Notes on Nonrepetitive Graph Colouring

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Abstract

A vertex colouring of a graph is *nonrepetitive on paths* if there is no path $v_1, v_2, \dots, v_{2t}$ such that v_i and v_{t+i} receive the same colour for all $i = 1, 2, \dots, t$. We determine the maximum density of a graph that admits a k-colouring that is nonrepetitive on paths. We prove that every graph has a subdivision that admits a 4-colouring that is nonrepetitive on paths. The best previous bound was 5. We also study colourings that are nonrepetitive on walks, and provide a conjecture that would imply that every graph with maximum degree Δ has a $f(\Delta)$-colouring that is nonrepetitive on walks. We prove that every graph with treewidth k and maximum degree Δ has a $O(k\Delta)$-colouring that is nonrepetitive on paths, and a $O(k\Delta^3)$-colouring that is nonrepetitive on walks.

1 Introduction

We consider simple, finite, undirected graphs G with vertex set $V(G)$, edge set $E(G)$, and maximum degree $\Delta(G)$. Let $[t] := \{1, 2, \dots, t\}$. A *walk* in G is a sequence $v_1, v_2, \dots, v_t$ of vertices of G, such that $v_i v_{i+1} \in E(G)$ for all $i \in [t-1]$. A *k-colouring* of G is a function f that assigns one of k colours to each vertex of G. A walk $v_1, v_2, \dots, v_{2t}$ is *repetitively* coloured by f if $f(v_i) = f(v_{t+i})$ for all $i \in [t]$. A walk $v_1, v_2, \dots, v_{2t}$ is *boring* if $v_i = v_{t+i}$ for all $i \in [t]$. Of course, a boring walk is repetitively coloured by every colouring. We say a colouring f is *nonrepetitive on walks* (or *walk-nonrepetitive*) if the only walks that

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are repetitively coloured by f are boring. Let $\sigma(G)$ denote the minimum k such that G has a k-colouring that is nonrepetitive on walks.

A walk $v_1, v_2, \ldots, v_t$ is a *path* if $v_i \neq v_j$ for all distinct $i, j \in [t]$. A colouring f is *nonrepetitive on paths* (or *path-nonrepetitive*) if no path of G is repetitively coloured by f. Let $\pi(G)$ denote the minimum k such that G has a k-colouring that is nonrepetitive on paths. Observe that a colouring that is path-nonrepetitive is *proper*, in the sense that adjacent vertices receive distinct colours. Moreover, a path-nonrepetitive colouring has no 2-coloured P_4 (a path on four vertices). A proper colouring with no 2-coloured P_4 is called a *star colouring* since each bichromatic subgraph is a star forest; see [1, 8, 17, 18, 25, 28]. The *star chromatic number* $\chi_{\text{st}}(G)$ is the minimum number of colours in a proper colouring of G with no 2-coloured P_4. Thus

$$\chi(G) \leqslant \chi_{\text{st}}(G) \leqslant \pi(G) \leqslant \sigma(G). \tag{1}$$

Path-nonrepetitive colourings are widely studied [2–5, 9, 10, 12, 13, 19, 21, 23, 24]; see the surveys by Grytczuk [20, 22]. Nonrepetitive edge colourings have also been considered [4, 6].

The seminal result in this field is by Thue [27], who in 1906 proved[1] that the n-vertex path P_n satisfies

$$\pi(P_n) = \begin{cases} n & \text{if } n \leqslant 2, \\ 3 & \text{otherwise.} \end{cases} \tag{2}$$

A result by Kündgen and Pelsmajer [23] (see Lemma 3.4) implies

$$\sigma(P_n) \leqslant 4 \ . \tag{3}$$

Currie [11] proved that the n-vertex cycle C_n satisfies

$$\pi(C_n) = \begin{cases} 4 & \text{if } n \in \{5, 7, 9, 10, 14, 17\}, \\ 3 & \text{otherwise.} \end{cases} \tag{4}$$

Let $\pi(\Delta)$ and $\sigma(\Delta)$ denote the maximum of $\pi(G)$ and $\sigma(G)$, taken over all graphs G with maximum degree $\Delta(G) \leqslant \Delta$. Now $\pi(2) = 4$ by (2) and (4). In general, Alon et al. [4] proved that

$$\frac{\alpha\Delta^2}{\log \Delta} \leqslant \pi(\Delta) \leqslant \beta\Delta^2, \tag{5}$$

for some constants α and β. The upper bound was proved using the Lovász Local Lemma, and the lower bound is attained by a random graph.

In Section 2 we study whether $\sigma(\Delta)$ is finite, and provide a natural conjecture that would imply an affirmative answer.

[1] The nonrepetitive 3-colouring of P_n by Thue [27] is obtained as follows. Given a nonrepetitive sequence over $\{1, 2, 3\}$, replace each 1 by the sequence 12312, replace each 2 by the sequence 131232, and replace each 3 by the sequence 1323132. Thue [27] proved that the new sequence is nonrepetitive. Thus arbitrarily long paths can be nonrepetitively 3-coloured.

