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ANAGRAM-FREE COLORINGS OF GRAPH SUBDIVISIONS*

TIM E. WILSON† AND DAVID R. WOOD†

Abstract. An *anagram* is a word of the form WP where W is a non-empty word and P is a permutation of W. A vertex coloring of a graph is *anagram-free* if no subpath of the graph is an anagram. Anagram-free graph coloring was independently introduced by Kamčev, Łuczak, and Sudakov [*Combin. Probab. Comput.*, 27 (2018), pp. 623–642] and ourselves [*Electron. J. Combin.*, 25 (2018), pp. 2–20]. In this paper we introduce the study of anagram-free colorings of graph subdivisions. We show that every graph has an anagram-free 8-colorable subdivision. The number of division vertices per edge is exponential in the number of edges. For trees, we construct anagram-free 10-colorable subdivisions with fewer division vertices per edge. Conversely, we prove lower bounds, in terms of division vertices per edge, on the anagram-free chromatic number for subdivisions of the complete graph and subdivisions of complete trees of bounded degree.

Key words. anagram-free coloring, vertex coloring, subdivision, trees

AMS subject classifications. 05C15, 05C05

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1. Introduction.

An *anagram* is a word of the form WP where W is a non-empty word and P is a permutation of W. A vertex coloring of a graph is *anagram-free* if the sequence of colors on every path in the graph is not an anagram. The *anagram-free chromatic number*, $\phi(G)$, of a graph G, is the minimum number of colors in an anagram-free coloring of G. Alon et al. [1] proposed anagram-free coloring as a subject of study as a generalization of square-free coloring. A *square* is a word of the form WW where W is a non-empty word. A graph coloring is *square-free* if the sequence of colors on every path in the graph is not a square. A square-free graph coloring is also called a *nonrepetitive coloring*. The *square-free chromatic number*, $\pi(G)$, of a graph G, is the minimum number of colors in a square-free coloring of G.

Square-free words and anagram-free words both originate from the study of the combinatorics of words. Square-free words are known as *nonrepetitive* words and anagram-free words are known as *abelian square-free* or *strongly nonrepetitive*. Both types of words can be arbitrarily long with a bounded number of distinct symbols. In particular, Thue [16] constructed arbitrarily long square-free words on three symbols. Keränen [12, 13] constructed arbitrarily long anagram-free words on four symbols. The longest square-free or anagram-free words on two symbols have length 3. The longest anagram-free words on three symbols have length 7 [5]. Words are equivalent to colorings of paths, so $\pi(P) \leqslant 3$ and $\phi(P) \leqslant 4$ for all paths P.

Square-free coloring was introduced by Alon et al. [1] and has since received much attention [3, 4, 6, 7, 8, 9, 10]. A central area of study has been to bound $\pi(G)$ by a function of maximum degree, $\Delta(G)$. Alon et al. [1] proved a result that implies $\pi(G) \leqslant c\Delta(G)^2$ for some constant c. Several subsequent works improved the value of c [8, 10], with the best known value being $c = 1 + o(1)$ [6]. Lower bounds for square-free coloring apply to anagram-free coloring because $\phi(G) \geqslant \pi(G)$ for all graphs G.

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†School of Mathematical Sciences, Monash University, Melbourne, Australia (timothy.e.wilson@monash.edu, david.wood@monash.edu).

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Indeed, a square is an anagram with the identity permutation, so for a coloring to be anagram-free it must also be square-free. Anagram-free colorings were recently introduced by Kamčev, Łuczak, and Sudakov [11] and Wilson and Wood [17], both proving, among other results, that ϕ is not bounded by a function of maximum degree.

In this paper we study ϕ on graph subdivisions, with a focus on constructing subdivisions with bounded anagram-free chromatic number. A *subdivision* of a graph, G, is a graph obtained from G by replacing each edge $vw \in E(G)$ by a path with endpoints vw. If an edge uv of G is replaced by a path $uw_1w_2 \dots w_{i-1}v$ of length i, then we say that uv was *subdivided i times* and call the vertices $w_1, \dots, w_{i-1}$ *division vertices*. The *k-subdivision* of G is the subdivision in which every edge of G is subdivided exactly k times. Similarly, a *$(\leqslant k)$-subdivision* of G is a subdivision in which every edge of G is subdivided at most k times. Graphs with many division vertices are locally paths or stars, so one would expect highly subdivided graphs to have relatively low anagram-free chromatic number. Square-free coloring has been studied on subdivisions of graphs, and here this intuition is known to hold. Grytczuk [7] showed that every graph has a subdivision, S, with $\pi(S) \leqslant 5$, with the bound later improved to 4 by Barát and Wood [2], and finally to 3 by Pezarski and Zmarz [15].

Before introducing our results, we summarize the known results for π and ϕ on trees. For a rooted tree, T, with root r, the *depth* of a vertex v in T is the distance between v and r. A *d-ary tree* is a rooted tree with at most d children per vertex. The *complete d-ary tree of height h* is the rooted tree such that every non-leaf vertex has d children and every leaf has depth h. The complete 2-ary tree is called the *complete binary tree*. Brešar et al. [3] studied square-free colorings of trees, showing that $\pi(T) \leqslant 4$ for every tree T, and that T has a subdivision, S, with $\pi(S) \leqslant 3$. By contrast, ϕ is unbounded on trees [11]. In particular, Kamčev, Łuczak, and Sudakov [11] prove the following bounds for the complete binary tree.

THEOREM 1 (Kamčev, Łuczak, and Sudakov [11]). *Let T_h be the complete binary tree of height h. Then*

$$\sqrt{\frac{h}{\log_2 h}} \leqslant \phi(T_h) \leqslant h+1.$$

The upper bound, $\phi(T) \leqslant h+1$, holds for every tree, T, of height h, and is obtained by coloring vertices by their depth. Wilson and Wood [17] show that this upper bound is almost best possible on general trees by proving that $\phi(T) \geqslant h$, where T is the $(h-1)^h$-ary tree of height h.

1.1. Subdivisions of trees. We now introduce the results in the present paper. Our results complement the bounds on ϕ for trees proved in [11, 17]. We construct anagram-free 8-colorable subdivisions of binary trees.

