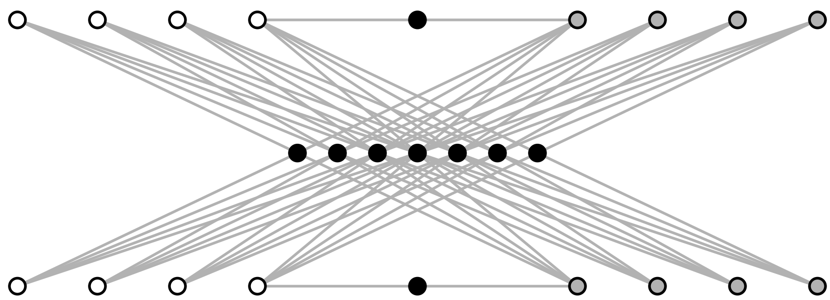
Combinatorial geometry of point sets with collinearities



Michael S. Payne

Department of Mathematics and Statistics

The University of Melbourne

Submitted in total fulfilment of the requirements
of the degree of Doctor of Philosophy

February 2014

Abstract

In this thesis we study various combinatorial problems relating to the geometry of point sets in the Euclidean plane. The unifying theme is that all the problems involve point sets that are not in general position, but have some collinearities. As well as giving rise to natural and interesting problems, the study of point sets with collinearities has important connections to other areas of mathematics such as number theory.

Dirac conjectured that every set P of n non-collinear points in the plane contains a point in at least $\frac{n}{2} - c$ lines determined by P, for some constant c. It is known that some point is in $\Omega(n)$ lines determined by P. We show that some point is in at least $\frac{n}{37}$ lines determined by P.

Erdős posed the problem to determine the maximum integer $f(n, \ell)$ such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n, \ell)$ points with no three collinear. First we prove that if $\ell \leqslant O(\sqrt{n})$ then $f(n, \ell) \geqslant \Omega(\sqrt{n/\ln \ell})$. Second we prove that if $\ell \leqslant O(n^{(1-\epsilon)/2})$ then $f(n, \ell) \geqslant \Omega(\sqrt{n \log_\ell n})$, which implies all previously known lower bounds on $f(n, \ell)$ and improves them when ℓ is not constant. Our results answer a symmetric version of the problem posed by Gowers, namely how many points are required to ensure there are q collinear points or q points in general position.

The visibility graph of a finite set of points in the plane has an edge between two points if the line segment between them contains no other points. We establish bounds on the edge- and vertex-connectivity of visibility graphs. We find that every minimum edge cut is the set of edges incident to a vertex of minimum degree. For vertex-connectivity, we prove that every visibility graph with n vertices and at most ℓ collinear vertices has connectivity at least $\frac{n-1}{\ell-1}$, which is tight. We also prove that the vertex-connectivity is at least half the minimum degree.

We study some questions related to bichromatic point sets in the plane. Given two disjoint point sets A and B in the plane, the bivisibility graph

has an edge between a point in A and a point in B if there are no other points on the line segment between them. We characterise the connected components of bivisibility graphs and give lower bounds on the number of edges and the maximum degree. We also show that all sufficiently large visibility graphs contain a given bipartite graph or many collinear points. Lastly we make some progress on a conjecture of Kleitman and Pinchasi about lower bounds on the number of bichromatic lines determined by a bichromatic point set.

An empty pentagon in a point set P in the plane is a set of five points in P in strictly convex position with no other point of P in their convex hull. We prove that every finite set of at least $328\ell^2$ points in the plane contains an empty pentagon or ℓ collinear points. This bound is optimal up to a constant factor.

Declaration

This is to certify that:

- this thesis comprises only my original work towards the PhD except where indicated in the Preface,
- due acknowledgement has been made in the text to all other material used, and
- this thesis is fewer than one hundred thousand words in length, exclusive of tables, maps, bibliographies and appendices.

Michael S. Payne

Preface

All work towards this thesis was carried out during the period of PhD candidature at the University of Melbourne. None of the work has been submitted for any other qualification. Except for results of other authors who are acknowledged as they are introduced, the results of Chapters 3 through 7 are to the best of my knowledge original contributions.

Most of this work is the result of academic collaboration which I gratefully acknowledge. Much of this work has been published in a peer reviewed journal or is publicly available as a preprint and currently under peer review. In each case I was primarily responsible for the planning, drafting and preparation of the work for publication.

Chapter 3 is the result of collaboration with my thesis advisor David Wood. It consists mostly of material from our preprint *Progress on Dirac's conjecture* [72] which is currently under peer review. Sections 3.2.1 and 3.2.2 contain additional material.

Chapter 4 is the result of collaboration with David Wood. It consists entirely of material from our paper *On the general position subset selection problem* [71], with some revisions.

Chapter 5 is the result of collaboration with Attila Pór, Pavel Valtr and David Wood. It consists entirely of material from our paper *On the connectivity of visibility graphs* [70], with some revisions.

The material of Chapter 6 is my own work and has not been published elsewhere. Parts of it can be considered as extensions to Chapters 3 and 5.

Chapter 7 is the result of collaboration with János Barát, Vida Dujmović, Gwenaël Joret, Ludmila Scharf, Daria Schymura, Pavel Valtr and David Wood. It consists entirely of material from our preprint *Empty pentagons in point sets with collinearities* [6] which is currently under peer review, with some revisions.

All figures were created by myself except for Figures 2.1, 5.1, 7.8 and 7.9

which were created by David Wood, and the remaining figures of Chapter 7 which were created by Ludmila Scharf and Daria Schymura.

Acknowledgements

I would like to thank my advisor David Wood for his time and encouragement, and all my co-authors with whom I enjoyed doing this work so much.

Thanks to the Group of Eight (Go8) and the German Academic Exchange Service (DAAD) for funding our project *Problems in geometric graph theory* which led in particular to the work in Chapter 7. Thanks to Jens Schmidt and Helmut Alt for organising the German side of the project, and for hosting me on various occasions at the Freie Universität Berlin. I am also grateful to the Australian Government for providing for my living expenses in the form of an Australian Postgraduate Award.

Thanks to my partner Anuradhi for her love and support and for sharing the life of the graduate student with me – it's been a lot of fun. Thanks also to Moritz and Kaie, and all my friends from Berlin, for their hospitality and company during my visits. And lastly thanks to my family for their support, and especially my parents for encouraging my curiosity at every stage.

Contents

List of Figures

Chapter 1

Summary

In this thesis, various problems in combinatorial geometry are studied. The unifying theme that runs through the problems is that they deal with finite point sets in the Euclidean plane that have some collinearities. This chapter contains a brief outline of the main contributions of the thesis. Many definitions, along with detailed discussion of the background to this work will be deferred until Chapter 2.

Dirac's Conjecture and Beck's Theorem

Chapters 3 and 4 deal with combinatorial problems about incidences between points and lines in the Euclidean plane.

Dirac [17] conjectured that every set P of n non-collinear points in the plane contains a point in at least $\frac{n}{2} - c_1$ lines determined by P, for some constant c_1. The following weakening was proved by Beck [7] and Szemerédi–Trotter [93]: every set P of n non-collinear points contains a point in at least $\frac{n}{c_2}$ lines determined by P, for some large positive constant c_2. In Chapter 3.1 we find new bounds on the constant c_2. Our main result is the following.

Theorem (3.3)**.** *Every set P of n non-collinear points in the plane contains a point in at least $\frac{n}{37}$ lines determined by P.*

In the same paper, Beck [7] proved that every set of n points in the plane with at most ℓ collinear determines at least $\frac{1}{c_3}n(n-\ell)$ lines, for some large positive constant c_3. This is one of a pair of related results that are now known as Beck's Theorem. In Chapter 3.2 we calculate the best known constant for Beck's Theorem, proving the following theorem.

Theorem (3.15)**.** *Every set P of n points with at most ℓ collinear determines at least $\frac{1}{93}n(n-\ell)$ lines.*

General position subset selection

In Chapter 4.1 we study the problem of selecting a set in general position from a set of points with collinearities, as originally posed by Erdős [28, 29]. Let $f(n,\ell)$ be the maximum integer such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n,\ell)$ points with no three collinear. We prove two main theorems.

Theorem (4.3)**.** *Let P be a set of n points with at most ℓ collinear. Then P contains a set of $\Omega(n/\sqrt{n\ln\ell+\ell^2})$ points in general position. In particular, if $\ell \leqslant O(\sqrt{n})$ then P contains a set of $\Omega(\sqrt{\frac{n}{\ln\ell}})$ points in general position.*

Theorem (4.5)**.** *Fix constants $\epsilon > 0$ and $d > 0$. Let P be a set of n points in the plane with at most ℓ collinear points, where $\ell \leqslant (dn)^{(1-\epsilon)/2}$. Then P contains a set of $\Omega(\sqrt{n\log_\ell n})$ points in general position.*

Theorem 4.5 implies all previously known lower bounds on $f(n,\ell)$ and improves them when ℓ is not constant. Theorem 4.3 provides an almost complete answer to a symmetric Ramsey style version of the general position subset selection problem posed by Gowers [40]. He asked for the minimum integer $\mathrm{GP}(q)$ such that every set of at least $\mathrm{GP}(q)$ points in the plane contains q collinear points or q points in general position. Gowers noted that $\mathrm{GP}(q) \geqslant \Omega(q^2)$, and Theorem 4.3 implies that $\mathrm{GP}(q) \leqslant O(q^2\ln q)$, so the asymptotic growth is determined up to a logarithmic factor.

In Chapter 4.2 we consider the more general problem of finding subsets with at most k collinear points in a point set with at most ℓ collinear, and prove analogous results in this setting too. Let $f(n, \ell, k)$ be the maximum integer such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n, \ell, k)$ points with at most k collinear.

Theorem (4.7)**.** *If $k \geqslant 3$ is constant and $\ell \leqslant O(\sqrt{n})$, then*

$$f(n, \ell, k) \geqslant \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}\right).$$

Theorem (4.9)**.** *Fix constants $d > 0$ and $\epsilon \in (0, 1)$. If $k \geqslant 3$ is constant and $4 \leqslant \ell \leqslant dn^{(1-\epsilon)/2}$ then*

$$f(n, \ell, k) \geqslant \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}(\ln n)^{1/k}\right).$$

There is a natural generalisation of Gowers' problem to finding subsets with at most k collinear. Let $\mathrm{GP}_k(q)$ be the minimum integer such that every set of at least $\mathrm{GP}_k(q)$ points in the plane contains q collinear points or q points with at most k collinear, for $k \geqslant 3$. It is not too hard to show that $\mathrm{GP}_k(q) \geqslant \Omega(q^2)$, and Theorem 4.7 implies that $\mathrm{GP}_k(q) \leqslant O(q^2)$, so in this case the asymptotic growth is determined up to a constant factor.

Connectivity of visibility graphs

In studying point sets and the lines they generate, it is often useful to consider the visibility graph of the point set. The *visibility graph* of a finite set of points in the plane has the points as vertices and an edge between two vertices if the line segment between them contains no other points in the set. In Chapter 5 we study visibility graphs in their own right, focussing on edge- and vertex-connectivity.

Unless all its vertices are collinear, a visibility graph has diameter at most 2, and so it follows by a result of Plesník [75] that its edge-connectivity equals its minimum degree. We strengthen the result of Plesník as follows.

Theorem (5.2)**.** *Let G be a graph with diameter* 2*. Then the edge-connectivity of G equals its minimum degree. Moreover, for all distinct vertices*

v and w in G, if $d := \min\{\deg(v), \deg(w)\}$ then there are d edge-disjoint vw-paths of length at most 4*.*

Furthermore, we characterise minimum edge-cuts in visibility graphs.

Theorem (5.6)**.** *Every minimum edge-cut in a non-collinear visibility graph is the set of edges incident to some vertex.*

For vertex-connectivity, we prove the following.

Theorem (5.11)**.** *Every non-collinear visibility graph with minimum degree δ has vertex-connectivity at least* $\frac{\delta}{2} + 1$*.*

Then we consider once again the parameter ℓ, the maximum number of collinear points.

Theorem (Corollary 5.15)**.** *Let G be the visibility graph of a set of n points with at most ℓ collinear. Then G has vertex-connectivity at least $\frac{n-1}{\ell-1}$, which is best possible.*

In the case that $\ell = 4$, we improve the bound in Theorem 5.11.