In Section 3 we study path- and walk-nonrepetitive colourings of graphs of bounded treewidth[2]. Kündgen and Pelsmajer [23] and Barát and Varjú [5] independently proved that graphs of bounded treewidth have bounded π. The best bound is due to Kündgen and Pelsmajer [23] who proved that $\pi(G) \leqslant 4^k$ for every graph G with treewidth at most k. Whether there is a polynomial bound on π for graphs of treewidth k is an open question. We answer this problem in the affirmative under the additional assumption of bounded degree. In particular, we prove a $\mathcal{O}(k\Delta)$ upper bound on π, and a $\mathcal{O}(k\Delta^3)$ upper bound on σ.

In Section 4 we will prove that every graph has a subdivision that admits a path-nonrepetitive 4-colouring; the best previous bound was 5. In Section 5 we determine the maximum density of a graph that admits a path-nonrepetitive k-colouring, and prove bounds on the maximum density for walk-nonrepetitive k-colourings.

2 Is $\sigma(\Delta)$ bounded?

Consider the following elementary lower bound on σ, where G^2 is the *square* graph of G. That is, $V(G^2) = V(G)$, and $vw \in E(G^2)$ if and only if the distance between v and w in G is at most 2. A proper colouring of G^2 is called a *distance*-2 colouring of G.

Lemma 2.1. *Every walk-nonrepetitive colouring of a graph G is distance-2. Thus $\sigma(G) \geqslant \chi(G^2) \geqslant \Delta(G) + 1$.*

Proof. Consider a walk-nonrepetitive colouring of G. Adjacent vertices v and w receive distinct colours, as otherwise v, w would be a repetitively coloured path. If u, v, w is a path, and u and w receive the same colour, then the non-boring walk u, v, w, v is repetitively coloured. Thus vertices at distance at most 2 receive distinct colours. Hence $\sigma(G) \geqslant \chi(G^2)$. In a distance-2 colouring, each vertex and its neighbours all receive distinct colours. Thus $\chi(G^2) \geqslant \Delta(G) + 1$. □

Hence $\Delta(G)$ is a lower bound on $\sigma(G)$. Whether high degree is the only obstruction for bounded σ is an open problem.

Open Problem 2.2. Is there a function f such that $\sigma(\Delta) \leqslant f(\Delta)$?

First we answer Open Problem 2.2 in the affirmative for $\Delta = 2$. The following lemma will be useful.

Lemma 2.3. *Fix a distance-2 colouring of a graph G. If $W = (v_1, v_2, \dots, v_{2t})$ is a repetitively coloured non-boring walk in G, then $v_i \neq v_{t+i}$ for all $i \in [t]$.*

Proof. Suppose on the contrary that $v_i = v_{t+i}$ for some $i \in [t-1]$. Since W is repetitively coloured, $c(v_{i+1}) = c(v_{t+i+1})$. Each neighbour of v_i receives a distinct colour. Thus $v_{i+1} = v_{t+i+1}$. By induction, $v_j = v_{t+j}$ for all $j \in [i, t]$. By the same argument, $v_j = v_{t+j}$ for all $j \in [1, i]$. Thus W is boring, which is the desired contradiction. □

[2]The *treewidth* of a graph G can be defined to be the minimum integer k such that G is a subgraph of a chordal graph with no clique on $k + 2$ vertices. Treewidth is an important graph parameter, especially in structural graph theory and algorithmic graph theory; see the surveys [7, 26].

Proposition 2.4. $\sigma(2) \leqslant 5$.

Proof. A result by Kündgen and Pelsmajer [23] implies that $\sigma(P_n) \leqslant 4$ (see Lemma 3.4). Thus it suffices to prove that $\sigma(C_n) \leqslant 5$. Fix a walk-nonrepetitive 4-colouring of the path $(v_1, v_2, \dots, v_{2n-4})$. Thus for some $i \in [1, n-2]$, the vertices v_i and v_{n+i-2} receive distinct colours. Create a cycle C_n from the sub-path $v_i, v_{i+1}, \dots, v_{n+i-2}$ by adding one vertex x adjacent to v_i and v_{n+i-2}. Colour x with a fifth colour. Observe that since v_i and v_{n+i-2} receive distinct colours, the colouring of C_n is distance-2. Suppose on the contrary that C_n has a repetitively coloured walk $W = y_1, y_2, \dots, y_{2t}$. If x is not in W, then W is a repetitively coloured walk in the starting path, which is a contradiction. Thus $x = y_i$ for some $i \in [t]$ (with loss of generality, by considering the reverse of W). Since x is the only vertex receiving the fifth colour and W is repetitive, $x = y_{t+i}$. By Lemma 2.3, W is boring. Hence the 5-colouring of C_n is walk-nonrepetitive. ☐

Below we propose a conjecture that would imply a positive answer to Open Problem 2.2. First consider the following lemma which is a slight generalisation of a result by Barát and Varjú [6]. A walk $v_1, v_2, \dots, v_t$ has *length* t and *order* $|\{v_i : 1 \leqslant i \leqslant t\}|$. That is, the order is the number of distinct vertices in the walk.