THEOREM 2. *Every binary tree, T, of height h, has a $(\leqslant 3^{h-1}-1)$-subdivision, S, with $\phi(S) \leqslant 8$.*

More generally, we construct anagram-free 10-colorable subdivisions of d-ary trees.

THEOREM 3. *Every d-ary tree, T, of height h, has a $\left(\leqslant 2d(d+1)^{h-1}\right)$-subdivision, S, with $\phi(S) \leqslant 10$.*

The number of division vertices per edge is exponential in the height for Theorems 2 and 3. This raises the question of whether better constructions exist. In particular, does every tree of bounded degree have an anagram-free c-colorable subdivision with the number of division vertices per edge growing slower than exponentially with height? We answer this question in the negative with the lower bound in the following theorem.

THEOREM 4. *The k-subdivision, S, of the complete d-ary tree of height h satisfies*

$$\sqrt{\frac{h}{\log_{\min\{d,(h(k+1))^2\}}(h(k+1))}} \leqslant \phi(S) \leqslant \frac{10h}{\log_{d+1}(k/2d)} + 14.$$

Theorem 4 implies that, for sufficiently large height h, the number of division vertices per edge in an anagram-free c-colorable subdivision of the complete d-ary tree is at least

$$k \geqslant \frac{d^{h/c^2}}{h} - 1,$$

which is exponential in h for fixed c. The upper bound in Theorem 4 is obtained by applying Theorem 3 to appropriate subtrees of the complete d-ary tree. The lower bound is a generalization of Theorem 1; see Theorem 12 for details.

1.2. Subdivisions of general graphs. We also study ϕ on subdivisions of general graphs and prove the following theorems in this direction. The first has fewer division vertices per edge, while the second has fewer colors.

THEOREM 5. *Every graph G has a $(\leqslant 3(2)^{2|E(G)|-1} - 1)$-subdivision, S, with $\phi(S) \leqslant 14$.*

THEOREM 6. *Every graph G has a $(\leqslant 45\left(\frac{75}{9}+1\right)^{2|E(G)|-1})$-subdivision, S, with $\phi(S) \leqslant 8$.*

The bound $\phi(S) \leqslant 8$ in Theorem 6 is our best bound on ϕ, notably better than the bound for subdivisions of trees (Theorem 3). On the other hand, Theorem 3 uses fewer division vertices. Indeed, if T is the complete d-ary tree, then the number of division vertices per edge is polynomial in $|E(T)|$.

To investigate the optimality, in terms of division vertices per edge, of Theorems 5 and 6, we prove a lower bound on $\phi(K_n)$, the complete graph on n vertices. Such results exist for π; in particular, Nešetřil, Ossona de Mendez, and Wood [14] proved the following theorem.

THEOREM 7 (Nešetřil, Ossona de Mendez, and Wood [14]). *For $k \geqslant 2$, the k-subdivision of K_n, denoted by S, satisfies*

$$\left(\frac{n}{2}\right)^{1/(k+1)} \leqslant \pi(S) \leqslant 9\left\lceil n^{1/(k+1)}\right\rceil.$$

Since $\phi(G) \geqslant \pi(G)$, the lower bound in Theorem 7 implies that $k \geqslant \log_c(n/2) - 1$ for every anagram-free c-colorable k-subdivision of K_n. We prove the following improvement.

THEOREM 8. *Let S be a $(\leqslant k)$-subdivision of K_n. If S is anagram-free c-colorable, then*

$$k \geqslant \left(c!\left(\frac{n}{c} - 1\right)\right)^{1/c} - c.$$

For fixed c, the bound in Theorem 8 is $k \geqslant \Omega\left(n^{1/c}\right)$, which is larger than the logarithmic bound implied by Theorem 7. Still, this lower bound is much less than the exponential upper bound implied by Theorem 5. We expect that both our upper and lower bounds on k can be significantly improved.

2. Basic observations. This section contains basic observations and definitions that will be used throughout the rest of the paper. A *color multiset* of size n on c colors is a multiset of size n with entries from $[c] := \{1, 2, \dots, c\}$. Let $\mathcal{M}_{n,c}$ be the set of all color multisets of size n on c colors, and let $\mathcal{M}_{\leqslant n,c}$ be the set of all color multisets of size at most n on c colors. For a colored graph G, define the following. Let $M(G)$ be the multiset of colors assigned to the vertices of G. For a subset, C, of the colors, let $M_C(G)$ be $M(G)$ restricted to C. Let $V_C(G)$ be the vertices of G that have a color from C.

Call a path *even* if it has an even number of vertices. Define LR to be the *split* of an even path, P, if $|L| = |R|$ and $P = LR$. Note that a colored path, P, is an anagram if and only if $M(L) = M(R)$. Equivalently, P is not an anagram if $M_C(L) \neq M_C(R)$ for some set of colors C. For a path P and set of colors C, define P *restricted to* C to be the word $\omega_C(P) := f(v_1)f(v_2)\dots f(v_x)$, where $v_1, v_2, \dots, v_x$ are the vertices in $V_C(P)$, in the order defined by P, and f is the vertex coloring of P. Similarly, for a word $W = w_1w_2\dots w_n$ and set of symbols C, define W *restricted to* C to be $\omega_C(W) := f(w_1)f(w_2)\dots f(w_n)$, where $f(w) = w$ if $w \in C$ and $f(w)$ is the empty character otherwise. We use these observations in the form of the following lemma.

LEMMA 9. *A path, P, colored by C, is an anagram if and only if for all $C' \subseteq C$, P restricted to C' is an anagram or the empty word.*

Proof. We first prove the forward implication. Let $C' \subseteq C$ be such that $\omega_{C'}(P)$ is nonempty since the empty case is trivial. Let LR be the split of P. Note that $M_{C'}(L) = M(\omega_{C'}(L))$ and $M_{C'}(R) = M(\omega_{C'}(R))$. Since P is an anagram,

$$M(\omega_{C'}(L)) = M_{C'}(L) = M_{C'}(R) = M(\omega_{C'}(R)).$$

Therefore $\omega_{C'}(P)$ is an anagram.

To prove the back implication, take $C' = C$. Then P restricted to C', which is all of P, is an anagram. ∎

The *midedge* of an even path P with split LR is the edge of P not contained in L or R. For a connected graph G, define the *distance* between an edge ab and a vertex v to be the minimum of $\operatorname{dist}(a, v)$ and $\operatorname{dist}(b, v)$.