Theorem (5.18)**.** *Let G be a visibility graph with minimum degree δ and at most four collinear vertices. Then G has vertex-connectivity at least* $\frac{2\delta+1}{3}$*.*

Theorem 5.18 is best possible for every δ since there are point sets with at most three collinear points whose visibility graph has connectivity $\frac{2\delta+1}{3}$. The construction is due to Alperin, Buhler, Chalcraft and Rosenberg (see Trimble [96] for an account of the authorship of the construction). It uses real points on an elliptic curve and takes advantage of the group structure that exists on these points. It is described at the end of Chapter 5.

Bivisibility graphs

In Chapter 6 we study a kind of bipartite visibility graph that was useful in the investigation of the connectivity of visibility graphs. Given two disjoint point sets A and B in the plane, the *bivisibility graph* has vertices $A \cup B$ and an edge between a point in A and a point in B if there are no other points of $A \cup B$ on the line segment between them.

The number of edges in a bivisibility graph is at least the number of lines with a point from both A and B. These are called *bichromatic lines.* We apply Theorem 3.15 and some other known results [65, 80] to obtain the following lower bound on the number of bichromatic lines.

Theorem (Corollary 6.10)**.** *Let P be a set of n red and n blue points in the plane with at most ℓ collinear. Then P determines at least $\frac{1}{186}n(2n-\ell)$ bichromatic lines.*

This also gives a lower bound on the maximum degree of a bivisibility graph.

Corollary (6.11)**.** *Let A be a set of n red points and B a set of n blue points in the plane, such that $A \cup B$ is not collinear. Then the bivisibility graph $\mathcal{B}(A, B)$ has maximum degree at least $n/94$.*

Another corollary is related to an important conjecture of Kára, Pór and Wood [48]. The Big-Line-Big-Clique Conjecture asserts, roughly, that every sufficiently large visibility graph contains a large clique or many collinear points. Applying a classical result of Kővári, Sós and Turán [53] yields the following bipartite subgraph version. A similar statement holds for bivisibility graphs.

Corollary (6.14)**.** *For all integers $t, \ell \geqslant 2$, there exists an integer n such that every visibility graph on n or more points contains a $K_{t,t}$ subgraph or ℓ collinear points.*

We also make some progress toward a conjecture of Kleitman and Pinchasi [50].

Theorem (6.20 and 6.23)**.** *Let P be a set of n red, and n or $n-1$ blue points in the plane. If neither colour class is collinear, then P determines at least $|P|-2$ bichromatic lines. Moreover, if $n \geqslant 10$, then P determines at least $|P|-1$ bichromatic lines, which is best possible.*

Kleitman and Pinchasi conjectured that, under these assumptions, P determines at least $|P| - 1$ bichromatic lines for all n.

Empty pentagons

In Chapter 7 we study special configurations of points within point sets with collinearities. We focus on empty convex k-gons, which are sets of k points in strictly convex position with no other point in the convex hull. It is known that a point set P, even in general position, need not contain an empty heptagon no matter how large P is [45]. On the other hand, sufficiently large point sets in general position always contain empty hexagons [39, 64]. It is an open question whether the same holds for sufficiently large point sets with no ℓ collinear points.

We study this question for the case of empty pentagons. Abel et al. [1] showed that sufficiently large point sets with no ℓ collinear always contain empty pentagons. Their bound on the necessary size of such a point set was doubly exponential in ℓ. We improve this bound as follows.

Theorem (7.1)**.** *Let P be a finite set of points in the plane. If P contains at least $328\ell^2$ points, then P contains an empty pentagon or ℓ collinear points.*

This is optimal up to a constant factor since the $(\ell-1)\times(\ell-1)$ grid contains no empty pentagon and no ℓ collinear points.

Chapter 2

Background

The topic of this thesis is the combinatorial geometry of finite sets of points in the Euclidean plane with collinearities. The problems studied are combinatorial in that they involve estimating the number or size of certain discrete substructures within these point sets.

Any two points in the plane determine a straight line, and a collinearity is when three or more points lie on a line. In terms of the space of possible coordinates for these sets of points, almost all (in the measure theoretic sense) finite point sets contain no collinearities. Another way of saying this is that a randomly chosen point set would have no collinearities with probability 1. In this sense point sets with collinearities are special.

By virtue of having collinearities, the coordinates of the points satisfy certain algebraic relationships. In investigating combinatorial properties of these point sets, the configurations that are extremal often seem to exhibit strong symmetries. It is not surprising then that such problems often lead to interesting links with algebra and number theory, though these links are not the focus of this thesis.

The methods employed here are rather more combinatorial and geometric in nature. Graph theory plays a prominent role, as do basic topological methods, ideas from convex geometry, and more. This mix of techniques is not uncommon in combinatorics, and many of these ideas will be introduced only as needed.

But let us begin by formally defining some terms that recur throughout the thesis. The Euclidean plane $\mathbb{R}^2$ is referred to simply as *the plane*. Let P be a finite set of points in the plane. In fact, P always denotes such a set. A line that contains at least two points in P is said to be *determined* by P. A set of points is *collinear* if it is contained in a line, otherwise it is *non-collinear*. A set of points in the plane is in *general position* if it contains no three collinear points. (Other notions of general position are possible, but this is the only one we consider.) So, if P is in general position, then every line determined by P contains exactly two points from P.

A *graph*, often denoted $G = (V, E)$, consists of a finite set V called the *vertices* of G together with a set E of two-element subsets of V called the *edges* of G. Less formally, the vertices of a graph represent some objects, while the edges represent some connection or relation between pairs of vertices. Two vertices connected by an edge are said to be *adjacent*. Defined as above, a graph has no loops (edges from a vertex to itself) or repeated edges. Graphs are abstract objects, but when studying geometric problems we are often interested in specific representations of them. One basic representation is a *drawing* of a graph in the plane, with points representing the vertices and arcs connecting two vertices representing edges. A graph is *planar* if it has a drawing without any edges intersecting (except at shared vertices) and a *plane graph* is a graph together with such a drawing. It is often convenient to conflate a graph and its elements with their representations, as already demonstrated in the last sentence, where edges were identified with their representing arcs. For the sake of readability, such abuse is used whenever it is unlikely to cause any confusion. Graph theory has a large amount of associated terminology, too much to define here. Our usage follows the standard text on the subject [16].

Graphs appear often in this thesis, usually arising from some geometric situation. The following is a prime example. Let P be a finite set of points in the plane. Two distinct points v and w in the plane are *visible* with respect to P if no point in P lies on the open line segment vw. The *visibility graph*[1] of P has vertex set P, and two vertices are adjacent if and only if they are visible with respect to P. In other words, the visibility graph is obtained by considering the lines determined by P, and two points are adjacent if they are consecutive on such a line.

2.1 Incidence geometry in the plane

Visibility graphs are both a useful tool and an interesting object of study in their own right. Their most famous application is probably in Székely's celebrated proof [91] of the Szemerédi-Trotter Theorem [93], though he did not use the term visibility graph. His proof partly inspired our work on incidence geometry presented in Chapter 3, and the Szemerédi–Trotter Theorem is a key ingredient in the results of Chapter 4. Indeed the proof is so short that we are able to explain it and its prerequisites completely in this section. We begin with Euler's Formula[2], continue to the Crossing Lemma, and then prove the Szemerédi–Trotter Theorem as well as an important related result known as Beck's Theorem.

One of the most basic results in discrete geometry is the invariance of the Euler characteristic, which is really a special case of deep results in topology (see for example [43]). Given a plane graph G, the regions of the compliment of G are known as *faces*.

Theorem 2.1 (Euler's Formula)**.** *For any connected plane graph G with n vertices, m edges and f faces,*

$$n - m + f = 2 \ .$$

There are many different proofs of Theorem 2.1. They have been collected by Eppstein [24], and one of the simplest is the following. It uses multigraphs,

[1]There are various other kinds of graph called visibility graphs, such as graphs defined by visibility among vertices of a polygon. We consider only the kind defined here.

[2]There are various results known as Euler's Formula. More specifically, we refer to Euler's Polyhedral Formula, also known as the Euler characteristic.

meaning loops and multiple edges are allowed.

Proof. Proceed by induction on the number of edges. If there are no edges, then G is a single vertex, and there is one face, so $n - m + f = 2$. Now suppose there is at least one edge e. If e joins two distinct vertices, contract it to a single vertex. This reduces n and m by 1, while leaving f unchanged, so $n-m+f$ is unchanged. On the other hand, if e is a loop, it separates two faces (by the Jordan curve theorem). Delete e and merge these two faces. This reduces m and f by 1, while leaving n unchanged, so $n-m+f$ is again unchanged. □

The *crossing number* of a graph G, denoted by $\mathrm{cr}(G)$, is the minimum number of crossings in a drawing of G. See [69, 92] for surveys on the crossing number. The following lower bound on $\mathrm{cr}(G)$ was first proved by Ajtai et al. [3] and Leighton [56] (with weaker constants). A simple proof with better constants can be found in [2]. The following version is due to Pach et al. [66].

Theorem 2.2 (Crossing Lemma). *For every graph G with n vertices and $m \geqslant \frac{103}{16}n$ edges,*

$$\mathrm{cr}(G) \geqslant \frac{1024\, m^3}{31827\, n^2} \quad .$$

The following well known proof gives weaker constants using only Theorem 2.1 [13].

Proof. Using the fact that each face has at least three edges, it follows from Theorem 2.1 that a planar graph has at most $3n$ edges. Starting with a drawing of G with the fewest possible crossings, and removing edges until the graph is planar, it follows that $\mathrm{cr}(G) \geqslant m - 3n$. Now consider a randomly chosen[3] induced subgraph H of G that includes each vertex independently with probability p. The expected number of vertices in H is pn, the expected number of edges is $p^2 m$, and the expected number of

[3]Since probabilistic arguments are not used in any of the new proofs in this thesis we omit any proper introduction. See a standard text such as [5].

crossings (in the sub-drawing of the original drawing) is $p^4 \operatorname{cr}(G)$. Hence by linearity of expectation, $p^4 \operatorname{cr}(G) \geqslant p^2 m - 3pn$. Setting $p = 4n/m$, which is less than 1 if G has at least $4n$ edges, it follows that $\operatorname{cr}(G) \geqslant m^3/64n^2$. $\square$

As already mentioned, Székely's proof [91] of the Szemeredi–Trotter Theorem [93] uses the Crossing Lemma and visibility graphs. In fact, in Chapter 3 we employ a slight strengthening of the Szemerédi–Trotter Theorem, and that is what we prove here. First we need a few more definitions. For $i \geqslant 2$, an *i-line* is a line containing exactly i points in P. Let s_i be the number of i-lines. Let G_i be the spanning subgraph of the visibility graph of P consisting of all edges in j-lines where $j \geqslant i$; see Figure 2.1 for an example. Note that since each i-line contributes $i-1$ edges, $|E(G_i)| = \sum_{j \geqslant i}(j-1)s_j$. Part (a) of the following version of the Szemerédi–Trotter Theorem gives a bound on $|E(G_i)|$, while part (b) is the well known version that bounds the number of j-lines for $j \geqslant i$.

