-queue layout. Heath and Pemmaraju [64] conjectured that the queue-number of a planar poset is at most its height (the maximum number of pairwise comparable elements). This was disproved by Knauer, Micek, and Ueckerdt [71] who presented a poset of height 2 and queue-number 4. Theorem 1 and results of Knauer, Micek and Ueckerdt imply that planar posets of height h have queue-number $O(h)$; see Theorem 6 in [71]. Heath and Pemmaraju [64] also conjecture that every poset of width w (the maximum number of pairwise incomparable elements) has queue-number at most w. The best known upper bounds are $O(w^2)$ for general posets and $3w - 2$ for planar posets [71].

6. It is natural to ask for the largest class of graphs with bounded queue-number. First note that Theorem 3 cannot be extended to the setting of an excluded topological minor, since graphs with bounded degree have arbitrarily high queue-number [63, 100]. However, it is possible that every class of graphs with strongly sub-linear separators has bounded queue-number. Here a class $\mathcal{G}$ of graphs has *strongly sub-linear separators* if $\mathcal{G}$ is closed under taking subgraphs, and there exists constants $c, \beta > 0$, such that every n-vertex graph in $\mathcal{G}$ has a balanced separator of order $cn^{1-\beta}$. Already the $\beta = \frac{1}{2}$ case looks challenging, since this would imply Theorem 3.

7. Is there a polynomial function f such that every graph with treewidth k has queue-number at most $f(k)$? The best lower and upper bounds on $f(k)$ are $k + 1$ and $2^k - 1$, both due to Wiechert [98].

8. Do the results in the present paper have algorithmic applications? Consider the method of Baker [9] for designing polynomial-time approximation schemes for problems on planar graphs. This method partitions the graph into BFS layers, such that the problem can be solved optimally on each layer (since the induced subgraph has bounded treewidth), and then combines the solutions from each layer. Our results (Theorem 11) give a more precise description of the layered structure of planar graphs (and other more general classes). It is conceivable that this extra structural information is useful when designing algorithms.

   Note that all our proofs lead to polynomial-time algorithms for computing the desired decomposition and queue layout. Pilipczuk and Siebertz [86] claim $O(n^2)$ time complexity for their decomposition. The same is true for Lemma 13: Given the colours of the vertices on F, we can walk down the BFS tree T in linear time and colour every vertex. Another linear-time enumeration of the faces contained in F finds the trichromatic triangle. It is easily seen that Lemma 21 has polynomial time complexity (given the embedding). Polynomial-time algorithms for our other results follow based on the linear-time algorithm of Mohar [79] to test if a given graph has Euler genus at most any fixed number g, and the polynomial-time algorithm of Demaine, Hajiaghayi, and Kawarabayashi [30] for computing the decomposition in the graph minor structure theorem (Theorem 23).

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