3. Subdivisions of trees. This section contains our results for trees. For every vertex v in a rooted tree T, define $A_T(v)$ to be the set of ancestors and descendants of v in T. A *branch vertex* is a vertex of a rooted tree with at least two children.

3.1. Subdivisions of binary trees.

THEOREM 2. *Every binary tree, T, of height h, has a $(\leqslant 3^{h-1} - 1)$-subdivision, S, with $\phi(S) \leqslant 8$.*

Proof. 2-color the edges of T with $\{1, 2\}$ such that for every branch vertex, $v \in V(T)$, with children c_1 and c_2, the edge vc_i receives color i. Color the remaining edges arbitrarily from $\{1, 2\}$. Let S be the subdivision of T such that edges at distance x from the root are subdivided $3^{h-x-1} - 1$ times. Note that edges incident with leaves of depth h are not subdivided.

Let r be the root of S. Label the vertices of S according to the edge 2-coloring of T as follows:

- Label division vertices with the color of the corresponding edge in T.
- Label r with 1.
- Label the original non-root vertices with the label of their parent edge in T.

Let $W = w_1 w_2 \dots$ be an anagram-free word on $\{1,2,3,4\}$. Define $V_\ell(S)$ to be the set of vertices with label ℓ in S. Color every vertex $v \in V(S)$ by (ℓ, w_x) where ℓ is the label of v and x is the number of vertices with label ℓ on the vr-path. We now show that this 8-coloring of S is anagram-free.

Let P be an even order path in S. Consider the case where there is some $\ell \in \{1,2\}$ such that $A_S(v) = V_\ell(P)$ for all vertices $v \in V_\ell(P)$. If $V_\ell(P) = \emptyset$, then P is not an anagram because, by construction, S restricted to a label is anagram-free. So now consider $V_\ell(P) \neq \emptyset$ and let $C' = \{\ell\} \times \{1,2,3,4\}$. Then $\omega_{C'}(P) = (\ell, w_y)(\ell, w_{y+1}) \dots (\ell, w_{y+|V_\ell(P)|})$, for some integer y, because the number of ℓ labelled vertices on the vr-path increments by 1 for all vertices $v \in V(P_\ell)$ along P. Therefore $\omega_{C'}(P)$ is a subword of W. Thus, by Lemma 9, P is not an anagram.

Now consider the case where for every $\ell \in \{1,2\}$ there exists a $v \in V_\ell(P)$ such that $A_S(v) \neq V_\ell(P)$. Let u be the minimum depth vertex in $V(P)$. Both labels have vertices that are not mutual ancestors or descendants. Thus, u has two children in T, $x, y \in V(T)$, and, in addition, $x, y \in V(P)$.

Partition $V(P)$ into $X := (V(P) \cap A_S(x)) \setminus \{u\}$ and $Y := V(P) \cap A_S(y)$. Let LR be the split of P such that, without loss of generality, $u \in V(R)$. Also without loss of generality, choose x and y such that $Y \subseteq V(R)$. Since $Y \cap V(L) = \emptyset$,

$$|X \cap V(L)| = |V(L)| = |V(R)| = |X \cap V(R)| + |Y \cap V(R)|.$$

Let z be the integer such that $3^z + 1$ is the order of the ux-path in S. We will prove an upper bound on $|X \cap V(R)|$ to show that the midedge of P is "close" to u. First, note that

$$|Y \cap V(R)| \geqslant 3^z + 2$$

because the edge uy was subdivided $3^z - 1$ times. Since $|X|$ is at most the length of a path from u to a leaf,

$$|X \cap V(L)| \leqslant 3^z + 3^{z-1} + \dots + 3^1 + 1 - |X \cap V(R)| = \frac{3}{2}3^z - \frac{1}{2} - |X \cap V(R)|.$$

Therefore

$$|X \cap V(R)| = |X \cap V(L)| - |Y \cap V(R)| \leqslant \frac{3}{2}3^z - \frac{1}{2} - |X \cap V(R)| - 3^z - 2.$$

Thus

$$|X \cap V(R)| \leqslant \frac{1}{4}3^z - \frac{3}{4}.$$

Without loss of generality, let x have label 1 and y have label 2. Indeed, the labels of x and y are distinct because edges ux and uy have different colors in T. Since all vertices on the ux-path (except possibly u) have label 1,

$$|V_1(L)| \geqslant 3^z - |X \cap V(R)| \geqslant \frac{3}{4}3^z + \frac{3}{4}.$$

To put an upper bound on $|V_1(R)|$, assume the worse case, that u has label 1. Then

$$\begin{aligned} |V_1(R)| &\leqslant |X \cap V_1(R)| + 1 + 3^{z-1} + 3^{z-2} + \dots + 3^1 + 1 \\ &\leqslant \frac{1}{4}3^z - \frac{3}{4} + 1 + \frac{3}{2}3^{z-1} - \frac{1}{2} \\ &= \frac{3}{4}3^z - \frac{5}{4}. \end{aligned}$$

It follows that $|V_1(R)| < |V_1(L)|$. Therefore P is not an anagram. □

3.2. Subdivisions of d-ary trees. The construction in Theorem 2 does not extend to a good bound on ϕ for subdivisions of complete d-ary trees. The obvious extension, using d labels for the edge coloring, shows that the complete d-ary tree has a $4d$-colorable subdivision. We prove the following result for complete d-ary trees.

THEOREM 10. *The complete d-ary tree, T, of height h, has a $\left(\leqslant 2d(d+1)^{h-1}\right)$-subdivision, S, with $\phi(S) \leqslant 10$.*

Proof. Let r be the root of T. For all $x \in [h]$ and $y \in [d]$, let $t_{x,y} = y(d+1)^{x-1}$. Define the labelling $\ell : E(T) \to [d]$ such that edges incident with the same parent vertex receive distinct labels. Let S be the subdivision of T such that every edge $e \in E(T)$ is subdivided $2t_{h-z,\ell(e)}$ times where z is the depth of e. Note that $z \in \{0, \ldots, h-1\}$.