Theorem 2.3 (Szemerédi–Trotter Theorem)**.** *Let α and β be positive constants such that every graph H with n vertices and $m \geqslant \alpha n$ edges satisfies*

$$\operatorname{cr}(H) \geqslant \frac{m^3}{\beta n^2} \quad .$$

Let P be a set of n points in the plane. Then

$$\textit{(a)} \quad \sum_{j \geqslant i}(j-1)s_j \leqslant \max\left\{\alpha n, \frac{\beta\, n^2}{2(i-1)^2}\right\} \quad ,$$

$$\textit{and (b)} \quad \sum_{j \geqslant i} s_j \leqslant \max\left\{\frac{\alpha n}{i-1}, \frac{\beta\, n^2}{2(i-1)^3}\right\} \quad .$$

Proof. Suppose $\sum_{j \geqslant i}(j-1)s_j = |E(G_i)| \geqslant \alpha n$. Applying the version of the Crossing Lemma assumed in the statement of Theorem 2.3 to G_i,

$$\begin{aligned}\operatorname{cr}(G_i) \geqslant \frac{|E(G_i)|^3}{\beta n^2} &= \frac{(\sum_{j \geqslant i}(j-1)s_j)^2 |E(G_i)|}{\beta n^2} \\ &\geqslant \frac{(i-1)^2(\sum_{j \geqslant i} s_j)^2 |E(G_i)|}{\beta n^2} \quad .\end{aligned}$$

On the other hand, since two lines cross at most once,

$$\operatorname{cr}(G_i) \leqslant \binom{\sum_{j \geqslant i} s_j}{2} \leqslant \frac{1}{2}\Big(\sum_{j \geqslant i} s_j\Big)^2 \quad .$$

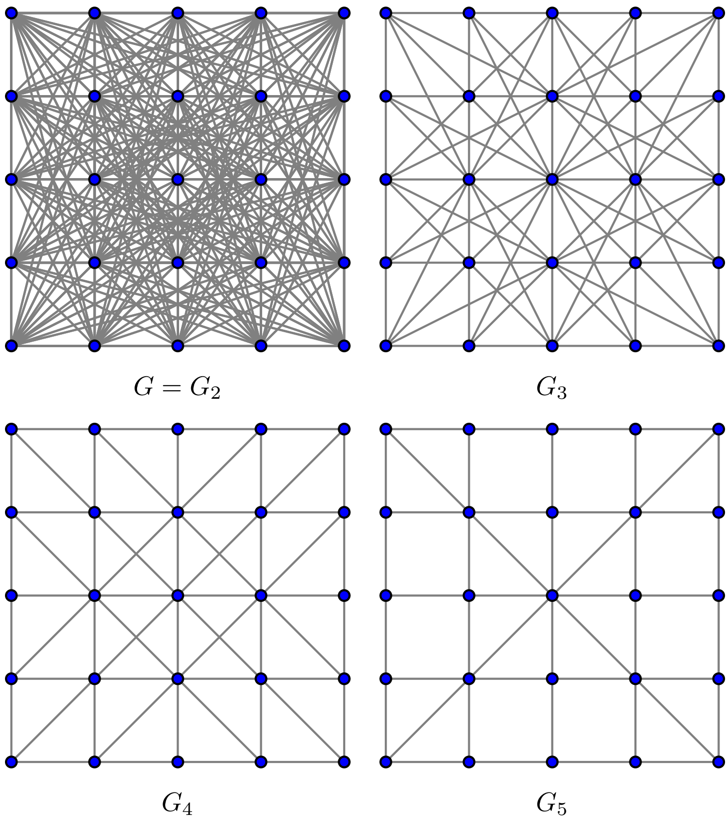


Figure 2.1: The graphs G_2, G_3, G_4, G_5 in the case of the 5×5 grid.

Combining these inequalities yields part (a). Part (b) follows directly from part (a). □

In 1951, Dirac [17] conjectured that every set P of n non-collinear points contains a point in at least $\frac{n}{2} - c_1$ lines determined by P, for some constant c_1. Ten years later Erdős [26] suggested a weakening of this conjecture, that there must be a point in at least n/c_2 lines determined by P, for some constant $c_2 > 0$. In the 1983 paper that established Theorem 2.3, Szemerédi and Trotter [93] proved this weakening as a consequence of their main theorem.

Independently and at the same time, Beck [7] also proved the weakened conjecture using a result similar to the Szemerédi–Trotter Theorem but somewhat weaker. Beck also used the following theorem, which is sometimes known as 'Beck's Theorem'. To distinguish it from the related result Theorem 2.6 below, we will call it Beck's Two Extremes Theorem[4].

Theorem 2.4 (Beck's Two Extremes Theorem)**.** *Let P be a set of n points not all collinear. Then either (a) some line contains $2^{-15}n$ points in P, or (b) P determines at least $2^{-15}n^2$ lines with at most 2^7 points.*

Here we give a short and simple proof of Beck's Two Extremes Theorem using the Szemerédi–Trotter Theorem. It is based on a well known proof (see for example [12]), with some refinements similar to those we use in Chapter 3. No attempt is made to optimise the constants.

Proof. Note that if $n < 2^{16}$ we are done since alternative (a) must hold. Suppose alternative (a) does not hold, so there are at most $2^{-15}n$ points on a line. Consider the number of pairs of points that determine lines with more than 2^7 points. We will apply Theorem 2.3(a) with $\alpha = 4$ and $\beta = 64$.

$$\begin{aligned}
\sum_{i=2^7+1}^{2^{-15}n} \binom{i}{2} s_i &= \frac{1}{2} \sum_{i=2^7+1}^{2^{-15}n} i(i-1)s_i \\
&= \frac{1}{2}\left(2^7 \sum_{i=2^7+1}^{2^{-15}n} (i-1)s_i + \sum_{j=2^7+1}^{2^{-15}n} \sum_{i=j}^{2^{-15}n} (i-1)s_i \right) \\
&\leqslant \frac{1}{2}\left(\frac{\beta n^2}{2\cdot 2^7} + \alpha n + \sum_{i=2^7}^{2^{-15}n} \frac{\beta n^2}{2i^2} + \alpha n \right) \\
&\leqslant 16n^2 \left(2^{-7} + \sum_{i=2^7} \frac{1}{i^2} \right) + 2^{-14}n^2 \\
&\leqslant 0.251n^2 \ .
\end{aligned}$$

This implies that the number of pairs of points that determine lines with at most 2^7 points is at least $\binom{n}{2} - 0.251n^2 = 0.249n^2 - 0.5n \geqslant 0.24899n^2$, since $n \geqslant 2^{16}$. Hence the number of such lines is at least $0.24899n^2/\binom{2^7}{2} \geqslant 2^{-15}n^2$. $\square$

[4]Here we follow Theran [95]

Erdős' weakening of Dirac's Conjecture is an immediate consequence. We include Beck's proof.

Theorem 2.5 (Weak Dirac Conjecture)**.** *Every set P of n non-collinear points contains a point in at least $2^{-15}n$ lines determined by P.*

Proof. If alternative (b) of Theorem 2.4 holds, then some point is in at least $2^{-15}n^2/n$ lines. On the other hand, if alternative (a) holds, then there is a line with at least $2^{-15}n$ points, and any point not on this line is in at least $2^{-15}n$ lines. □

In Chapter 3.1 we make some progress towards Dirac's Conjecture, showing that 2^{-15} can be improved to $1/37$ in the above theorem.

Apart from Theorem 2.5, Beck's [7] other main result was the following theorem, settling a conjecture of Erdős [27]. It is also a consequence of Theorem 2.4. We include Beck's proof for completeness.

Theorem 2.6 (Beck's Theorem)**.** *Let P be a set of n points with at most ℓ collinear. Then P determines at least $2^{-31}n(n-\ell)$ lines.*

Proof. If alternative (b) of Theorem 2.4 holds then we are done. So suppose that alternative (a) holds and the longest line L contains $\ell \geqslant 2^{-15}n$ points in P. There are $n-\ell$ points not on L, so let H be a set of $h = 2^{-15}(n-\ell)$ of them. Counting lines with one point in L and at least one in H (and subtracting overcounts) yields the following lower bound on the number of lines.

$$h\ell - \binom{h}{2} \geqslant h\left(\ell - \frac{h}{2}\right) \geqslant \frac{n-\ell}{2^{15}}\left(\frac{n}{2^{15}} - \frac{n-\ell}{2^{16}}\right) \geqslant \frac{n-\ell}{2^{15}} \cdot \frac{n}{2^{16}} \ .$$

□

The constant in the above theorem is even weaker than what is typically given. This is because our version of Theorem 2.4 was designed to improve the constant in Theorem 2.5. However, in Chapter 3.2 we will see that the constant in this version can easily be improved to 2^{-16}. Much more careful analysis allows us to improve the constant to 1/93.

To illustrate the broad importance of the Szemerédi–Trotter Theorem, we pause to mention a notable application in number theory. Given a finite set of real numbers A, the *sum set* $A+A$ is the set of sums of pairs of numbers in A, and the *product set* $A \cdot A$ is the set of products of pairs of numbers in A. It is a natural question how small these sets may be. When A is an arithmetic progression, $|A+A| = 2|A|-1$, which is minimal. When A is a geometric progression, $|A \cdot A| = 2|A|-1$, which is minimal. However, as one might suspect, it is not possible for both the sum set and the product set of A to be small (linear in $|A|$). Erdős and Szemerédi [35] proved that, for some $c, \delta > 0$, either the sum set or the product set has size at least $c|A|^{1+\delta}$. Later Elekes [21] improved this to $\frac{2}{5}|A|^{5/4}$. His simple proof applied the Szemerédi–Trotter Theorem to a set of points and lines constructed from the set A. Solymosi [84] improved the bound further through a more sophisticated application of the Szemerédi–Trotter Theorem. Erdős and Szemerédi [35] conjectured that, for all $\epsilon > 0$, either the sum set or the product set has size at least $c(\epsilon)|A|^{2-\epsilon}$, for some constant $c(\epsilon) > 0$.

2.2 Independent sets in hypergraphs

The Szemerédi–Trotter Theorem again turns out to be useful in Chapter 4 when we study the size of the largest general position subset in a point set with collinearities. We use it to bound the number of collinear triples of points in a point set as a function of its size and the maximum number of collinear points. By focussing on collinear triples, we are able to apply known results about uniform hypergraphs.

A *hypergraph* consists of a set of vertices along with a set of subsets of the vertex set called *hyperedges*. A hypergraph is *k-uniform* if all the hyperedges have cardinality k. Thus a graph is a 2-uniform hypergraph.

For a point set P in the plane, let $H(P)$ be the 3-uniform hypergraph having vertex set P and a hyperedge for each collinear triple in P. Even if P has more than three points on some line, $H(P)$ captures all the information about collinearities in P. Collinear subsets of P are those subsets L such that every triple of points in L is a hyperedge in $H(P)$.

An *independent set* in a hypergraph is a set of vertices that does not contain any hyperedges. The *independence number* of a hypergraph H, denoted $\alpha(H)$, is the size of the largest independent set in H. Since an independent set in $H(P)$ corresponds to a subset of P with no three collinear points, independent sets are precisely the subsets of P in general position. The independence number of $H(P)$ is therefore the size of the largest subset of P in general position.

Our general approach in Chapter 4 is to combine the bound on the number of hyperedges obtained from the Szemerédi–Trotter Theorem with known bounds on the independence number of 3-uniform hypergraphs. The bound on the number of hyperedges encapsulates some of the geometric restrictions on P, while the bounds on independence numbers are purely combinatorial. They are generally proven using the probabilistic method. Since we use these bounds as black boxes, we will only introduce them as needed.

2.3 Visibility graphs

We have seen some ways in which visibility graphs have played a role in the development of discrete geometry. The study of visibility graphs in their own right is a relatively recent development. Much of this study has focussed on questions related to the clique and chromatic number of visibility graphs. The *chromatic number* of a graph is the least number of colours required to colour the vertices so that no two adjacent vertices receive the same colour. A *clique* in a graph is a complete subgraph, that is, a set of vertices among which every possible edge is present. The *clique number* of a graph is the size of the largest clique. Clearly the chromatic number of a graph is at least the clique number.

Kára, Pór and Wood [48] asked whether the chromatic number of a visibility graph is bounded from above by a function of its clique number. They showed that visibility graphs with clique number at most 3 are 3-colourable. They also made the following important conjecture.

Conjecture 2.7 (Big-Line-Big-Clique Conjecture). *For all integers $k, \ell \geqslant 2$, there exists an integer n such that every visibility graph on n or more vertices contains a clique of size k or ℓ collinear points.*

So far this conjecture has only been proven for $k \leqslant 5$ (see Section 2.4 below). The most obvious approach for the general conjecture fails. Turán's Theorem [97] says that the maximum number of edges in a graph on n vertices with no clique of size k is $\frac{(k-1)n^2}{2k}$. However, for each n, Sylvester [87–90] constructed a set of n points with no four collinear whose visibility graph has $\frac{n^2}{3} + O(n)$ edges. For large n this is less than the number of edges required by Turán's Theorem. See [48, 59, 73, 78] for more results and conjectures about the clique and chromatic number of visibility graphs. Further related results can be found in [1, 20].

Faced with the difficulty of proving Conjecture 2.7 and other related conjectures, we decided to study more basic properties of visibility graphs so as to deepen our understanding of their structure. In Chapter 5 we investigate the connectivity properties of visibility graphs. A graph is *connected* if there exists a path between any two vertices in the graph. A graph is *k-vertex-connected* if it has more than k vertices and it remains connected whenever fewer than k vertices are deleted. Since a complete graph cannot be disconnected by removing vertices, this means that K_n is $(n-1)$-vertex-connected. A graph is *k-edge-connected* if it remains connected whenever fewer than k edges are deleted. A *vertex (edge) cut* in a graph is a set of vertices (edges) whose removal disconnects the graph. Thus a (non-complete) graph is k-connected if its cuts all have size at least k. Menger's Theorem gives a very useful characterisation of k-connectivity.

Theorem 2.8 (Menger's Theorem). *A graph is k-vertex-connected (k-edge-connected) if and only if there exist k internally vertex-disjoint (edge-disjoint) paths between each pair of distinct vertices.*

The *degree* of a vertex v in a graph is the number of edges that contain v. The *minimum (maximum) degree* of a graph G is the minimum (maximum) degree of a vertex in G. Our main results in Chapter 5 give bounds on the connectivity of visibility graphs in terms of the minimum degree.