Proposition 2.5. *Suppose that in some coloured graph, there is a repetitively coloured non-boring walk. Then there is a repetitively coloured non-boring walk of order k and length at most $2k^2$.*

Proof. Let k be the minimum order of a repetitively coloured non-boring walk. Let $W = v_1, v_2, \dots, v_{2t}$ be a repetitively coloured non-boring walk of order k and with t minimum. If $2t \leqslant 2k^2$, then we are done. Now assume that $t > k^2$. By the pigeonhole principle, there is a vertex x that appears at least $k+1$ times in $v_1, v_2, \dots, v_t$. Thus there is a vertex y that appears at least twice in the set $\{v_{t+i} : v_i = x, i \in [t]\}$. As illustrated in Figure 1, $W = AxBxCA'yB'yC'$ for some walks A, B, C, A', B', C' with $|A| = |A'|$, $|B| = |B'|$, and $|C| = |C'|$. Consider the walk $U := AxCA'yC'$. If U is not boring, then it is a repetitively coloured non-boring walk of order at most k and length less than $2t$, which contradicts the minimality of W. Otherwise U is boring, implying $x = y$, $A = A'$, and $C = C'$. Thus $B \neq B'$ since W is not boring, implying $xBxB'$ is a repetitively coloured non-boring walk of order at most k and length less than $2t$, which contradicts the minimality of W. ☐

We conjecture the following strengthening of Proposition 2.5.

Conjecture 2.6. *Let G be a graph. Consider a path-nonrepetitive distance-2 colouring of G with c colours, such that G contains a repetitively coloured non-boring walk. Then G contains a repetitively coloured non-boring walk of order k and length at most $h(c) \cdot k$, for some function h that only depends on c.*

Theorem 2.7. *If Conjecture 2.6 is true, then there is a function f for which $\sigma(\Delta) \leqslant f(\Delta)$. That is, every graph G has a walk-nonrepetitive colouring with $f(\Delta(G))$ colours.*

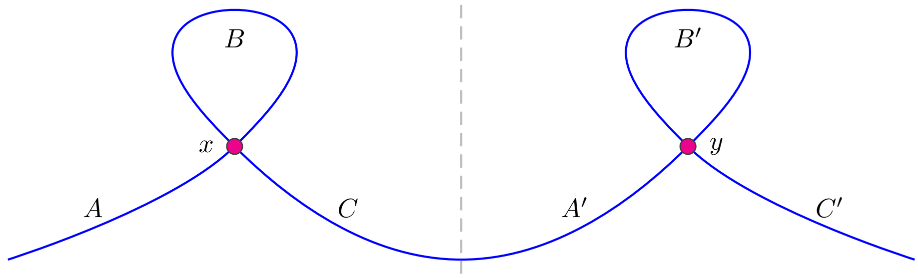

Figure 1: Illustration for the proof of Proposition 2.5.

Theorem 2.7 is proved using the Lovász Local Lemma [16].

Lemma 2.8 ([16])**.** *Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_r$ be a partition of a set of 'bad' events $\mathcal{A}$. Suppose that there are sets of real numbers $\{p_i \in [0,1) : i \in [r]\}$, $\{x_i \in [0,1) : i \in [r]\}$, and $\{D_{ij} \geqslant 0 : i,j \in [r]\}$ such that the following conditions are satisfied by every event $A \in \mathcal{A}_i$:*

- *the probability* $\mathbf{P}(A) \leqslant p_i \leqslant x_i \prod_{j=1}^{r}(1-x_j)^{D_{ij}}$ *, and*
- *A is mutually independent of $\mathcal{A} \setminus (\{A\} \cup \mathcal{D}_A)$, for some $\mathcal{D}_A \subseteq \mathcal{A}$ with $|\mathcal{D}_A \cap \mathcal{A}_j| \leqslant D_{ij}$ for all $j \in [r]$.*

Then

$$\mathbf{P}\left(\bigwedge_{A\in\mathcal{A}} \overline{A}\right) \geqslant \prod_{i=1}^{r}(1-x_i)^{|\mathcal{A}_i|} > 0 \ .$$

That is, with positive probability, no event in $\mathcal{A}$ occurs.

Proof of Theorem 2.7. Let f_1 be a path-nonrepetitive colouring of G with $\pi(G)$ colours. Let f_2 be a distance-2 colouring of G with $\chi(G^2)$ colours. Note that $\pi(G) \leqslant \beta\Delta^2$ for some constant β by Equation (5), and $\chi(G^2) \leqslant \Delta(G^2)+1 \leqslant \Delta^2+1$ by a greedy colouring of G^2. Hence f_1 and f_2 together define a path-nonrepetitive distance-2 colouring of G. The number of colours $\pi(G)\cdot\chi(G^2)$ is bounded by a function solely of $\Delta(G)$. Consider this initial colouring to be fixed. Let c be a positive integer to be specified later. For each vertex v of G, choose a third colour $f_3(v) \in [c]$ independently and randomly. Let f be the colouring defined by $f(v) = (f_1(v), f_2(v), f_3(v))$ for all vertices v.