Let $\ell_T : V(T) \to \{\text{black}, \text{white}\}$ be a proper vertex 2-coloring of T. Define the labelling $\ell_S : V(S) \to \{\text{black}, \text{white}, \text{red}, \text{green}\}$ as follows. If $v \in V(S)$ is an original vertex, then $\ell_S(v) := \ell_T(v)$. Otherwise, let $v' \in V(S)$ be the closest original vertex to v and $e \in E(T)$ be the edge such that v is a division vertex of e. If v' is the parent of e, then $\ell_S(v) :=$ red; otherwise, $\ell_S(v) :=$ green. Note that v' is well defined because all edges of T have an even number of division vertices. See Figure 1 for an example of this construction.

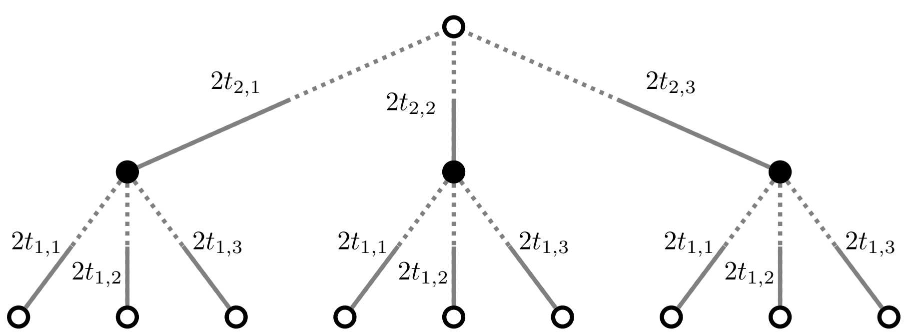

FIG. 1. *S for the complete 3-ary tree of height 2. The edges represent a number of division vertices denoted by their label. Each edge has a dotted half representing its red division vertices and a solid half representing its green division vertices.*

Define the *red-depth* of a vertex $v \in V(S)$ to be the number of red vertices on the vr-path in S, and define *green-depth* analogously. Let $W = w_1w_2\ldots$ be a long anagram-free word on $\{1,2,3,4\}$. Define the vertex coloring f as follows. If $v \in V(S)$ is an original vertex, then color v by $\ell_S(v)$. Otherwise, let i be the $\ell_S(v)$-depth of v and define $f(v) := (w_i, \ell_S(v))$. A vertex has label black or white if and only if it is an original vertex. Thus, f is a 10-coloring of S.

Let P be a path in S, and assume for the sake of contradiction that P is an anagram. P contains at least one division vertex because the original vertices have a proper coloring in T, and all edges not incident with leaves have at least one division vertex. Let u be the vertex with minimum depth in P.

First, consider the case where u is an endpoint of P. In this case, $V(P) \subseteq A_S(v)$ for all vertices $v \in V(P)$. Without loss of generality, P contains a red division vertex. Therefore the red-depth increments by one for red vertices along P. It follows that the sequence of colors along the red vertices of P is a subword of W. Thus, $\omega_{\text{red}}(P)$ is not an anagram. Therefore, by Lemma 9, P is not an anagram.

The remaining case is where u is not an endpoint of P. In this case, u is an original vertex. For all $e \in E(T)$, let D_e be the division vertices of e. Say that P *hits* e if $D_e \cap V(P) \neq \emptyset$ and that P *contains* e if $D_e \subseteq V(P)$. Let α be the largest edge in T (the edge with most division vertices in S) hit by P and β be the second largest edge in T hit by P. Since $t_{x,y} > t_{x',y'}$ for all y, y', and $x > x'$, the edges of T are larger nearer the root. Thus, both α and β are adjacent to u. Let v_α and v_β be the endpoints of P denoted such that the uv_α-path hits D_α.

Let $C' = \{\text{red}, \text{green}\} \times \{1, 2, 3, 4\}$ and define W_L, W_α, W_β, and W_R so that the concatenation $W_L W_\alpha W_\beta W_R = \omega_{C'}(P)$ and

- W_L is the subword corresponding to the division vertices in $V(uv_\alpha\text{-path}) \setminus D_\alpha$,
- W_α is the subword corresponding to the vertices $V(P) \cap D_\alpha$,
- W_β is the subword corresponding to the vertices $V(P) \cap D_\beta$, and
- W_R is the subword corresponding to the remaining division vertices of P.

Note that each of W_L and W_R may be the empty word.

Let x_α, y_α, and y_β be such that $|D_\alpha| = 2t_{x_\alpha, y_\alpha}$ and $|D_\beta| = 2t_{x_\alpha, y_\beta}$. First,

$$|W_L| \leqslant b := 2 \sum_{i=1}^{x_\alpha - 1} t_{i,d}$$

because $|W_L|$ is at most the number of division vertices on the longest path from the child of α to a leaf of S. Similarly, $|W_R| \leqslant b$. For all $x \in [h]$,

$$t_{x,1} = 1 + \sum_{i=1}^{x-1} t_{i,d}$$

because, by induction on x,

$$t_{x,1} = (d+1)t_{x-1,1} = (d+1)\left(1 + \sum_{i=1}^{x-2} t_{i,d}\right) = (d+1) + \sum_{i=2}^{x-1} t_{i,d} = 1 + \sum_{i=1}^{x-1} t_{i,d}.$$

Therefore

$$|D_\alpha| = 2t_{x_\alpha, y_\alpha} \geqslant 2t_{x_\alpha,1} = 2 + 2 \sum_{i=1}^{x_\alpha - 1} t_{i,d} > b.$$

Similarly, $|D_\beta| > b$. Also,

$$|D_\alpha| = 2y_\alpha t_{x_\alpha,1} \geqslant 2y_\beta t_{x_\alpha,1} + 2t_{x_\alpha,1} = 2t_{x_\alpha, y_\beta} + 2 + 2 \sum_{i=1}^{x_\alpha - 1} t_{i,d} > |D_\beta| + b.$$

Recall that the vertex coloring of T is a proper 2-coloring and that $V(P)$ contains an original vertex. The shortest anagram in a proper 2-coloring has four vertices. Therefore, by Lemma 9, both L and R contain at least two original vertices. Thus, P contains at least three edges of T. This implies that at least one of W_L and W_R is not the empty word. Thus at least one of α and β is contained in P. Let LR be the split of P with $v_\alpha \in V(L)$.