A graph is *bipartite* if it has chromatic number at most 2. In other words, the vertex set can be partitioned into two parts so that each edges contains a vertex in each part. In Chapter 6 we study a kind of bipartite visibility graph that was useful in Chapter 5. Given two disjoint point sets in the plane A and B, the *bivisibility graph* has vertex set $A \cup B$, and an edge between a vertex in A and another in B if they are visible with respect to $A \cup B$. The sets A and B are often thought of as being coloured with two different colours. We begin by characterising the connected components of bivisibility graphs, and then turn our attention to lower bounds on the number of edges. The number of edges in a bivisibility graph is at least the number of *bichromatic lines*, that is, lines containing a point from both A and B.

The study of bichromatic lines in bichromatic point sets has some history. We use some results of Pach and Pinchasi [65] and Purdy and Smith [80] to adapt our optimised version of Beck's Theorem to give a lower bound on the number of bichromatic lines. One corollary of this is a linear lower bound on the maximum degree of non-collinear bivisibility graphs. Another corollary is a bivisibility version of the Big-Line-Big-Clique Conjecture, which says that sufficiently large bivisibility graphs with no ℓ collinear points contain large complete bipartite subgraphs. Since bivisibility graphs are subgraphs of visibility graphs, a similar statement holds for visibility graphs too. Unlike the Big-Line-Big-Clique Conjecture, this corollary follows directly from well known results in extremal graph theory.

Finally we turn our attention to general linear lower bounds on the number of bichromatic lines, not depending on the maximum number of collinear points. The monochromatic version of this problem has a longer history. In 1948, de Bruijn and Erdős [15] proved that every non-collinear set of n points in the plane determines at least n lines. In fact, they proved this result in a more general combinatorial setting.

Theorem 2.9 (de Bruijn and Erdős)**.** *Let S be a set of cardinality n and $\{S_1, \ldots, S_k\}$ a collection of subsets of S such that each pair of elements in S is contained in exactly one S_i. Then either $S = S_i$ for some i, or $k \geqslant n$.*

As noted by de Bruijn and Erdős, the special case where S is a set of points in the plane and the S_i are the collinear subsets of S is easier to prove than the general theorem. It follows by induction from the well-known Sylvester-Gallai Theorem (actually first proven by Melchior [60]), which says that every finite non-collinear set of points in the plane determines a line with just two points. In the case of bichromatic lines, Kleitman and Pinchasi [50] conjectured that if P is a set of n red, and n or $n-1$ blue points in the plane and neither colour class is collinear, then P determines at least $|P|-1$ bichromatic lines. As motivation, Kleitman and Pinchasi note that together with the following theorem of Motzkin [63], their conjecture would imply the plane case of Theorem 2.9.

Theorem 2.10 (Motzkin)**.** *Every non-collinear set of red and blue points in the plane determines a monochromatic line.*

We make some progress toward the conjecture of Kleitman and Pinchasi, but also show that, unlike Theorem 2.9, it is not true in a purely combinatorial setting. A similar combinatorial version of the problem has been studied by Meshulam [61].

Theorem 2.11 (Meshulam)**.** *Let $X_1, \ldots X_c$ be disjoint sets of cardinality n (these are colour classes), let $S = \bigcup_i X_i$ and let $\{S_1, \ldots, S_k\}$ be a collection of subsets of S such that each pair of elements in S is contained in exactly one S_i (these are 'lines'). Then either $S = S_i$ for some i or $|\{i : \forall j \;\; S_i \not\subset X_j\}| \geqslant (c-1)n$ (this counts non-monochromatic 'lines').*

In the bichromatic case with $c = 2$ we have at least n bichromatic lines, roughly half the number conjectured by Kleitman and Pinchasi under the stronger assumption that no colour class is collinear. It is an interesting question whether the lower bound of Theorem 2.11 can be improved under this assumption.

2.4 Convex configurations

We now require some further geometric definitions. A set X in the plane is *convex* if for every pair of points in X, the straight line segment between them is also contained in X. For a set of points P in the plane, the *convex hull* of P, denoted $\text{conv}(P)$, is the smallest convex set containing P. P is in *convex position* if every point of P lies on the boundary of $\text{conv}(P)$. Another classical result in discrete geometry is the Erdős–Szekeres Theorem [33].

Theorem 2.12 (Erdős–Szekeres Theorem)**.** *For every integer k there is a minimum integer* $\text{ES}(k)$ *such that every set of at least* $\text{ES}(k)$ *points in general position in the plane contains k points in convex position.*

Erdős [27] asked whether a similar result held for *empty k-gons*, that is, k points in convex position with no other points inside their convex hull. Horton [45] answered this question in the negative by showing that there are arbitrarily large point sets in general position that contain no empty heptagon. On the other hand, Harborth [42] showed that every set of at least 10 points in general position contains an empty pentagon. More recently, Nicolás [64] and Gerken [39] independently settled the question for $k = 6$ by showing that sufficiently large point sets in general position always contain empty hexagons; see also [52, 98].

In order to address similar questions for point sets with collinearities, it is helpful to refine the definition of convex position. A point $x \in P$ is a *corner* of P if $\text{conv}(P \setminus \{x\}) \neq \text{conv}(P)$. The set P is in *strictly convex position* if every point in P is a corner of P. By way of contrast, a set in convex position, but not necessarily in strictly convex position, is said to be in *weakly convex position.* Thus a set in strictly convex position is also in weakly convex position. A *weakly* (respectively *strictly*) *convex k-gon* is a set of k points in weakly (respectively strictly) convex position.

It is well known that the Erdős–Szekeres theorem generalises for point sets with collinearities; see [1] for proofs. One generalisation states that every set of at least $\text{ES}(k)$ points contains a weakly convex k-gon. For strictly convex

k-gons, it is necessary to consider point sets with bounded collinearities, since a collinear point set has at most two points in strictly convex position. In this case the generalisation states that for all integers k and ℓ there exists a minimum integer $\mathrm{ES}(k, \ell)$ such that every set of at least $\mathrm{ES}(k, \ell)$ points in the plane contains ℓ collinear points or a strictly convex k-gon.

In Chapter 7 we study the problem of finding strictly convex empty k-gons in point sets with no ℓ collinear. Horton's negative result [45] for empty heptagons also applies in this setting. For $k \geqslant 7$ there may be no empty k-gons even in a very large point set with bounded collinearities. On the other hand, Abel et al. [1] showed that every finite set of at least $\mathrm{ES}\left(\frac{(2\ell-1)^{\ell}-1}{2\ell-2}\right)$ points in the plane contains an empty pentagon or ℓ collinear points. The case $k = 6$ remains open for $\ell \geqslant 4$, and it is not clear how to adapt the proofs of Nicolás [64] and Gerken [39] to deal with collinearities. Our contribution is to improve on the result of Abel et al., showing that every finite set of at least $328\ell^2$ points contains an empty pentagon or ℓ collinear points.

Note that since the vertices of an empty pentagon form a clique in the visibility graph, this establishes the $k \leqslant 5$ case of the Big-Line-Big-Clique Conjecture (2.7). In the other direction, Wood [101] asked whether the visibility graphs of point sets with no empty pentagon have bounded clique or chromatic number. Cibulka et al. [11] answered this question in the negative by constructing a family of sets with no empty pentagon but arbitrarily large clique (and thus also chromatic) number.

Chapter 3

Dirac's Conjecture and Beck's Theorem

3.1 Dirac's Conjecture

In 1951, Gabriel Dirac [17] made the following conjecture, which remains unresolved:

Conjecture 3.1 (Dirac's Conjecture)**.** *There is a constant c_1 such that every set P of n non-collinear points contains a point in at least $\frac{n}{2} - c_1$ lines determined by P.*

See reference [4] for examples showing that the $\frac{n}{2}$ bound would be tight. Note that if P is non-collinear and contains at least $\frac{n}{2}$ collinear points, then Dirac's Conjecture holds. Thus we may assume that P contains at most $\frac{n}{2}$ collinear points, and $n \geqslant 5$. In 1961, Erdős [26] proposed the following weakened conjecture.

Conjecture 3.2 (Weak Dirac Conjecture)**.** *There is a constant c_2 such that every set P of n non-collinear points contains a point in at least $\frac{n}{c_2}$ lines determined by P.*

In 1983, the Weak Dirac Conjecture was proved independently by Beck [7] and Szemerédi and Trotter [93], in both cases with c_2 unspecified and very large. We prove the Weak Dirac Conjecture with c_2 much smaller. (See references [30, 32, 49, 57, 79] for more on Dirac's Conjecture.)

Theorem 3.3. *Every set P of n non-collinear points contains a point in at least $\frac{n}{37}$ lines determined by P.*

Theorem 3.3 is a consequence of the following theorem. The points of P together with the lines determined by P are called the *arrangement* of P.

Theorem 3.4. *For every set P of n points in the plane with at most $\frac{n}{37}$ collinear points, the arrangement of P has at least $\frac{n^2}{37}$ point-line incidences.*

Proof of Theorem 3.3. Let P be a set of n non-collinear points in the plane. If P contains at least $\frac{n}{37}$ collinear points, then every other point is in at least $\frac{n}{37}$ lines determined by P (one through each of the collinear points). Otherwise, by Theorem 3.4, the arrangement of P has at least $\frac{n^2}{37}$ incidences, and so some point is incident with at least $\frac{n}{37}$ lines determined by P. $\square$

The proof of Theorem 3.4 takes inspiration from the well known proof of Beck's Two Extremes Theorem (2.4) [12] as a corollary of the Szemerédi–Trotter Theorem (2.3) [93], and also from the simple proof of the Szemerédi–Trotter Theorem due to Székely [91], which in turn is based on the Crossing Lemma (2.2). These proofs were discussed in Chapter 2.

The proof of Theorem 3.4 also employs Hirzebruch's Inequality [44]. As before, s_i is the number of lines containing i points in P.

Theorem 3.5 (Hirzebruch's Inequality)**.** *Let P be a set of n points with at most $n-3$ collinear. Then*

$$s_2 + \frac{3}{4}s_3 \geqslant n + \sum_{i \geqslant 5}(2i-9)s_i \ .$$

Hirzebruch's Inequality is rather interesting in that it does not follow from Euler's formula like many other results discussed here. Instead, it is a consequence of deep results in algebraic geometry and it applies in a much broader setting than the real plane. In particular, it is also valid for arrangements of points in the complex plane. In 1995, Erdős and Purdy [31] asked for a combinatorial proof of the inequality, a fascinating question that remains open.

Theorem 3.4 follows from the Crossing Lemma (2.2) and the following general result by setting $\alpha = \frac{103}{16}$, $\beta = \frac{31827}{1024}$, $c = 71$, and $\delta = \epsilon$, in which case $\delta \geqslant \frac{1}{36.158}$. The value of δ is readily calculated numerically since

$$\begin{aligned}\sum_{i \geqslant c} \frac{i+1}{i^3} &= \sum_{i \geqslant 1} \frac{i+1}{i^3} - \sum_{i=1}^{c-1} \frac{i+1}{i^3} \\ &= \zeta(2) + \zeta(3) - \sum_{i=1}^{c-1} \frac{i+1}{i^3} \\ &= 2.847\ldots - \sum_{i=1}^{c-1} \frac{i+1}{i^3} \quad ,\end{aligned}$$

where ζ is the Riemann zeta function.

Theorem 3.6. *Let α and β be positive constants such that every graph H with n vertices and $m \geqslant \alpha n$ edges satisfies*

$$\operatorname{cr}(H) \geqslant \frac{m^3}{\beta n^2} \quad .$$

Fix an integer $c \geqslant 8$ and a real $\epsilon \in (0, \frac{1}{2})$. Let $h := \frac{c(c-2)}{5c-18}$. Then for every set P of n points in the plane with at most ϵn collinear points, the arrangement of P has at least δn^2 point-line incidences, where

$$\delta = \frac{1}{h+1} \left(1 - \epsilon\alpha - \frac{\beta}{2} \left(\frac{(c-h-2)(c+1)}{c^3} + \sum_{i \geqslant c} \frac{i+1}{i^3} \right) \right) \quad .$$

Proof. Let $J := \{2, 3, \ldots, \lfloor \epsilon n \rfloor\}$. Considering the visibility graph G of P and its subgraphs G_i (as defined in Chapter 2), let k be the minimum integer such that $|E(G_k)| \leqslant \alpha n$. If there is no such k then let $k := \lfloor \epsilon n \rfloor + 1$. An integer $i \in J$ is *large* if $i \geqslant k$, and is *small* if $i \leqslant c$. An integer in J that is neither large nor small is *medium*.