Let $h := h(\pi(G)\cdot\chi(G^2))$ from Conjecture 2.6. A non-boring walk $v_1, v_2, \dots, v_{2t}$ of order i is *interesting* if its length $2t \leqslant hi$, and $f_1(v_j) = f_1(v_{t+j})$ and $f_2(v_j) = f_2(v_{t+j})$ for all $j \in [t]$. For each interesting walk W, let A_W be the event that W is repetitively coloured by f. Let $\mathcal{A}_i$ be the set of events A_W, where W is an interesting walk of order i. Let $\mathcal{A} = \bigcup_i \mathcal{A}_i$.

We will apply Lemma 2.8 to prove that, with positive probability, no event A_W occurs. This will imply that there exists a colouring f_3 such that no interesting walk is repetitively coloured by f. A non-boring non-interesting walk $v_1, v_2, \dots, v_{2t}$ of order i satisfies (a) $2t > hi$, or (b) $f_1(v_j) \neq f_1(v_{t+j})$ or $f_2(v_j) \neq f_2(v_{t+j})$ for some $j \in [t]$. In case (a), by the assumed truth of Conjecture 2.6, W is not repetitively coloured by f. In case (b), $f(v_j) \neq f(v_{t+j})$ and W is not repetitively coloured by f. Thus no non-boring walk is repetitively coloured by f, as desired.

Consider an interesting walk $W = v_1, v_2, \dots, v_{2t}$ of order i.

We claim that $v_\ell \neq v_{t+\ell}$ for all $\ell \in [t]$. Suppose on the contrary that $v_\ell = v_{t+\ell}$ for some $\ell \in [t]$. Since W is not boring, $v_j \neq v_{t+j}$ for some $j \in [t]$. Thus $v_j = v_{t+j}$ and $v_{j+1} \neq v_{t+j+1}$ for some $j \in [t]$ (where v_{t+t+1} means v_1). Since W is interesting, $f_2(v_{j+1}) = f_2(v_{t+j+1})$, which is a contradiction since v_{j+1} and v_{t+j+1} have a common neighbour v_j $(= v_{t+j})$. Thus $v_j \neq v_{t+j}$ for all $j \in [t]$, as claimed.

This claim implies that for each of the i vertices x in W, there is at least one other vertex y in W, such that $f_3(x) = f_3(y)$ must hold for W to be repetitively coloured. Hence at most $c^{i/2}$ of the c^i possible colourings of W under f_3, lead to repetitive colourings of W under f. Thus the probability $\mathbf{P}(A_W) \leqslant p_i := c^{-i/2}$, and Lemma 2.8 can be applied as long as

$$c^{-i/2} \leqslant x_i \prod_j (1 - x_j)^{D_{ij}} \quad , \tag{6}$$

Every vertex is in at most $hj\Delta^{hj}$ interesting walks of order j. Thus an interesting walk of order i shares a vertex with at most $hij\Delta^{hj}$ interesting walks of order j. Thus we can take $D_{ij} := hij\Delta^{hj}$. Define $x_i := (2\Delta^h)^{-i}$. Note that $x_i \leqslant \frac{1}{2}$. So $1 - x_i \geqslant \mathbf{e}^{-2x_i}$. Thus to prove (6) it suffices to prove that

$$\begin{aligned}
& c^{-i/2} \leqslant x_i \prod_j \mathbf{e}^{-2x_j D_{ij}} \quad , \\
\Longleftarrow\quad & c^{-i/2} \leqslant (2\Delta^h)^{-i} \prod_j \mathbf{e}^{-2(2\Delta^h)^{-j} hij\Delta^{hj}} \quad , \\
\Longleftarrow\quad & c^{-1/2} \leqslant (2\Delta^h)^{-1} \prod_j \mathbf{e}^{-2(2)^{-j} hj} \quad , \\
\Longleftarrow\quad & c^{-1/2} \leqslant (2\Delta^h)^{-1} \mathbf{e}^{-2h\sum_j j2^{-j}} \quad , \\
\Longleftarrow\quad & c^{-1/2} \leqslant (2\Delta^h)^{-1} \mathbf{e}^{-4h} \quad , \\
\Longleftarrow\quad & c \geqslant 4(\mathbf{e}^4\Delta)^{2h} \quad .
\end{aligned}$$

Choose c to be the minimum integer that satisfies this inequality, and the lemma is applicable. We obtain a c-colouring f_3 of G such that f is nonrepetitive on walks. The number of colours in f is at most $h\lceil 4(\mathbf{e}^4\Delta)^{2h}\rceil$, which is a function solely of Δ. $\square$

3 Trees and Treewidth

We start this section by considering walk-nonrepetitive colourings of trees.

Theorem 3.1. *Let T be a tree. A colouring c of T is walk-nonrepetitive if and only if c is path-nonrepetitive and distance-2.*

Proof. For every graph, every walk-nonrepetitive colouring is path-nonrepetitive (by definition) and distance-2 (by Lemma 2.1).