Consider the case where α is not contained in P. Since $|W_\beta| = |D_\beta| > b \geqslant |W_R|$, W_R is a subword of $\omega_{C'}(R)$. This implies that L only contains one original vertex, which is a contradiction. Thus, P is not an anagram.

Now consider the case where α is contained in P. Then $|W_\alpha| = |D_\alpha|$. Since exactly half the division vertices of each edge are labelled red,

$$\begin{aligned}|\omega_{\text{red}}(W_\alpha)| = |\omega_{\text{green}}(W_\alpha)| &= \frac{|W_\alpha|}{2},\\ |\omega_{\text{green}}(W_\beta)| &\leqslant \frac{|W_\beta|}{2},\\ |\omega_{\text{red}}(W_L)| &\leqslant \frac{b}{2},\\ |\omega_{\text{green}}(W_R)| &\leqslant \frac{b}{2}.\end{aligned}$$

If all vertices corresponding to $\omega_{\text{green}}(W_\alpha)$ are in L, then

$$|\omega_{\text{green}}(L)| \geqslant |\omega_{\text{green}}(W_\alpha)| > \frac{|D_\beta| + b}{2} \geqslant |\omega_{\text{green}}(W_\beta)| + |\omega_{\text{green}}(W_R)| \geqslant |\omega_{\text{green}}(R)|.$$

Thus P is not an anagram. If all vertices corresponding to $\omega_{\text{red}}(W_\alpha)$ are in R, then

$$|\omega_{\text{red}}(R)| \geqslant |\omega_{\text{red}}(W_\alpha)| > \frac{b}{2} \geqslant |\omega_{\text{red}}(W_L)| \geqslant |\omega_{\text{red}}(L)|.$$

Thus P is not an anagram. This covers all cases since $v_\alpha \in V(L)$. ∎

Theorem 3 is a simple corollary of Theorem 10.

THEOREM 3. *Every d-ary tree, T, of height h, has a $\left(\leqslant 2d(d+1)^{h-1}\right)$-subdivision, S, with $\phi(S) \leqslant 10$.*

Proof. Apply Theorem 10 to the complete d-ary tree of height h, and take the appropriate subgraph of the resulting subdivision. ∎

The next section shows that the exponential upper bound on the number of division vertices per edge in Theorem 3 is necessary.

3.3. Lower bounds. This subsection extends Theorem 1, for complete binary trees, by Kamčev, Łuczak, and Sudakov [11]. We generalize their method of proof to obtain a result about subdivisions of high degree trees. The following definitions are extensions of those found in their original paper.

Let T be a rooted tree with root r. The *effective vertices* of T are its leaves and branch vertices. The *effective root* of T is the closest effective vertex to r, including r. The *effective height* of T is the minimum, over the leaves of T, of the number of branch vertices on each root to leaf path.

Call T *essentially i-monochromatic* if all of its effective vertices are colored i. Call T *essentially monochromatic* if it is essentially i-monochromatic for some i. For $d \geqslant 2$, a *d-branch tree* is a rooted tree such that every branch vertex has at least d children.

LEMMA 11. *For all integers $a_1, \dots, a_c \geqslant 0$ and $d \geqslant 2$, every d-branch tree with vertices colored by $[c]$ and effective height at least $\sum_{i=1}^{c} a_i$ contains an essentially i-monochromatic d-branch subtree of effective height at least a_i for some $i \in [c]$.*

Proof. We proceed by induction on $\sum_{i=1}^{c} a_i$. The base case, $a_1 = \cdots = a_c = 0$, is satisfied by taking a single vertex as the required d-branch subtree.

Let T be a d-branch tree of effective height $a_1 + \cdots + a_c \geqslant 1$ with vertices colored by $[c]$. Without loss of generality, its effective root, v, has color 1. Let $v_1, \dots, v_d$

be children of v. Let T_j be the subtree rooted at v_j. Note that T_j has effective height at least $(a_1 - 1) + a_2 + \cdots + a_c$. If, for some $j \in [d]$ and $i \in \{2, \ldots, c\}$, T_j contains an essentially i-monochromatic subtree of effective height a_i, then we are done. Otherwise, by induction, each T_j contains an essentially 1-monochromatic d-branch subtree of effective height $a_1 - 1$. These subtrees, together with v, are an essentially 1-monochromatic d-branch subtree of T, as required. ∎

We now prove a lower bound on ϕ by using an essentially monochromatic subtree to find anagrams in sufficiently large trees.

THEOREM 12. *Let T be a d-branch tree of effective height at least h' and height at most $h \geqslant \max\{2, \sqrt{d}\}$. Then*

$$\phi(T) \geqslant c := \left\lceil \sqrt{\frac{h'}{\log_d h}} \,\right\rceil.$$

Proof. If $c \leqslant 1$, the theorem follows trivially, so assume $c > 1$. Let T be colored with x colors where $1 \leqslant x \leqslant c - 1$. Our goal is to show that T contains an anagram. For $i \in [x]$, define $a_i \in \{\lfloor h'/x \rfloor, \lceil h'/x \rceil\}$ such that $\sum_{i=1}^{x} a_i = h'$. By Lemma 11, and without loss of generality, T contains an essentially 1-monochromatic d-branch subtree, S, of effective height at least $\lfloor h'/x \rfloor$.

Let r be the root of S. There are at least $d^{\lfloor h'/x \rfloor}$ paths from r to the leaves of S, and the coloring of each path defines a multiset of order at most $h+1$. Since each path shares the color of r, there are at most h^x distinct multisets that can occur on the paths. Since $x \leqslant c - 1$,

$$\#\text{multisets} \leqslant h^x < h^{(c^2/x)-2}.$$

Since $h \geqslant \sqrt{d}$,

$$h^{(c^2/x)-2} \leqslant \frac{1}{d} h^{(c^2/x)}.$$

Therefore

$$\#\text{multisets} < \frac{1}{d} h^{(c^2/x)} = \frac{1}{d} \left(h^{\frac{1}{\log_d h}} \right)^{(h'/x)} = d^{(h'/x)-1} \leqslant d^{\lfloor h'/x \rfloor} \leqslant \#\text{paths}.$$

So there is a multiset that occurs on two different paths, P_1 and P_2, from r to the leaves of S. Let v be the lowest common vertex of P_1 and P_2, and let ℓ_i be the leaf endpoint of P_i. Since $M(P_1) = M(P_2)$, by definition, $M(P_1 - P_2) = M(P_2 - P_1)$. Since S is essentially 1-monochromatic, the vertices v, ℓ_1, and ℓ_2 have color 1. Thus, $((P_1 - P_2) \setminus \{\ell_1\})((P_2 - P_1) \setminus \{v\})$ is an anagram. ∎

3.4. Bounds for subdivisions of the complete d-ary tree. We now use Theorem 10 to prove an upper bound on ϕ for some subdivision of a given tree.