Recall that an i-line is a line containing i points in P. An *i-pair* is a pair of points in an i-line. A *small pair* is an i-pair for some small i. Define *medium pairs* and *large pairs* analogously, and let P_S, P_M and P_L denote the number of small, medium and large pairs respectively. An *i-incidence* is an incidence between a point of P and an i-line. A *small incidence* is an i-incidence for some small i. Define *medium incidences* analogously, and let I_S and I_M denote the number of small and medium incidences respectively. Let I denote the total number of incidences. Thus,

$$I = \sum_{i \in J} i s_i \ .$$

The proof proceeds by establishing an upper bound on the number of small pairs in terms of the number of small incidences. Analogous bounds are proved for the number of medium pairs, and the number of large pairs. Combining these results gives the desired lower bound on the total number of incidences.

For the bound on small pairs, Hirzebruch's Inequality (3.5) is useful. Since we may assume fewer than $\frac{n}{2}$ points are collinear, and thus $n \geqslant 5$, there are no more than $n-3$ collinear points. Therefore, Hirzebruch's Inequality implies that $hs_2 + \frac{3h}{4}s_3 - hn - h\sum_{i \geqslant 5}(2i-9)s_i \geqslant 0$ since $h > 0$. Thus,

$$\begin{aligned}
P_S &= s_2 + 3s_3 + 6s_4 + \sum_{i=5}^{c} \binom{i}{2} s_i \\
&\leqslant (h+1)s_2 + \left(\frac{3h}{4} + 3\right) s_3 + 6s_4 + \sum_{i=5}^{c} \binom{i}{2} s_i - hn - h\sum_{i=5}^{c}(2i-9)s_i \\
&\leqslant \frac{h+1}{2} \cdot 2s_2 + \frac{h+4}{4} \cdot 3s_3 + \frac{3}{2} \cdot 4s_4 + \sum_{i=5}^{c} \left(\frac{i-1}{2} - 2h + \frac{9h}{i}\right) i s_i - hn \ .
\end{aligned}$$

Setting $X := \max\left\{\frac{h+1}{2}, \frac{h+4}{4}, \frac{3}{2}, \max_{5 \leqslant i \leqslant c}\left(\frac{i-1}{2} - 2h + \frac{9h}{i}\right)\right\}$ implies that

$$P_S \leqslant X I_S - hn \ . \tag{1}$$

The above inequality is strongest when X is minimised by determining the optimal value of h as follows. Let $\gamma(h,i) := \frac{i-1}{2} - 2h + \frac{9h}{i}$. The second partial derivative of $\gamma(h,i)$ with respect to i is positive for $i > 0$, so $\gamma(h,i)$ is maximised for $i = 5$ or $i = c$, and the other values of i can be ignored. Thus X is bounded from below by five linear functions of h. Notice that for fixed c, $\frac{h+1}{2}$ increases with h, while $\gamma(h,c)$ decreases with h. Therefore X is at least the value of these functions at their intersection point, which occurs at $h = \frac{c(c-2)}{5c-18}$. Using the fact that $c \geqslant 8$, it can be checked that this intersection point satisfies the other three constraints[1], and is therefore the optimal solution.

To bound the number of medium pairs, consider a medium $i \in J$. Since i is not large, $\sum_{j \geqslant i}(j-1)s_j > \alpha n$. Hence, using parts (a) and (b) of the Szemerédi–Trotter Theorem (2.3),

$$\sum_{j \geqslant i} js_j = \sum_{j \geqslant i}(j-1)s_j + \sum_{j \geqslant i} s_j \leqslant \frac{\beta n^2}{2(i-1)^2} + \frac{\beta n^2}{2(i-1)^3} = \frac{\beta n^2 i}{2(i-1)^3} \ . \tag{2}$$

Given the factor X in the bound on the number of small pairs in (1), it helps to introduce the same factor in the bound on the number of medium pairs. It is convenient to define $Y := c - 1 - 2X$.

$$\begin{aligned} P_M - XI_M &= \left(\sum_{i=c+1}^{k-1} \binom{i}{2} s_i \right) - X \left(\sum_{i=c+1}^{k-1} is_i \right) \\ &= \frac{1}{2} \sum_{i=c+1}^{k-1} (i-1-2X)\, is_i \\ &= \frac{1}{2} \sum_{i=c+1}^{k-1} (i-c+Y)\, is_i \\ &= \frac{1}{2} \left(\sum_{i=c+1}^{k-1} \sum_{j=i}^{k-1} js_j \right) + \frac{Y}{2} \left(\sum_{i=c+1}^{k-1} is_i \right) \ . \end{aligned}$$

Applying (2) yields

$$P_M - XI_M \leqslant \frac{\beta\, n^2}{4} \left(Y \frac{c+1}{c^3} + \sum_{i \geqslant c} \frac{i+1}{i^3} \right) \ . \tag{3}$$

It remains to bound the number of large pairs:

$$P_L = \sum_{i=k}^{\lfloor \epsilon n \rfloor} \binom{i}{2} s_i \leqslant \frac{\epsilon n}{2} \sum_{i \geqslant k} (i-1)s_i = \frac{\epsilon n}{2} |E(G_k)| \leqslant \frac{\epsilon \alpha\, n^2}{2} \ . \tag{4}$$

[1]A simple way to do this is to note that $h(c)$ increases with c for $c \geqslant 8$ and so $h \geqslant \frac{24}{11}$. Then compare $X = \frac{h+1}{2}$ to the other three constraints.

Combining (1), (3) and (4),

$$\begin{aligned}\binom{n}{2} &= \frac{1}{2}(n^2 - n) \\ &\leqslant P_S + P_M + P_L \\ &\leqslant XI_S - hn + XI_M + \frac{\beta\, n^2}{4}\left(Y\frac{c+1}{c^3} + \sum_{i\geqslant c}\frac{i+1}{i^3}\right) + \frac{\epsilon\alpha\, n^2}{2} \quad .\end{aligned}$$

Thus,

$$I \geqslant I_S + I_M \geqslant \frac{1}{2X}\left(1 - \epsilon\alpha - \frac{\beta}{2}\left(Y\frac{c+1}{c^3} + \sum_{i\geqslant c}\frac{i+1}{i^3}\right)\right)n^2 + \frac{2h-1}{2X}n \quad .$$

The result follows since $h \geqslant 1$. □

It is worth noting that the methods used in the proof of Theorem 3.6 can be used to obtain good lower bounds on the number of edges in a visibility graph. The main difference is that edges ($\sum(i-1)s_i$) are counted instead of incidences ($\sum is_i$). For instance, we can prove the following result.

Theorem 3.7. *Let P be a set of n points in the plane with at most $\frac{n}{50}$ collinear. Then the visibility graph of P has at least $\frac{n^2}{50}$ edges.*

For point sets with at most $o(n)$ collinear points, the following is the best asymptotic result we have obtained.

Theorem 3.8. *Let P be a set of n points in the plane with at most ℓ collinear. Then the visibility graph of P has at least $\frac{n^2}{39} - O(\ell n)$ edges.*

3.2 Beck's Theorem

In his work on the Weak Dirac Conjecture, Beck proved the following theorem [7].

Theorem 3.9 (Beck's Theorem)**.** *There is a constant $c_3 > 0$ such that every set P of n points with at most ℓ collinear determines at least $c_3 n(n-\ell)$ lines.*

In Chapter 2 we gave a relatively simple proof of Beck's Theorem (2.6) with $c_3 = 2^{-31}$. Here our aim is to find tighter bounds on c_3. First we use Theorem 3.6 and some well known lemmas to show that $c_3 \geqslant \frac{1}{98}$. A more tailored approach using similar methods is then employed to show $c_3 \geqslant \frac{1}{93}$.

The first tool we need is a classical inequality due to Melchior [60]. The proof uses Euler's formula applied to the projective dual configuration. Melchior's Inequality was later rediscovered by Kelly and Moser [49].

Theorem 3.10 (Melchior's Inequality)**.** *Let P be a set of n non-collinear points. Then*

$$s_2 \geqslant 3 + \sum_{i \geqslant 4} (i-3) s_i \ .$$

We will use the following straightforward corollary of Melchior's Inequality. As before, I is the total number of incidences in the arrangement of P. Let E be the total number of edges in the visibility graph of P, and let L be the total number of lines in the arrangement of P.

Lemma 3.11. *If P is not collinear, then $3L \geqslant 3 + I$, and $2L \geqslant 3 + E$.*

Proof. Melchior's Inequality is often written $\sum_{i \geqslant 2} (i-3) s_i \leqslant -3$, which is to say $3 + \sum i s_i \leqslant 3 \sum s_i$. Since $I = E + L$, it also follows that $2L \geqslant 3 + E$. □

It is interesting to note that since $I \geqslant 2L$ and $E \geqslant L$, all these parameters are within a constant factor of each other.

When there is a large number of collinear points, the following lemma becomes useful.

Lemma 3.12. *Let P be a set of n points in the plane such that some line contains exactly ℓ points in P. Then the visibility graph of P contains at least $\ell(n-\ell)$ edges.*

Proof. Let S be the set of ℓ collinear points in P. For each point $v \in S$ and for each point $w \in P \setminus S$, count the edge incident to w in the direction of v. Since S is collinear and w is not in S, no edge is counted twice. Thus $E \geqslant |S| \cdot |P \setminus S| = \ell(n-\ell)$. □

We note in passing that Lemmas 3.11 and 3.12 can be used to improve the proof of Theorem 2.6 and yield a constant of 2^{-16}. However by using Theorem 3.6 as well we can already do much better.

Theorem 3.13. *Every set P of n points with at most ℓ collinear determines at least $\frac{1}{98}n(n-\ell)$ lines.*

Proof. Assume ℓ is the size of the largest collinear subset of P. If $\ell \geqslant \frac{n}{49}$ then $E \geqslant \frac{1}{49}n(n-\ell)$ by Lemma 3.12 and thus $L > \frac{1}{98}n(n-\ell)$ by Lemma 3.11. On the other hand, suppose $\ell \leqslant \frac{n}{49}$. Setting $\alpha = \frac{103}{16}$, $\beta = \frac{31827}{1024}$, $\frac{\epsilon}{2} = \frac{\delta}{3}$ and $c = 67$ in Theorem 3.6 gives $\epsilon \geqslant \frac{1}{49}$ and $\delta \geqslant \frac{1}{32.57}$. So $I \geqslant \frac{1}{32.57}n^2 \geqslant \frac{1}{32.57}n(n-\ell)$ and thus $L > \frac{1}{98}n(n-\ell)$ by Lemma 3.11. $\square$

3.2.1 Further improvement

A more direct approach similar to the methods used in the proof of Theorem 3.6 can be used to improve Theorem 3.13 slightly to yield $\frac{1}{93}n(n-\ell)$ lines. We use the following more general result, which again employs Hirzebruch's Inequality (3.5).

Theorem 3.14. *Let α and β be positive constants such that every graph H with n vertices and $m \geqslant \alpha n$ edges satisfies*

$$\operatorname{cr}(H) \geqslant \frac{m^3}{\beta n^2} \quad .$$

Fix an integer $c \geqslant 29$. Then for every set P of n points in the plane with at most ℓ collinear points, the arrangement of P has at least

$$\left(\frac{1}{2} - \frac{\beta}{4}\left(\frac{1}{c} + \sum_{i=c} \frac{1}{i^2}\right)\right) \frac{4c-16}{c^2+3c-18} n^2 - \frac{\alpha}{2} \frac{4c-16}{c^2+3c-18} \ell n$$

lines with at most c points.