Now fix a path-nonrepetitive distance-2 colouring c of T. Suppose on the contrary that T has a repetitively coloured non-boring walk. Let $W = (v_1, v_2, \dots, v_{2t})$ be a repetitively coloured non-boring walk in T of minimum length. Some vertex is repeated in W, as otherwise W would be a repetitively coloured path. By considering the reverse of W, without loss of generality, $v_i = v_j$ for some $i \in [1, t-1]$ and $j \in [i+2, 2t]$. Choose i and j to minimise $j-i$. Thus v_i is not in the sub-walk $(v_{i+1}, v_{i+2}, \dots, v_{j-1})$. Since T is a tree, $v_{i+1} = v_{j-1}$. Thus $i+1 = j-1$, as otherwise $j-i$ is not minimised. That is, $v_i = v_{i+2}$. Assuming $i \neq t-1$, since W is repetitively coloured, $c(v_{t+i}) = c(v_{t+i+2})$, which implies that $v_{t+i} = v_{t+i+2}$ because c is a distance-2 colouring. Thus, even if $i = t-1$, deleting the vertices $v_i, v_{i+1}, v_{t+i}, v_{t+i+1}$ from W, gives a walk $(v_1, v_2, \dots, v_{i-1}, v_{i+2}, \dots, v_{t+i-1}, v_{t+i+2}, \dots, v_{2t})$ that is also repetitively coloured. This contradicts the minimality of the length of W. $\square$

Note that Theorem 3.1 implies that Conjecture 2.6 is vacuously true for trees. Also, since every tree T has a path-nonrepetitive 4-colouring [23] and a distance-2 $(\Delta(T)+1)$-colouring, Theorem 3.1 implies the following result, where the lower bound is Lemma 2.1.

Corollary 3.2. *Every tree T satisfies $\Delta(T) + 1 \leqslant \sigma(T) \leqslant 4(\Delta(T) + 1)$.*

In the remainder of this section we prove the following polynomial upper bounds on π and σ in terms of the treewidth and maximum degree of a graph.

Theorem 3.3. *Every graph G with treewidth k and maximum degree $\Delta \geqslant 1$ satisfies $\pi(G) \leqslant ck\Delta$ and $\sigma(G) \leqslant ck\Delta^3$ for some constant c.*

We prove Theorem 3.3 by a series of lemmas. The first is by Kündgen and Pelsmajer [23][3].

Lemma 3.4 ([23])**.** *Let P^+ be the pseudograph obtained from a path P by adding a loop at each vertex. Then $\sigma(P^+) \leqslant 4$.*

Now we introduce some definitions by Kündgen and Pelsmajer [23]. A *levelling* of a graph G is a function $\lambda : V(G) \to \mathbb{Z}$ such that $|\lambda(v) - \lambda(w)| \leqslant 1$ for every edge $vw \in E(G)$. Let $G_{\lambda=k}$ and $G_{\lambda>k}$ denote the subgraphs of G respectively induced by $\{v \in V(G) : \lambda(v) = k\}$ and $\{v \in V(G) : \lambda(v) > k\}$. The *$k$-shadow* of a subgraph H of G is the set of vertices in $G_{\lambda=k}$ adjacent to some vertex in H. A levelling λ is *shadow-complete* if the k-shadow of every component of $G_{\lambda>k}$ induces a clique. Kündgen and Pelsmajer [23] proved the following lemma for repetitively coloured paths. We show that the same proof works for repetitively coloured walks.

[3]The 4-colouring in Lemma 3.4 is obtained as follows. Given a nonrepetitive sequence on $\{1, 2, 3\}$, insert the symbol 4 between consecutive block of length two. For example, from the sequence 123132123 we obtain 1243143241243.

Lemma 3.5. *For every levelling λ of a graph G, there is a 4-colouring of G, such that every repetitively coloured walk $v_1, v_2, \dots, v_{2t}$ satisfies $\lambda(v_j) = \lambda(v_{t+j})$ for all $j \in [t]$.*

Proof. The levelling λ can be thought of as a homomorphism from G into P^+, for some path P. By Lemma 3.4, P^+ has a 4-colouring that is nonrepetitive on walks. Colour each vertex v of G by the colour assigned to $\lambda(v)$ (thought of as a vertex of P^+). Suppose $v_1, v_2, \dots, v_{2t}$ is a repetitively coloured walk in G. Thus $\lambda(v_1), \lambda(v_2), \dots, \lambda(v_{2t})$ is a repetitively coloured walk in P^+. Since the 4-colouring of P^+ is nonrepetitive on walks, $\lambda(v_1), \lambda(v_2), \dots, \lambda(v_{2t})$ is boring. That is, $\lambda(v_j) = \lambda(v_{t+j})$ for all $j \in [t]$. □

Lemma 3.6 ([23])**.** *If λ is a shadow-complete levelling of a graph G, then*

$$\pi(G) \leqslant 4 \cdot \max_k \pi(G_{\lambda=k}).$$

Now we generalise Lemma 3.6 for walks.