COROLLARY 1. *For every $k \geqslant 0$ and every complete d-ary tree of height h', T, there exists a $(\leqslant k)$-subdivision, S, such that*

$$\phi(S) \leqslant c := 10 \left\lceil \frac{h'}{\log_{d+1}(k/2d)} \right\rceil.$$

Proof. Let $x := c/10$, and let $B \subseteq E(T)$ be the set of edges with depths $i\lceil h'/x \rceil - 1$ for $i \in \{0, \ldots, x-1\}$, recalling that the depth of an edge is the minimum depth of its endpoints. Let $F := T - B$, and note that F is a forest where each component is a complete d-ary tree of height at most $\lceil h'/x \rceil$. Let $\mathcal{C}$ be the set of components of F. Root each component, $C \in \mathcal{C}$, at the vertex $r \in V(C)$ with minimum depth in T. The depth of r is $i\lceil h'/x \rceil$ for some $i \in \{0, \dots, x-1\}$. Define the *depth* of C to be i.

By the definition of c and x,

$$\log_{d+1}\left(\frac{k}{2d}\right) \geqslant \frac{h'}{x}.$$

This implies

$$k \geqslant 2d(d+1)^{\frac{h'}{x}} \geqslant 2d(d+1)^{\left\lceil \frac{h'}{x} \right\rceil - 1}.$$

Therefore, by Theorem 10, for every $C \in \mathcal{C}$, there exist a $(\leqslant k)$-subdivision, S_C, with $\phi(S_C) \leqslant 10$ since C has height at most $\lceil h'/x \rceil$. Anagram-free color S_C using colors $\{10i+1, \dots, 10(i+1)\}$ where i is the depth of C. Let $S = B + \cup_{C \in \mathcal{C}} S_C$. Note that S is a $(\leqslant k)$-subdivision of T with a $10x$ coloring. We now show that this coloring of S is anagram-free.

Let P be a subpath of S. Let $j \in \{0, \dots, x-1\}$ be the minimum depth of component $C \in \mathcal{C}$ such that S_C has a non-empty intersection with P. By the construction of S, P intersects with exactly one $C' \in \mathcal{C}'$ of depth j. Therefore P restricted to the colors of C' corresponds to a subpath of C', and, since C' is anagram-free, the restriction is not an anagram. Therefore, by Lemma 9, P is not an anagram. ∎

The following lemma generalizes results for $(\leqslant k)$-subdivisions to k-subdivisions. Note that the k-subdivision a graph, G, is a subdivision of every $(\leqslant k)$-subdivision of G.

LEMMA 13. *If S is a subdivision of G, then $\phi(S) \leqslant \phi(G) + 4$.*

Proof. Fix an anagram-free $\phi(G)$-coloring of G, and apply the coloring to the original vertices of S. The graph induced by the division vertices of S is a forest of paths. Color all of these paths with an anagram-free coloring on four new colors. By Lemma 9, this coloring of S is anagram-free. ∎

We now prove Theorem 4, introduced in section 1.1.

THEOREM 4. *The k-subdivision, S, of the complete d-ary tree of height h' satisfies*

$$\sqrt{\frac{h'}{\log_{\min\{d,(h'(k+1))^2\}}(h'(k+1))}} \leqslant \phi(S) \leqslant \frac{10h'}{\log_{d+1}(k/2d)} + 14.$$

Proof. Theorem 12 proves the lower bound. Corollary 1 and Lemma 13 prove the upper bound. ∎

4. Subdivisions of general graphs. Now we construct subdivisions of arbitrary graphs with bounded anagram-free chromatic number. Let $t = t_1, t_2, \dots$ be a sequence of positive integers. A subdivision, S, of a bipartite graph G, is a *t-sequence-subdivision of* G if there is a bijection, $\ell : V(G) \to [|V(G)|]$, that satisfies the following two conditions. The first condition is that there is a proper 2-coloring of G, with colors white and black, such that $\ell(u) > \ell(v)$ for every white vertex $u \in V(G)$ and black vertex $v \in V(G)$. The second condition requires some definitions. For every edge, $e \in E(G)$, define $w(e)$ to be the white vertex incident with e and $b(e)$ be the black vertex incident with e. Define the bijection, $\ell' : E(G) \to [|E(G)|]$, that orders edges in $E(G)$, first by the label of their white endpoint and second by the label of their black endpoint. That is, $\ell'(x) > \ell'(y)$ for edges $x, y \in E(G)$ if $\ell(w(x)) > \ell(w(y))$ or if $\ell(w(x)) = \ell(w(y))$ and $\ell(b(x)) > \ell(b(y))$. Note that ℓ' is determined by ℓ. The second condition on ℓ is that every edge, $e \in E(G)$, has $3t_{\ell'(e)}$ division vertices.

Let G be a graph, t be a sequence of positive integers, and S be a t-sequence-subdivision G with corresponding vertex and edge labellings ℓ and ℓ'. We now define the functions X, Y, and Z to divide subdivision vertices of every edge of G into disjoint subpaths. For each edge $uv \in E(G)$, with u colored white, define $X(uv)$, $Y(uv)$, and $Z(uv)$ such that $uX(uv)Y(uv)Z(uv)v$ is the path replacing uv in the subdivision S, with $|V(X(e))| = |V(Y(e))| = |V(Z(e))| = t_{\ell'(e)}$. Define sets of these subpaths, $\mathcal{X} := X(E(G))$, $\mathcal{Y} := Y(E(G))$, and $\mathcal{Z} := Z(E(G))$. A vertex coloring of S is *discriminating* if the following conditions hold.