Proof. Define small, medium and large pairs and lines as in the proof of Theorem 3.6. Then using Hirzebruch's Inequality (3.5),

$$P_S = s_2 + 3s_3 + 6s_4 + \sum_{i=5}^{c} \binom{i}{2} s_i$$

$$\leqslant (h+1)s_2 + \left(\frac{3h}{4}+3\right)s_3 + 6s_4 + \sum_{i=5}^{c}\binom{i}{2}s_i - hn - h\sum_{i=5}^{c}(2i-9)s_i$$
$$\leqslant (h+1)s_2 + \frac{3}{4}(h+4)s_3 + 6s_4 + \sum_{i=5}^{c}\left(\frac{i(i-1)}{2} - h(2i-9)\right)s_i - hn \ .$$

Using the fact that $c \geqslant 29$ and similar arguments to those used in the proof of Theorem 3.6, it is advantageous to set $h := \frac{c^2-c-2}{4c-16}$. This gives

$$\max\left\{h+1, \frac{3}{4}(h+4), 6, \max_{5\leqslant i\leqslant c}\left(\frac{i(i-1)}{2} - h(2i-9)\right)\right\} = h+1 =: X \ ,$$

and thus,

$$P_S \leqslant XL_S - hn \ .$$

For medium i, the assumed Crossing Lemma and part (a) of the Szemerédi-Trotter Theorem (2.3), imply that

$$\sum_{j\geqslant i}(j-1)s_j \leqslant \frac{\beta n^2}{2(i-1)^2} \ .$$

Thus,

$$\begin{aligned}
P_M &= \frac{1}{2}\sum_{i=c+1}^{k} i(i-1)s_i \\
&= \frac{1}{2}\left(c\sum_{i=c+1}^{k}(i-1)s_i + \sum_{j=c+1}^{k}\sum_{i=j}^{k}(i-1)s_i\right) \\
&\leqslant \frac{1}{2}\left(\frac{\beta n^2}{2c} + \sum_{i=c+1}\frac{\beta n^2}{2(i-1)^2}\right) \\
&\leqslant \frac{\beta n^2}{4}\left(\frac{1}{c} + \sum_{i=c}\frac{1}{i^2}\right) \ .
\end{aligned}$$

As in the proof of Theorem 3.6, we have $P_L \leqslant \ell\alpha n/2$. Adding it all up gives

$$\binom{n}{2} \leqslant XL_S + \frac{\beta n^2}{4}\left(\frac{1}{c} + \sum_{i=c}\frac{1}{i^2}\right) + \frac{\ell\alpha n}{2} - hn \ ,$$

so

$$\left(\frac{1}{2} - \frac{\beta}{4}\left(\frac{1}{c} + \sum_{i=c}\frac{1}{i^2}\right)\right)n^2 + \left(h - \frac{1}{2} - \frac{\ell\alpha}{2}\right)n \leqslant XL_S \ ,$$

and since $X = h + 1 = \frac{c^2+3c-18}{4c-16}$,

$$\left(\frac{1}{2} - \frac{\beta}{4}\left(\frac{1}{c} + \sum_{i=c} \frac{1}{i^2}\right)\right) \frac{4c-16}{c^2+3c-18} n^2 - \frac{\ell\alpha}{2} \frac{4c-16}{c^2+3c-18} n \leqslant L_S \ .$$

□

For constant ℓ, we may observe that the number of lines determined by P is $\Omega(n^2)$. Theorem 3.14 yields the best coefficient of n^2. Setting $c = 66$ gives at least $\frac{1}{70}n^2 - \frac{1}{5}\ell n$ lines with at most 66 points.

Lemma 3.12 together with Theorem 3.14 may be used to improve the constant in Beck's theorem further to $\frac{1}{93}$.

Theorem 3.15. *Every set P of n points with at most ℓ collinear determines at least $\frac{1}{93}n(n-\ell)$ lines.*

Proof. We may assume that ℓ is the size of the longest line. If $\ell \geqslant \epsilon n$ for some constant ϵ, then by Lemmas 3.11 and 3.12, $L > \epsilon n(n-\ell)/2$. On the other hand, Theorem 3.14 says $L_S \geqslant An^2 - Bn\ell$ for some $A(c)$ and $B(c)$ evident in the theorem. Observe that

$$\begin{aligned} & \frac{2A}{1+2B} \geqslant \epsilon \\ \Longrightarrow \quad & A \geqslant \epsilon/2 + B\epsilon - \epsilon^2/2 \\ \Longrightarrow \quad & An \geqslant \epsilon n/2 + (B - \epsilon/2)\epsilon n \\ \Longrightarrow \quad & An \geqslant \epsilon n/2 + (B - \epsilon/2)\ell \\ \Longrightarrow \quad & An^2 - Bn\ell \geqslant \epsilon n(n-\ell)/2 \ . \end{aligned}$$

So maximising $\frac{2A(c)}{1+2B(c)}$ yields the best possible ϵ. Setting $c = 76$ gives $\epsilon \leqslant 1/46.2$. Thus the constant for Beck's Theorem is at least $1/92.4$ □

3.2.2 Lines with few points

Beck's Theorem is often stated as a bound on the number of lines with few points. In his original paper, Beck [7] mentioned briefly in a footnote that Lemma 3.11 implies the following.

Observation 3.16 (Beck)**.** *If P is not collinear, then at least half the lines determined by P contain 3 points or less.*

Proof. By Lemma 3.11,

$$3s_2 + 3s_3 + 3\sum_{i\geqslant 4} s_i > \sum_{i\geqslant 2} is_i \geqslant 2s_2 + 2s_3 + 4\sum_{i\geqslant 4} s_i \ .$$

Thus

$$2(s_2 + s_3) > \sum_{i\geqslant 2} s_i \ ,$$

as desired. ☐

Corollary 3.17. *Every set P of n points with at most ℓ collinear determines at least $\frac{1}{186}n(n-\ell)$ lines each with at most 3 points.*

Hirzebruch's Inequality (3.5) may be used to find lower bounds on the number of lines with at most c points in a similar way to Observation 3.16.

Observation 3.18. *The number of lines with at most c points for $c \geqslant 4$ is at least $\frac{2c-7}{2c-6}$ times the total number of lines.*

Proof. If there are $n-2$ collinear points then there is only one line with more than c points and at least $n-1$ lines with less than c points. We may assume $n \geqslant 5$, so the lemma holds. If there are at most $n-3$ collinear points then by Hirzebruch's Inequality,

$$\sum_{i=2}^{c} s_i \geqslant s_2 + \frac{3}{4}s_3 > \sum_{i\geqslant 5}(2i-9)s_i \geqslant (2c-7)\sum_{i\geqslant c+1} s_i \ .$$

Thus,

$$(2c-6)\sum_{i=2}^{c} s_i > (2c-7)\sum_{i\geqslant 2} s_i \ .$$

☐

Chapter 4

General position subset selection

4.1 Original problem

Recall that a set of points in the plane is in *general position* if it contains no three collinear points. The general position subset selection problem asks, given a finite set of points in the plane with at most ℓ collinear, how big is the largest subset in general position? That is, determine the maximum integer $f(n, \ell)$ such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n, \ell)$ points in general position. Throughout this chapter we assume $\ell \geqslant 3$. Furthermore, as the results in this chapter are all asymptotic in n, it will be made explicit whenever ℓ is a constant not dependent on n. Otherwise ℓ is allowed to grow as a function of n.

The problem was originally posed by Erdős, first for the case $\ell = 3$ [28], and later in a more general form [29]. Füredi [36] showed that the density version of the Hales–Jewett theorem [37] implies that $f(n, \ell) \leqslant o(n)$ for all ℓ, and that a result of Phelps and Rödl [74] on independent sets in partial Steiner triple systems implies that

$$f(n, 3) \geqslant \Omega(\sqrt{n \ln n}) \ .$$

Until recently, the best known lower bound for $\ell \geqslant 4$ was $\sqrt{2n/(\ell-2)}$, proved by a greedy selection algorithm. Lefmann [55] showed that for constant ℓ,

$$f(n,\ell) \geqslant \Omega(\sqrt{n \ln n}) \ .$$

(In fact, his results are more general, see Section 4.2.)

In relation to the general position subset selection problem (and its relatives), Brass, Moser and Pach [9, p. 318] write, "To make any further progress, one needs to explore the geometric structure of the problem." We do this by using the Szemerédi–Trotter Theorem (2.3).

We give improved lower bounds on $f(n,\ell)$ when ℓ is not constant, with the improvement being most significant for values of ℓ around $\sqrt{n}$. Our first result (Theorem 4.3) says that if $\ell \leqslant O(\sqrt{n})$ then $f(n,\ell) \geqslant \Omega(\sqrt{\frac{n}{\ln \ell}})$. Our second result (Theorem 4.5) says that if $\ell \leqslant O(n^{(1-\epsilon)/2})$ then $f(n,\ell) \geqslant \Omega(\sqrt{n \log_\ell n})$. For constant ℓ, this implies Lefmann's lower bound on $f(n,\ell)$ mentioned above.

Our main tool is the following lemma.

Lemma 4.1. *Let P be a set of n points in the plane with at most ℓ collinear. Then the number of collinear triples in P is at most $c(n^2 \ln \ell + \ell^2 n)$ for some constant c.*

Proof. For $2 \leqslant i \leqslant \ell$, let s_i be the number of lines containing exactly i points in P. The Szemerédi–Trotter Theorem (2.3) implies that for some constant $c \geqslant 1$, and for all $i \geqslant 2$,

$$\sum_{j \geqslant i} s_j \leqslant c\left(\frac{n^2}{i^3} + \frac{n}{i}\right) \ .$$

Thus the number of collinear triples is

$$\begin{aligned} \sum_{i=2}^{\ell} \binom{i}{3} s_i &\leqslant \sum_{i=2}^{\ell} i^2 \sum_{j=i}^{\ell} s_j \\ &\leqslant \sum_{i=2}^{\ell} c i^2 \left(\frac{n^2}{i^3} + \frac{n}{i}\right) \end{aligned}$$

$$\leqslant c\sum_{i=2}^{\ell}\left(\frac{n^2}{i}+in\right)$$
$$\leqslant c(n^2\ln\ell+\ell^2 n) \ .$$

□

Note that Lefmann [54] proved Lemma 4.1 for the case of the $\sqrt{n}\times\sqrt{n}$ grid via a direct counting argument. A similar statement to Lemma 4.1 with $\ell=\sqrt{n}$ also appears in the book by Tao and Vu [94, Corollary 8.8].

To apply Lemma 4.1 it is useful to consider the 3-uniform hypergraph $H(P)$ determined by a set of points P, with vertex set P, and an edge for each collinear triple in P. A subset of P is in general position if and only if it is an independent set in $H(P)$. The size of the largest independent set in a hypergraph H is denoted $\alpha(H)$. Spencer [85] proved the following lower bound on $\alpha(H)$.

Lemma 4.2 (Spencer)**.** *Let H be an r-uniform hypergraph with n vertices and m edges. If $m<n/r$ then $\alpha(H)>n/2$. If $m\geqslant n/r$ then*

$$\alpha(H)>\frac{r-1}{r^{r/(r-1)}}\frac{n}{(m/n)^{1/(r-1)}} \ .$$

Lemmas 4.1 and 4.2 imply our first result.

Theorem 4.3. *Let P be a set of n points with at most ℓ collinear. Then P contains a set of $\Omega(n/\sqrt{n\ln\ell+\ell^2})$ points in general position. In particular, if $\ell\leqslant O(\sqrt{n})$ then P contains a set of $\Omega(\sqrt{\frac{n}{\ln\ell}})$ points in general position.*

Proof. Let m be the number of edges in $H(P)$. By Lemma 4.1, $m/n\leqslant c(n\ln\ell+\ell^2)$ for some constant c. Now apply Lemma 4.2 with $r=3$. If $m<n/3$ then $\alpha(H(P))>n/2$, as required. Otherwise,

$$\alpha(H(P))>\frac{2n}{3^{3/2}(m/n)^{1/2}}\geqslant\frac{2n}{3^{3/2}\sqrt{c(n\ln\ell+\ell^2)}}=\frac{2}{3\sqrt{3c}}\frac{n}{\sqrt{n\ln\ell+\ell^2}} \ .$$

□

Note that Theorem 4.3 also shows that if $\ell^2/\ln\ell \geqslant n$ then $f(n,\ell) \geqslant \Omega(n/\ell)$. This improves upon the greedy bound mentioned in the introduction, and is within a constant factor of optimal, since there are point sets with at most ℓ collinear that can be covered by n/ℓ lines.

Theorem 4.3 answers, up to a logarithmic factor, a symmetric Ramsey style version of the general position subset selection problem posed by Gowers [40]. He asked for the minimum integer $\mathrm{GP}(q)$ such that every set of at least $\mathrm{GP}(q)$ points in the plane contains q collinear points or q points in general position. Gowers noted that $\Omega(q^2) \leqslant \mathrm{GP}(q) \leqslant O(q^3)$. Theorem 4.3 with $\ell = q-1$ and $n = \mathrm{GP}(q)$ implies that $\Omega(\sqrt{\mathrm{GP}(q)/\ln(q-1)}) \leqslant q$ and so $\mathrm{GP}(q) \leqslant O(q^2 \ln q)$.

The bound $\mathrm{GP}(q) \geqslant \Omega(q^2)$ comes from the $q \times q$ grid, which contains no $q+1$ collinear points, and no more than $2q+1$ in general position, since each row can have at most 2 points. Determining the maximum number of points in general position in the $q \times q$ grid is known as the *no-three-in-line problem*, first posed by Dudeney in 1917 [18]. See [41] for the best known bound and for more on its history.