Lemma 3.7. *If H is a subgraph of a graph G, and λ is a shadow-complete levelling of G, then*

$$\sigma(H) \;\leqslant\; 4\,\chi(H^2) \cdot \max_k \sigma(G_{\lambda=k}) \;\leqslant\; 4(\Delta(H)^2 + 1) \cdot \max_k \sigma(G_{\lambda=k}).$$

Proof. Let c_1 be the 4-colouring of G from Lemma 3.5. Let c_2 be an optimal walk-nonrepetitive colouring of each level $G_{\lambda=k}$. Let c_3 be a proper $\chi(H^2)$-colouring of H^2. The second inequality in the lemma follows from the first since $\chi(H^2) \leqslant \Delta(H)^2 + 1$. Let $c(v) := (c_1(v), c_2(v), c_3(v))$ for each vertex v of H. We claim that c is nonrepetitive on walks in H.

Suppose on the contrary that $W = v_1, \dots, v_{2t}$ is a non-boring walk in H that is repetitively coloured by c. Then W is repetitively coloured by each of c_1, c_2, and c_3. Thus $\lambda(v_i) = \lambda(v_{t+i})$ for all $i \in [t]$ by Lemma 3.5. Let W_k be the sequence (allowing repetitions) of vertices $v_i \in W$ such that $\lambda(v_i) = k$. Since $v_i \in W_k$ if and only if $v_{t+i} \in W_k$, each sequence W_k is repetitively coloured. That is, if $W_k = x_1, \dots, x_{2s}$ then $c(x_i) = c(x_{s+i})$ for all $i \in [s]$.

Let k be the minimum level containing a vertex in W. Let v_i and v_j be consecutive vertices in W_k with $i < j$. If $j = i + 1$ then $v_i v_j$ is an edge of W. Otherwise there is walk from v_i to v_j in $G_{\lambda>k}$ (since k was chosen minimum), implying $v_i v_j$ is an edge of G (since λ is shadow-complete). Thus W_k forms a walk in $G_{\lambda=k}$ that is repetitively coloured by c_2. Hence W_k is boring. In particular, some vertex $v_i = v_{t+i}$ is in W_k. Since W is not boring, $v_j \neq v_{t+j}$ for some $j \in [t]$. Without loss of generality, $i < j$ and $v_\ell = v_{t+\ell}$ for all $\ell \in [i, j-1]$. Thus v_j and v_{t+j} have a common neighbour $v_{j-1} = v_{t+j-1}$ in H, which implies that $c_3(v_j) \neq c_3(v_{t+j})$. But $c(v_j) = c(v_{t+j})$ since W is repetitively coloured, which is the desired contradiction. □

Note that some dependence on $\Delta(H)$ in Lemma 3.7 is unavoidable, since $\sigma(H) \geqslant \chi(H^2) \geqslant \Delta(H) + 1$.

Lemma 3.7 enables the following strengthening of Corollary 3.2.

Lemma 3.8. *Every tree T satisfies $\Delta(T)+1 \leqslant \sigma(T) \leqslant 4\,\Delta(T)$.*

Proof. Let r be a leaf vertex of T. Let $\lambda(v)$ be the distance from r to v in T. Then λ is a shadow-complete levelling of T in which each level is an independent set. A greedy algorithm proves that $\chi(T^2) \leqslant \Delta(T)+1$. Thus Lemma 3.7 implies that $\sigma(T) \leqslant 4\,\Delta(T)+4$. Observe that the proof of Lemma 3.7 only requires $c_3(v) \neq c_3(w)$ whenever v and w are in the same level and have a common parent. Since r is a leaf, each vertex has at most $\Delta(T)-1$ children. Thus a greedy algorithm produces a $\Delta(T)$-colouring with this property. Hence $\sigma(T) \leqslant 4\,\Delta(T)$. □

A *tree-partition* of a graph G is a partition of its vertices into sets (called *bags*) such that the graph obtained from G by identifying the vertices in each bag is a forest (after deleting loops and replacing parallel edges by a single edge)[4].

Lemma 3.9. *Let G be a graph with a tree-partition in which every bag has at most ℓ vertices. Then G is a subgraph of a graph G' that has a shadow-complete levelling in which each level satisfies*

$$\pi(G'_{\lambda=k}) \leqslant \sigma(G'_{\lambda=k}) \leqslant \ell.$$

Proof. Let G' be the graph obtained from G by adding an edge between all pairs of nonadjacent vertices in a common bag. Let F be the forest obtained from G' by identifying the vertices in each bag. Root each component of F. Consider a vertex v of G' that is in the bag that corresponds to node x of F. Let $\lambda(v)$ be the distance between x and the root of the tree component of F that contains x. Clearly λ is a levelling of G'. The k-shadow of each connected component of $G'_{\lambda>k}$ is contained in a single bag, and thus induces a clique on at most ℓ vertices. Hence λ is shadow-complete. By colouring the vertices within each bag with distinct colors, we have $\pi(G'_{\lambda=k}) \leqslant \sigma(G'_{\lambda=k}) \leqslant \ell$. □

Lemmas 3.6, 3.7 and 3.9 imply:

Lemma 3.10. *If a graph G has a tree-partition in which every bag has at most ℓ vertices, then $\pi(G) \leqslant 4\ell$ and $\sigma(G) \leqslant 4\ell(\Delta(G)^2+1)$.*

Wood [30] proved[5] that every graph with treewidth k and maximum degree $\Delta \geqslant 1$ has a tree-partition in which every bag has at most $\frac{5}{2}(k+1)(\frac{7}{2}\Delta-1)$ vertices. With Lemma 3.10 this proves the following quantitative version of Theorem 3.3.