(1) The original vertices of S are colored by the proper 2-coloring of G, and these two colors only occur on the original vertices.
(2) Every anagram in S contains at least one original vertex.
(3) For all $Q \in \{X, Y, Z\}$, there exists a nonempty set of colors, $C(Q)$, that occur only on the vertices of paths in $Q(E(G))$.
(4) For all $Q \in \{X, Y, Z\}$ and $q \in E(G)$,

$$\sum_{e \in E(G): \ell'(e) < \ell'(q)} |V_{C(Q)}(Q(e))| < |V_{C(Q)}(Q(q))|.$$

Note that whether S has a discriminating vertex coloring depends on the sequence t. For example, the sequence $t_i = 1$, for all i, causes condition (4) to fail for sufficiently large G.

THEOREM 14. *Let S be a t-sequence-subdivision of a graph G with sequence t. Every discriminating vertex coloring of S is anagram-free.*

Proof. Let ℓ and ℓ' be the associated vertex and edge labellings of G, respectively. Let f be a discriminating vertex coloring of S.

Let P be a path in S, and assume for the sake of contradiction that P is an anagram. By condition (2), $V(P)$ contains at least one original vertex. Since G is properly 2-colored, all subpaths of G that are anagrams have order at least 4. The 2-coloring of G is applied to the original vertices of S. Thus, by Lemma 9, P contains at least four original vertices. Therefore P has at least one subpath from each of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$. Let $x, y, z \in E(G)$ be the edges maximizing ℓ' such that $V(P) \cap V(X(x)) \neq \emptyset$, $V(P) \cap V(Y(y)) \neq \emptyset$, and $V(P) \cap V(Z(z)) \neq \emptyset$.

A path, P', *partially intersects* P if $V(P') \nsubseteq V(P)$ and $V(P') \cap V(P) \neq \emptyset$. There are at most two paths in $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ that partially intersect P since every division vertex has degree 2 in S. Therefore at least one of $X(x)$, $Y(y)$, and $Z(z)$ is a subpath of P. Define $q \in \{x, y, z\}$ and $Q \in \{X, Y, Z\}$ such that $Q(q) \in \{X(x), Y(y), Z(z)\}$ is a subpath of P. Since f is a discriminating coloring,

$$\sum_{e \in E(G): \ell'(e) < \ell'(q)} |V_{C(Q)}(Q(e))| < |V_{C(Q)}(Q(q))|.$$

Therefore, by the maximality of $\ell'(q)$, there are more vertices in $Q(q)$ colored by $C(Q)$ than there are vertices colored by $C(Q)$ in the rest of of P. Thus $|V_{C(Q)}(Q(q))| > \frac{1}{2}|V_{C(Q)}(P)|$. Let LR be the split of P. By Lemma 9, $V_{C(Q)}(L) = V_{C(Q)}(R) = \frac{1}{2}|V_{C(Q)}(P)|$. Thus, both L and R intersect $Q(q)$. Therefore the midedge of P is an edge of $Q(q)$. Since the midedge of P is unique, exactly one of $X(x)$, $Y(y)$, and $Z(z)$ is a subpath of P.

Since G is properly 2-colored, every subpath of G that is an anagram has a white endpoint and a black endpoint. Therefore one of the endmost original vertices of P is white; call this vertex α. Since P partially intersects exactly two of $X(x)$, $Y(y)$, and $Z(z)$, there is a black vertex $\beta \in N_\alpha(G)$ such that $\alpha\beta \in \{x, y, z\}$, where $N_\alpha(G)$ is the neighborhood of α. Recall that both L and R contain at least two original vertices and the midpoint of P is in $Q(q)$. Therefore neither endpoint of q is an endmost original vertex of P, and so $\alpha \neq w(q)$. Also, there is a black vertex, $\gamma \in N_\alpha(G)$, such that the division vertices of $\alpha\gamma$ are all in P. Since $\alpha\beta \in \{x, y, z\}$ and $\alpha\beta \neq q$, there is an $A \in \{X, Y, Z\}$ such that $A(\alpha\beta) \in \{X(x), Y(y), Z(z)\}$ for some $A \neq Q$. It follows that $\ell'(\alpha\beta) > \ell'(q)$ because $A(q)$ is a subpath of P and $\ell'(\alpha\beta)$ is maximal. Therefore $\ell(\alpha) > \ell(w(q))$, and so $\ell'(\alpha\gamma) > \ell'(q)$. This contradicts the maximality of $\ell'(q)$ because $Q(\alpha\gamma)$ is a subpath of P. ◻

We now use Theorem 14 to prove Theorem 5.

THEOREM 5. *Every graph G' has a ($\leqslant 3(2)^{2|E(G')|-1}$)-subdivision, S, with $\phi(S) \leqslant 14$.*

Proof. Let G be the 1-subdivision of G', and note that G has a proper 2-coloring. Define the sequence t by $t_i = 2^{i-1}$ for $i \geqslant 1$. Let S be a t-sequence-subdivision of G. Since G has $2|E(G')|$ edges and $3t_{2|E(G')|} = 3(2)^{2|E(G')|-1}$, S satisfies the bound on division vertices per edge required by the theorem. Let ℓ and ℓ' be the associated vertex and edge labellings of G.

Let f be the vertex coloring of S defined as follows. Color the original vertices of S with the proper 2-coloring of G that corresponds to ℓ. Assign a disjoint set of four colors to each of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$. Color each of the paths in $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ with an anagram-free 4-coloring with their assigned set of four colors.

We now show that f is discriminating. Conditions (1) and (3) are satisfied trivially. Condition (2) is satisfied because each of the paths in $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ is anagram-free and they use their own set of colors. Thus, every anagram in S contains an original vertex. Condition (4) is satisfied because for all $Q \in \{X, Y, Z\}$ and $q \in E(G)$, $|V_{C(Q)}(Q(q))| = |V(Q(q))|$, and

$$\sum_{e \in E(G) : \ell'(e) < \ell'(q)} |V(Q(e))| = 2^{\ell'(q)-2} + \cdots + 1 = 2^{\ell'(q)-1} - 1 < 2^{\ell'(q)-1} = |V(Q(q))|.$$

Therefore f is an anagram-free 14-coloring of S. ◻

We use Theorem 5 to bound ϕ on subdivisions of graphs in terms of division vertices per edge.