As an aside, note that Pach and Sharir [67] proved a result somewhat similar to Lemma 4.1 for the number of triples in P determining a fixed angle $\alpha \in (0,\pi)$. Their proof is similar to that of Lemma 4.1 in its use of the Szemerédi–Trotter theorem. Also, Elekes [22] employed Lemma 4.2 to prove a similar result to Theorem 4.3 for the problem of finding large subsets with no triple determining a given angle $\alpha \in (0,\pi)$. Pach and Sharir and Elekes did not allow the case $\alpha = 0$, that is, collinear triples. This may be because their work did not consider the parameter ℓ, without which the case $\alpha = 0$ is exceptional since P could be entirely collinear, and all triples could determine the same angle.

The following lemma of Sudakov [86, Proposition 2.3] is a corollary of a result by Duke, Lefmann and Rödl [19].

Lemma 4.4 (Sudakov)**.** *Let H be a 3-uniform hypergraph on n vertices with m edges. Let $t \geqslant \sqrt{m/n}$ and suppose there exists a constant $\epsilon > 0$ such that* *the number of edges containing any fixed pair of vertices of H is at most $t^{1-\epsilon}$. Then $\alpha(H) \geqslant \Omega\left(\frac{n}{t}\sqrt{\ln t}\right)$.*

Lemmas 4.1 and 4.4 can be used to prove our second result.

Theorem 4.5. *Fix constants $\epsilon > 0$ and $d > 0$. Let P be a set of n points in the plane with at most ℓ collinear points, where $\ell \leqslant (dn)^{(1-\epsilon)/2}$. Then P contains a set of $\Omega(\sqrt{n \log_\ell n})$ points in general position.*

Proof. Let m be the number of edges in $H(P)$. By Lemma 4.1, for some constant $c \geqslant 1$,

$$m \leqslant c\ell^2 n + cn^2 \ln \ell < cdn^2 + cn^2 \ln \ell \leqslant (d+1)cn^2 \ln \ell \ .$$

Define $t := \sqrt{(d+1)cn \ln \ell}$. Thus $t \geqslant \sqrt{m/n}$. Each pair of vertices in H is in less than ℓ edges of H, and

$$\ell \leqslant (dn)^{(1-\epsilon)/2} < ((d+1)cn \ln \ell)^{(1-\epsilon)/2} = t^{1-\epsilon} \ .$$

Thus the assumptions in Lemma 4.4 are satisfied. So H contains an independent set of size $\Omega(\frac{n}{t}\sqrt{\ln t})$. Moreover,

$$\begin{aligned}
\frac{n}{t}\sqrt{\ln t} &= \sqrt{\frac{n}{(d+1)c \ln \ell}}\ \sqrt{\ln \sqrt{(d+1)cn \ln \ell}} \\
&\geqslant \sqrt{\frac{n}{(d+1)c \ln \ell}}\ \sqrt{\frac{1}{2}\ln n} \\
&= \sqrt{\frac{1}{2(d+1)c}}\ \sqrt{\frac{n \ln n}{\ln \ell}} \\
&= \Omega(\sqrt{n \log_\ell n}) \ .
\end{aligned}$$

Thus P contains a subset of $\Omega(\sqrt{n \log_\ell n})$ points in general position. □

4.2 Generalised problem

In this section we consider a natural generalisation of the general position subset selection problem. Given $k < \ell$, Erdős [29] asked for the maximum integer $f(n, \ell, k)$ such that every set of n points in the plane with at most ℓ collinear contains a subset of $f(n, \ell, k)$ points with at most k collinear. Thus $f(n, \ell) = f(n, \ell, 2)$. We prove results similar to Theorems 4.3 and 4.5 in this generalised setting.

Brass [8] considered this question for constant $\ell = k + 1$, and showed that

$$o(n) \geqslant f(n, k+1, k) \geqslant \Omega(n^{(k-1)/k} (\ln n)^{1/k}) \ .$$

This can be seen as a generalisation of the results of Füredi [36] for $f(n, 3, 2)$. As in Füredi's work, the lower bound comes from a result on partial Steiner systems [82], and the upper bound comes from the density Hales–Jewett theorem [38]. Lefmann [55] further generalised these results for constant ℓ and k by showing that

$$f(n, \ell, k) \geqslant \Omega(n^{(k-1)/k} (\ln n)^{1/k}) \ .$$

The density Hales–Jewett theorem also implies the general bound $f(n, \ell, k) \leqslant o(n)$ for all ℓ and k.

The result of Lefmann may be generalised to include the dependence of $f(n, \ell, k)$ on ℓ for constant $k \geqslant 3$, analogously to Theorems 4.3 and 4.5 for $k = 2$. The first result we need is a generalisation of Lemma 4.1. It is proved in the same way.

Lemma 4.6. *Let P be a set of n points in the plane with at most ℓ collinear. Then, for $k \geqslant 4$, the number of collinear k-tuples in P is at most $c(\ell^{k-3} n^2 + l^{k-1} n)$ for some absolute constant c.*

Lemmas 4.2 and 4.6 imply the following theorem which is proved in the same way as Theorem 4.3.

Theorem 4.7. *If $k \geqslant 3$ is constant and $\ell \leqslant O(\sqrt{n})$, then*

$$f(n, \ell, k) \geqslant \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}\right).$$

For $\ell = \sqrt{n}$ and constant $k \geqslant 3$, Theorem 4.7 implies

$$f(n, \sqrt{n}, k) \geqslant \Omega\left(\frac{n^{(k-1)/k}}{n^{(k-2)/2k}}\right) = \Omega\left(n^{(2k-2-k+2)/2k}\right) = \Omega(\sqrt{n}) \ .$$

This answers completely a generalised version of Gowers' question [40], namely, to determine the minimum integer $\mathrm{GP}_k(q)$ such that every set of at least $\mathrm{GP}_k(q)$ points in the plane contains q collinear points or q points with at most k collinear, for $k \geqslant 3$. Thus $\mathrm{GP}_k(q) \leqslant O(q^2)$. The bound $\mathrm{GP}_k(q) \geqslant \Omega(q^2)$ comes from the following construction. Let $m := \lfloor (q-1)/k \rfloor$ and let P be the $m \times m$ grid. Then P has at most m points collinear, and $m < q$. If S is a subset of P with at most k collinear, then S has at most k points in each row. So $|S| \leqslant km \leqslant q - 1$.

Theorem 4.5 can be generalised using Lemma 4.6 and a theorem of Duke, Lefmann and Rödl [19] (the one that implies Lemma 4.4).

Theorem 4.8 (Duke, Lefmann and Rödl)**.** *Let H be a k-uniform hypergraph with maximum degree $\Delta(H) \leqslant t^{k-1}$ where $t \gg k$. Let $p_j(H)$ be the number of pairs of edges of H sharing exactly j vertices. If $p_j(H) \leqslant nt^{2k-j-1-\gamma}$ for $j = 2, \ldots, k-1$ and some constant $\gamma > 0$, then*

$$\alpha(H) \geqslant C(k, \gamma)\frac{n}{t}(\ln n)^{1/(k-1)}$$

for some constant $C(k, \gamma) > 0$.

Theorem 4.9. *Fix constants $d > 0$ and $\epsilon \in (0, 1)$. If $k \geqslant 3$ is constant and $4 \leqslant \ell \leqslant dn^{(1-\epsilon)/2}$ then*

$$f(n, \ell, k) \geqslant \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}(\ln n)^{1/k}\right) .$$

Proof. Given a set P of n points with at most ℓ collinear, a subset with at most k collinear points corresponds to an independent set in the $(k+1)$-uniform hypergraph $H_{k+1}(P)$ of collinear $(k+1)$-tuples in P. By Lemma 4.6, the number of edges in $H_{k+1}(P)$ is $m \leqslant c(n^2\ell^{k-2} + nl^k)$ for some constant c. The first term dominates since $\ell \leqslant o(\sqrt{n})$. For n large enough, $m/n \leqslant 2cn\ell^{k-2}$.

To limit the maximum degree of $H_{k+1}(P)$, discard vertices of degree greater than $2(k+1)m/n$. Let $\tilde{n}$ be the number of such vertices. Considering the sum of degrees, $(k+1)m \geqslant \tilde{n}2(k+1)m/n$, and so $\tilde{n} \leqslant n/2$. Thus discarding these vertices yields a new point set P' such that $|P'| \geqslant n/2$ and

$\Delta(H_{k+1}(P')) \leqslant 4(k+1)cn\ell^{k-2}$. Note that an independent set in $H_{k+1}(P')$ is also independent in $H_{k+1}(P)$.

Set $t := (4(k+1)cn\ell^{k-2})^{1/k}$, so $m \leqslant \frac{1}{2(k+1)}nt^k$ and $\Delta(H_{k+1}(P')) \leqslant t^k$, as required for Theorem 4.8. By assumption, $\ell \leqslant dn^{(1-\epsilon)/2}$. Thus

$$\ell \leqslant d\left(\frac{t^k\ell^{2-k}}{4(k+1)c}\right)^{\frac{1-\epsilon}{2}} .$$

Hence $\ell^{\frac{2}{1-\epsilon}+k-2} \leqslant \frac{d^{2/(1-\epsilon)}t^k}{4(k+1)c}$, implying $\ell \leqslant C_1(k)t^{\frac{k}{\frac{2}{1-\epsilon}+k-2}} = C_1(k)t^{\frac{1-\epsilon}{1-\epsilon+\frac{2\epsilon}{k}}}$ for some constant $C_1(k)$. Define $\epsilon' := 1 - \frac{1-\epsilon}{1-\epsilon+\frac{2\epsilon}{k}}$, so $\epsilon' > 0$ (since $\epsilon < 1$) and $\ell \leqslant C_1(k)t^{1-\epsilon'}$. To bound $p_j(H_{k+1}(P'))$ for $j = 2, \dots, k$, first choose one edge (which determines a line), then choose the subset to be shared, then choose points from the line to complete the second edge of the pair. Thus for $\gamma := \epsilon'/2$ and sufficiently large n,

$$\begin{aligned} p_j(H_{k+1}(P')) &\leqslant m\binom{k+1}{j}\binom{\ell-k-1}{k+1-j} \\ &\leqslant C_2(k)nt^k\ell^{k+1-j} \\ &\leqslant C_2(k)(C_1(k))^{k+1-j}nt^kt^{(1-\epsilon')(k+1-j)} \\ &\leqslant nt^{2(k+1)-j-1-\gamma} . \end{aligned}$$

Hence the second requirement of Theorem 4.8 is satisfied. Thus

$$\begin{aligned} \alpha(H_{k+1}(P')) &\geqslant \Omega\left(\frac{n}{t}(\ln t)^{1/k}\right) \\ &\geqslant \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}\left(\ln((n\ell^{k-2})^{1/k})\right)^{1/k}\right) \\ &\geqslant \Omega\left(\frac{n^{(k-1)/k}}{\ell^{(k-2)/k}}(\ln n)^{1/k}\right) . \end{aligned}$$

□

4.3 Conjectures

Theorem 4.7 suggests the following conjecture, which would completely answer Gowers' question [40], showing that $\mathrm{GP}(q) = \Theta(q^2)$. It is true for the $\sqrt{n} \times \sqrt{n}$ grid [25, 41].

Conjecture 4.10. $f(n, \sqrt{n}) \geqslant \Omega(\sqrt{n})$.

A natural variation of the general position subset selection problem is to colour the points of P with as few colours as possible, such that each colour class is in general position. A straightforward application of the Lovász Local Lemma shows that under this requirement, n points with at most ℓ collinear are colourable with $O(\sqrt{\ell n})$ colours[1]. The following conjecture would imply Conjecture 4.10. It is also true for the $\sqrt{n} \times \sqrt{n}$ grid [100].

Conjecture 4.11. *Every set P of n points in the plane with at most $\sqrt{n}$ collinear can be coloured with $O(\sqrt{n})$ colours such that each colour class is in general position.*

The following proposition is somewhat weaker than Conjecture 4.11.

Proposition 4.12. *Every set P of n points in the plane with at most $\sqrt{n}$ collinear can be coloured with $O(\sqrt{n}\ln^{3/2} n)$ colours such that each colour class is in general position.*

Proof. Colour P by iteratively selecting a largest subset in general position and giving it a new colour. Let $P_0 := P$. Let C_i be a largest subset of P_i in general position and let $P_{i+1} := P_i \setminus C_i$. Define $n_i := |P_i|$. Applying Lemma 4.1 to P_i shows that $H(P_i)$ has $O(n_i^2 \ln \ell + \ell^2 n_i)$ edges. Thus the average degree of $H(P_i)$ is at most $O(n_i \ln \ell + \ell^2)$ which is $O(n \ln n)$ since $n_i \leqslant n$ and $\ell \leqslant \sqrt{n}$.