Theorem 3.11. *Every graph G with treewidth k and maximum degree $\Delta \geqslant 1$ satisfies $\pi(G) \leqslant 10(k+1)(\frac{7}{2}\Delta-1)$ and $\sigma(G) \leqslant 10(k+1)(\frac{7}{2}\Delta-1)(\Delta^2+1)$.*

[4]The proof by Kündgen and Pelsmajer [23] that $\pi(G) \leqslant 4^k$ for graphs with treewidth at most k can also be described using tree-partitions; cf. [15, 29].

[5]The proof by Wood [30] is a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski [14].

4 Subdivisions

The results of Thue [27] and Currie [11] imply that every path and every cycle has a subdivision H with $\pi(H) = 3$. Brešar et al. [9] proved that every tree has a subdivision H such that $\pi(H) = 3$. Which graphs have a subdivision H with $\pi(H) = 3$ is an open problem [20]. Grytczuk [20] proved that every graph has a subdivision H with $\pi(H) \leqslant 5$. Here we improve this bound as follows.

Theorem 4.1. *Every graph G has a subdivision H with $\pi(H) \leqslant 4$.*

Proof. Without loss of generality G is connected. Say $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$. As illustrated in Figure 2, let H be the subdivision of G obtained by subdividing every edge $v_iv_j \in E(G)$ (with $i < j$) $j - i - 1$ times. The distance of every vertex in H from v_0 defines a levelling of H such that the endpoints of every edge are in consecutive levels. By Lemma 3.5, there is a 4-colouring of H, such that for every repetitively coloured path $x_1, x_2, \dots, x_t, y_1, y_2, \dots, y_t$ in H, x_j and y_j have the same level for all $j \in [t]$. Hence there is some j such that x_{j-1} and x_{j+1} are at the same level. Thus x_j is an original vertex v_i of G. Without loss of generality x_{j-1} and x_{j+1} are at level $i - 1$. There is only one original vertex at level i. Thus y_j, which is also at level i, is a division vertex. Now y_j has two neighbours in H, which are at levels $i - 1$ and $i + 1$. Thus y_{j-1} and y_{j+1} are at levels $i - 1$ and $i + 1$, which contradicts the fact that x_{j-1} and x_{j+1} are both at level $i - 1$. Hence we have a 4-colouring of H that is nonrepetitive on paths. $\square$

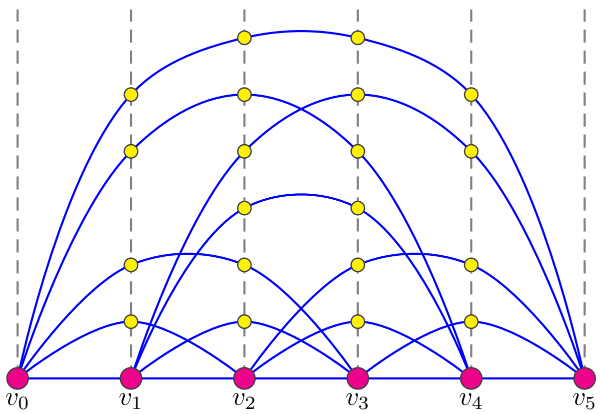

Figure 2: The subdivision H with $G = K_6$.

It is possible that every graph has a subdivision H with $\pi(H) \leqslant 3$. If true, this would provide a striking generalisation of the result of Thue [27] discussed in Section 1.

5 Maximum Density

In this section we study the maximum number of edges in a nonrepetitively coloured graph.

Proposition 5.1. *The maximum number of edges in an n-vertex graph G with $\pi(G) \leqslant c$ is $(c-1)n - \binom{c}{2}$.*

Proof. Say G is an n-vertex graph with $\pi(G) \leqslant c$. Fix a c-colouring of G that is nonrepetitive on paths. Say there are n_i vertices in the i-th colour class. Every cycle receives at least three colours. Thus the subgraph induced by the vertices coloured i and j is a forest, and has at most $n_i + n_j - 1$ edges. Hence the number of edges in G is at most

$$\sum_{1 \leqslant i < j \leqslant c} (n_i + n_j - 1) = \sum_{1 \leqslant i \leqslant c} (c-1)n_i - \binom{c}{2} = (c-1)n - \binom{c}{2}.$$

This bound is attained by the graph consisting of a complete graph K_{c-1} completely connected to an independent set of $n-(c-1)$ vertices, which obviously has a c-colouring that is nonrepetitive on paths. $\square$

Now consider the maximum number of edges in a coloured graph that is nonrepetitive on walks. First note that the example in the proof of Proposition 5.1 is repetitive on walks. Since $\sigma(G) \geqslant \Delta(G) + 1$ and $|E(G)| \leqslant \frac{1}{2}\Delta(G)|V(G)|$, we have the trivial upper bound,

$$|E(G)| \leqslant \tfrac{1}{2}(\sigma(G) - 1)|V(G)|.$$

This bound is tight for $\sigma = 2$ (matchings) and $\sigma = 3$ (cycles), but is not known to be tight for $\sigma \geqslant 4$.