THEOREM 15. *For every graph G and $k \in \mathbb{Z}^+$, there exists a ($\leqslant 3(4)^{\lceil |E(G)|/k \rceil}$)-subdivision, S, of G with $\phi(S) \leqslant 2 + 12k$.*

Proof. Take k subgraphs of G with an equitable number of edges per subgraph. Subdivide and color them using Theorem 5. Merge these subdivisions to obtain an anagram-free $2 + 12k$ coloring of G. ◻

We now optimize our use of Theorem 14 to improve the upper bound on ϕ.

THEOREM 6. *Every graph G' has a ($\leqslant 45\left(1 + \frac{75}{9}\right)^{2|E(G')|-1}$)-subdivision, S, with $\phi(S) \leqslant 8$.*

Proof. Let G be the 1-subdivision of G', and note that G has a proper 2-coloring. Define the sequence t with $t_1 = 8$ and

$$t_n = 15 + \left\lfloor \frac{25}{3} \sum_{i=1}^{n-1} t_i \right\rfloor . \tag{1}$$

Let S be a t-sequence-subdivision of G. It is straightforward to verify that $t_n \leqslant 15\left(1 + \frac{75}{9}\right)^{n-1}$. Thus, S satisfies the limit on division vertices per edge required by the theorem. Let ℓ and ℓ' be the associated vertex and edge labellings of G.

Define the coloring $f : V(S) \to \{1, 2, 3, 4, 5, 6, \text{white}, \text{black}\}$ as follows. Original vertices are colored white or black according to ℓ. For every $e \in E(G)$, define $P_e = v_1 \dots v_{3t_{\ell'(e)}}$ to be the division vertices of e. Let W be an anagram-free word on $\{1, 2, 3, 4\}$ of length $3\ell'(e)$, and color P_e as follows. For all $v_i \in V(P_e)$, if $W_i \in \{1, 2, 3\}$, then $f(v_i) := W_i$. Otherwise, $f(v_i) := 4$ if $v_i \in V(X(e))$, $f(v_i) := 5$ if $v_i \in V(Y(e))$, and $f(v_i) := 6$ if $v_i \in V(Z(e))$.

We now show that f is discriminating. Condition (1) is satisfied trivially. Condition (2) is satisfied because P_e is colored by an anagram-free word for all $e \in E(G)$. Condition (3) is satisfied by $C(X) = \{4\}$, $C(Y) = \{5\}$, and $C(Z) = \{6\}$. We now show that condition (4) is satisfied.

Let $Q \in \{X, Y, Z\}$ and $q \in E(G)$. The same symbol cannot occur twice in a row. Thus, $|V_{C(Q)}(Q(q))| \leqslant \frac{5}{9}|V(Q(q))|$ since $|V(Q(q))| \geqslant 8$. Therefore

$$\sum_{e \in E(G):\ell'(e)<\ell'(q)} |V_{C(Q)}(Q(e))| \leqslant \frac{5}{9} \sum_{e \in E(G):\ell'(e)<\ell'(q)} |V(Q(e))|.$$

Every anagram-free word of length 8 contains at least four distinct symbols. Therefore $|V_{C(Q)}(Q(q))| \geqslant \frac{1}{15}|V(Q(q))|$. By (1),

$$\frac{5}{9} \sum_{e \in E(G):\ell'(e)<\ell'(q)} |V(Q(e))| = \frac{5}{9} \sum_{i=1}^{n-1} t_i \leqslant \frac{1}{15} t_n - 1.$$

Therefore

$$\sum_{e \in E(G):\ell'(e)<\ell'(q)} |V_{C(Q)}(Q(e))| \leqslant \frac{5}{9} \sum_{i=1}^{n-1} t_i < \frac{1}{15} t_n = \frac{1}{15}|V(Q(e))| \leqslant |V_{C(Q)}(Q(q))|.$$

Thus condition (4) is satisfied. Thus, f is an anagram-free 8-coloring of S. ∎

Theorem 6 uses naive bounds on the density of symbols in anagram-free words. Better bounds on density would improve the base of the exponential in Theorem 6.

4.1. Subdivisions of complete graphs. Recall that $\mathcal{M}_{k,c}$ is the set of color multisets on c symbols of size k and that $\mathcal{M}_{\leqslant k,c}$ is the set of color multisets of c symbols of size at most k.

THEOREM 8. *Let S be a $(\leqslant k)$-subdivision of K_n. If S is anagram-free c-colorable, then*

$$k \geqslant \left(c! \left(\frac{n}{c} - 1 \right) \right)^{1/c} - c.$$

Proof. Suppose for the sake of contradiction that

$$k < \left(c!\left(\frac{n}{c}-1\right)\right)^{1/c} - c. \tag{2}$$

Fix an anagram-free coloring of S. Color each edge $e \in E(K_n)$ with the color multiset of the subdivision vertices of e in S, and color each vertex of K_n with its color in S. Note that there are

$$|\mathcal{M}_{\leqslant k,c}| = \sum_{i=0}^{k} \binom{i+c-1}{c-1} = \binom{k+c}{c} \leqslant \frac{(k+c)^c}{c!}$$

possibilities for the color of each edge. Let $x := \lceil n/c \rceil$, and let G be a vertex-monochromatic K_x subgraph of K_n. Note that

$$|E(G)| = \frac{x}{2}(x-1) \geqslant \frac{x}{2}\left(\frac{n}{c}-1\right).$$

Therefore, by (2),

$$|E(G)| \geqslant \frac{x}{2}\left(\frac{n}{c}-1\right) > \frac{x}{2}\frac{(k+c)^c}{c!} \geqslant \frac{x}{2}|\mathcal{M}_{\leqslant k,c}| \geqslant \frac{x}{2}\#\text{colors}.$$

So there is a set of more than $x/2$ edges that have the same color. Therefore there is a vertex, $v \in V(G)$, that is incident with at least two edges, $\alpha, \beta \in E(G)$, with the same color. Let u be the other endpoint of α, P_α be the path induced by the division vertices of α, and P_β be the path induced by the division vertices of β. Then $uP_\alpha vP_\beta$ is an anagram in S. ∎

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