We have the following lower bound.

Proposition 5.2. *For all $p \geqslant 1$, there are infinitely many graphs G with $\sigma(G) \leqslant 4p$ and*

$$|E(G)| \geqslant \tfrac{1}{8}(3\sigma(G) - 4)|V(G)| - \tfrac{1}{9}\sigma(G)^2.$$

Proof. Let G be the lexicographic product of a path and K_p; that is, G is the graph with a levelling λ in which each level induces K_p, and every edge is present between consecutive levels. Let c_1 be the 4-colouring of G from Lemma 3.5. If v is the j-th vertex in its level, where $j \in [p]$, then let $c(v) := (c_1(v), j)$. The number of colours is $4p$. Applying Lemma 3.5, it is easily verified that c is nonrepetitive on walks. Hence $\sigma(G) \leqslant 4p$. Now we count the edges: $|E(G)| = \frac{1}{2}(3p-1)|V(G)| - p^2$. As a lower bound, $\sigma(G) \geqslant \Delta(G)+1 = 3p$. Thus $|E(G)| \geqslant \frac{1}{2}(3\sigma(G)/4 - 1)|V(G)| - (\sigma(G)/3)^2$. $\square$

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A Corrigendum (12 December 2014)

The authors are extremely grateful to Joseph Antonides, Claire Spychalla, Nicole Yamzon who identified and corrected an error in this paper (in January 2014). The error occurs in the proof of Theorem 4.1 when $t = 2$. The proof claimed that x_{j-1} and x_{j+1} are at the same level for some j. Since only $x_1, \dots, x_t$ are defined, we must have $j - 1 \geqslant 1$ and

$j + 1 \leqslant t$, implying $t \geqslant 3$. Thus, this claim does not make sense if $t = 2$. In particular, if v_a, v_{b+1}, v_b, v_c is a path in G with $a < b < c - 1$, and p is the division vertex on $v_a v_{b+1}$ adjacent to v_{b+1}, and q is the division vertex on $v_b v_c$ adjacent to v_b, then q, v_b, v_{b+1}, p is a repetitively coloured path in H.

Antonides, Spychalla and Yamzon proposed the following correction. Let G be a connected graph with $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$. Let H be the subdivision of G obtained by subdividing every edge $v_i v_j \in E(G)$ (with $i < j$) $2(j - i) - 1$ times. The distance of every vertex in H from v_0 defines a levelling of H such that the endpoints of every edge are in consecutive levels. (Think of $v_0, v_1, \dots, v_{n-1}$ on a horizontal line, with a vertical line through each v_i, and an additional vertical line between v_i and v_{i+1}. Each edge is subdivided at each point it crosses a vertical line.)

Consider the 4-colouring of H given by Lemma 3.5. Suppose on the contrary that H contains a repetitively coloured path $x_1, x_2, \dots, x_t, y_1, y_2, \dots, y_t$. By Lemma 3.5, x_j and y_j have the same level for all $j \in [t]$.

First suppose that $t = 2$. Since x_1 and y_1 have the same level, and x_2 is adjacent to both x_1 and y_1, it must be that x_2 is an original vertex (since division vertices only have two neighbours, and they are on distinct levels). Similarly, y_1 is an original vertex. This is a contradiction, since no two original vertices are adjacent to H. Now assume that $t \geqslant 3$.

Now suppose that x_{j-1} and x_{j+1} are at the same level for some $j \in [2, t-1]$. Thus x_j is an original vertex of G. Say x_j is at level i. Without loss of generality x_{j-1} and x_{j+1} are at level $i - 1$. There is only one original vertex at each level. Thus y_j, which is also at level i, is a division vertex. Now y_j has two neighbours in H, which are at levels $i - 1$ and $i + 1$. Thus y_{j-1} and y_{j+1} are at levels $i - 1$ and $i + 1$, which contradicts the fact that x_{j-1} and x_{j+1} are both at level $i - 1$. Now assume that for all $j \in [2, t-1]$, the vertices x_{j-1} and x_{j+1} are at distinct levels.

Say x_1 is at level i. Without loss of generality, x_2 is at level $i + 1$ (since no edge has both endpoints in the same level). It follows that x_j is at level $i + j - 1$ for all $j \in [1, t]$. In particular, x_t is at level $i + t - 1$. Now, y_1 is at level i (the same level as x_1). Since $x_t y_1$ is an edge, and every edge goes between consecutive levels, $|(i + t - 1) - i| = 1$, implying $t = 2$, which is a contradiction. Hence we have a 4-colouring of H that is nonrepetitive on paths.

Finally, note that reference [6] appeared in *Ars Combin.* 87:377–383, 2008; reference [30] appeared in *European J. Combinatorics* 30:1245–1253, 2009; and Andrzej Pezarski and Michał Zmarz [Non-Repetitive 3-Coloring of Subdivided Graphs, *Electronic J. Combin.* 16(1):#N15, 2009] solved the problem posed at the end of Section 4 by proving that every graph has a nonrepetitively 3-colourable subdivision.