Applying Lemma 4.2 gives $|C_i| = \alpha(H(P_i)) > cn_i/\sqrt{n \ln n}$ for some constant $c > 0$. Thus $n_i \leqslant n(1 - c/\sqrt{n \ln n})^i$. It is well known (and not difficult to show) that if a sequence of numbers m_i satisfies $m_i \leqslant m(1 - 1/x)^i$ for some $x > 1$ and if $j > x \ln m$, then $m_j \leqslant 1$. Hence if $k \geqslant \sqrt{n \ln n} \ln n/c$ then $n_k \leqslant 1$, so the number of colours used is $O(\sqrt{n}\ln^{3/2} n)$. □

[1] Colouring each point of P with one of c colours uniformly at random, the probability of a particular collinear triple being monochromatic is $1/c^2$. These events are independent unless the triples intersect. Consider all lines determined by P that contain a fixed point p, and let k_i be the number of points on the ith line. Then the number of triples containing p is $\sum_i \binom{k_i - 1}{2} \leqslant \ell \sum_i k_i \leqslant \ell n$. Thus each triple intersects at most $3\ell n$ others. By the Local Lemma there exists a proper colouring as long as $12\ell n \leqslant c^2$.

The problem of determining the correct asymptotics of $f(n,\ell)$ (and $f(n,\ell,k)$) for constant ℓ remains wide open. The Szemerédi–Trotter theorem is essentially tight for the $\sqrt{n}\times\sqrt{n}$ grid [68], but says nothing for point sets with bounded collinearities. For this reason, the lower bounds on $f(n,\ell)$ for constant ℓ remain essentially combinatorial. Finding a way to bring geometric information to bear in this situation is an interesting challenge.

Conjecture 4.13. *If ℓ is constant, then $f(n,\ell) \geqslant \Omega(n/\operatorname{polylog}(n))$.*

The point set that gives the upper bound $f(n,\ell) \leqslant o(n)$ (from the density Hales–Jewett theorem) is the generic projection to the plane of the $\lfloor \log_\ell n \rfloor$-dimensional $\ell\times\ell\times\cdots\times\ell$ integer lattice (henceforth $[\ell]^d$ where $d := \lfloor \log_\ell(n) \rfloor$). The problem of finding large general position subsets in this point set for $\ell = 3$ is known as Moser's cube problem [62, 77], and the best known asymptotic lower bound is $\Omega(n/\sqrt{\ln n})$ [10, 77].

In the colouring setting, the following conjecture is equivalent to Conjecture 4.13 by an argument similar to that of Proposition 4.12.

Conjecture 4.14. *For constant $\ell \geqslant 3$, every set of n points in the plane with at most ℓ collinear can be coloured with $O(\operatorname{polylog}(n))$ colours such that each colour class is in general position.*

Conjecture 4.14 is true for $[\ell]^d$, which can be coloured with $O(d^{\ell-1})$ colours as follows. For each $x \in [\ell]^d$, define a signature vector in $\mathbb{Z}^\ell$ whose entries are the number of entries in x equal to $1, 2, \dots \ell$. The number of such signatures is the number of partitions of d into at most ℓ parts, which is $O(d^{\ell-1})$. Give each set of points with the same signature its own colour. To see that this is a proper colouring, suppose that $\{a,b,c\} \subset [\ell]^d$ is a monochromatic collinear triple, with b between a and c. Permute the coordinates so that the entries of b are non-decreasing. Consider the first coordinate i in which a_i, b_i and c_i are not all equal. Then without loss of generality, $a_i < b_i$. But this implies that a has more entries equal to a_i than b does, contradicting the assumption that the signatures are equal.

Chapter 5

Connectivity of visibility graphs

In this chapter we study the edge- and vertex-connectivity of visibility graphs. A graph G on at least $k+1$ vertices is k-vertex-connected (k-edge-connected) if G remains connected whenever fewer than k vertices (edges) are deleted. Menger's Theorem says that this is equivalent to the existence of k vertex-disjoint (edge-disjoint) paths between each pair of vertices. Let $\kappa(G)$ and $\lambda(G)$ denote the vertex- and edge-connectivity of a graph G. Let $\delta(G)$ denote the minimum degree of G. We have $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$.

If a visibility graph G has n vertices, at most ℓ of which are collinear, then $\delta(G) \geqslant \frac{n-1}{\ell-1}$. We will show that both edge- and vertex-connectivity are at least $\frac{n-1}{\ell-1}$ (Theorem 5.4 and Corollary 5.15). Since there are visibility graphs with $\delta = \frac{n-1}{\ell-1}$ these lower bounds are best possible.

We will refer to visibility graphs whose vertices are not all collinear as *non-collinear visibility graphs*. Non-collinear visibility graphs have diameter 2 [48], and it is known that graphs of diameter 2 have edge-connectivity equal to their minimum degree [75]. We strengthen this result to show that if a graph has diameter 2 then for any two vertices v and w with $\deg(v) \leqslant \deg(w)$, there are $\deg(v)$ edge-disjoint vw-paths of length at most 4

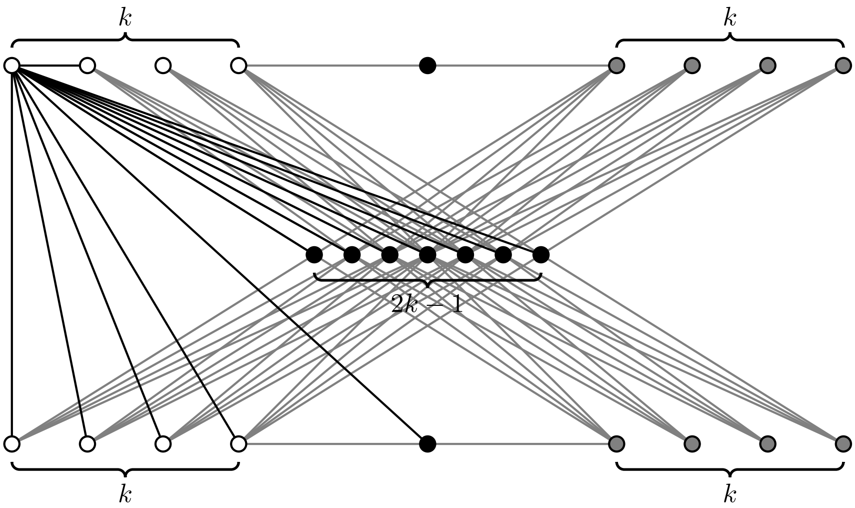


Figure 5.1: A visibility graph with vertex-connectivity $\frac{2\delta+1}{3}$. The black vertices are a cut set. The minimum degree $\delta = 3k + 1$ is achieved, for example, at the top left vertex. Not all edges are drawn.

(Theorem 5.2). We also characterise minimum edge cuts in visibility graphs as the sets of edges incident to a vertex of minimum degree (Theorem 5.6).

With regard to vertex-connectivity, our main result is that $\kappa \geqslant \frac{\delta}{2} + 1$ for all non-collinear visibility graphs (Theorem 5.11). This bound is qualitatively stronger than the bound $\kappa \geqslant \frac{n-1}{\ell-1}$ since it is always within a factor of 2 of being optimal. In the special case of at most four collinear points, we improve this bound to $\kappa \geqslant \frac{2\delta+1}{3}$ (Theorem 5.18). We conjecture that $\kappa \geqslant \frac{2\delta+1}{3}$ for all visibility graphs. This bound would be best possible since, for each integer k, there is a visibility graph with a vertex cut of size $2k+1$, but minimum degree $\delta = 3k + 1$. Therefore the vertex-connectivity is at most $2k + 1 = \frac{2\delta+1}{3}$. Figure 5.1 shows the case $k = 4$.

A central tool in this chapter, which is also of independent interest, is a kind of bipartite visibility graph. Let A and B be disjoint sets of points in the plane. The *bivisibility graph* $\mathcal{B}(A, B)$ of A and B has vertex set $A \cup B$, where points $v \in A$ and $w \in B$ are adjacent if and only if they are visible with respect to $A \cup B$. The following simple observation is used several times in this chapter, and highlights the importance of bivisibility graphs.

Observation 5.1. *Let G be a visibility graph. Let $\{A, B, C\}$ be a partition of $V(G)$ such that C separates A and B. If $\mathcal{B}(A, B)$ contains t pairwise non-crossing edges, then $|C| \geqslant t$ since there must be a distinct vertex in C on each such edge.*

Finally, one lemma in particular stands out as being of independent interest. Lemma 5.7 says that for any two properly coloured non-crossing geometric graphs that are separated by a line (except for some degenerate cases), there exists an edge joining them such that the union is a properly coloured non-crossing geometric graph.

5.1 Edge connectivity

Non-collinear visibility graphs have diameter at most 2 [48]. This is because even if two points cannot see each other, they can both see the point closest to the line containing them. Plesník [75] proved that the edge-connectivity of a graph with diameter at most 2 equals its minimum degree. Thus the edge-connectivity of a non-collinear visibility graph equals its minimum degree. There are several other known conditions that imply that the edge-connectivity of a graph is equal to the minimum degree; see for example [14, 76, 99]. Here we prove the following strengthening of the result of Plesník.

Theorem 5.2. *Let G be a graph with diameter 2. Then the edge-connectivity of G equals its minimum degree. Moreover, for all distinct vertices v and w in G, if $d := \min\{\deg(v), \deg(w)\}$ then there are d edge-disjoint vw-paths of length at most 4, including at least one of length at most 2.*

Proof. First suppose that v and w are not adjacent. Let C be the set of common neighbours of v and w. For each vertex $c \in C$, take the path (v, c, w). Let A be a set of $d - |C|$ neighbours of v not in C. Let B be a set of $d - |C|$ neighbours of w not in C. Let M_1 be a maximal matching in the bipartite subgraph of G induced by A and B. Call these matched vertices A_1 and B_1. For each edge $ab \in M_1$, take the path (v, a, b, w). Let A_2 and B_2 respectively be the subsets of A and B consisting of the unmatched vertices. Let $D := V(G)\setminus(A_2 \cup B_2 \cup \{v, w\})$. Let M_2 be an arbitrary pairing of vertices in A_2 and B_2. For each pair $ab \in M_2$, take the path (v, a, x, b, w), where x is a common neighbour of a and b (which exists since G has diameter 2). Since x is adjacent to a, $x \neq w$, and by the maximality of M_1, $x \notin B_2$. Similarly, $x \neq v$ and $x \notin A_2$, and so $x \in D$. Thus there are three types of paths, namely (v, C, w), (v, A_1, B_1, w), and (v, A_2, D, B_2, w). Paths within each type are edge-disjoint. Even though D contains A_1 and B_1, edges between each pair of sets from $\{A_1, B_1, A_2, B_2, C, D, \{v\}, \{w\}\}$ occur in at most one of the types, and all edges are between distinct sets from this collection. Hence no edge is used twice, so all the paths are edge-disjoint. The total number of paths is $|C| + |A_1| + |A_2| = d$. This finishes the proof if v and w are not adjacent. If G does contain the edge vw then take this as the first path, then remove it and find $d - 1$ paths in the same way as above. $\square$

Note that the lengths of the paths found in Theorem 5.2 cannot be improved, as shown by the following example. For integers $\gamma \geqslant 1$ and $\delta \geqslant 3$, let G be the graph obtained from a 5-cycle (v, w, x, y, z) by replacing x by a $(\delta - 1)$-clique X, replacing y by a γ-clique Y, replacing z by a $(\delta - 1)$-clique Z, and replacing edges between these vertices with complete bipartite subgraphs. Each vertex in X is adjacent to w and to each vertex in Y. Each vertex in Z is adjacent to v and to each vertex in Y. Thus G has minimum degree δ and diameter 2. Note that $\deg(v) = \deg(w) = \delta$. In fact, by choosing γ large, we can make v and w the only vertices of degree δ and every other vertex have arbitrarily large degree. Consider a set S of δ edge-disjoint paths between v and w. One path in S is the edge vw, while every other path has length at least 4. Thus the paths found in Theorem 2 are best possible. A further example, in which v and w are not adjacent, can be constructed by taking two disjoint $(\delta + 1)$-cliques and identifying a vertex from each. Suppose v and w come from different cliques and are not the identified vertex. Then there are $\delta - 1$ vw-paths of length 4 and one of length 2. Alternatively, one can take $\delta - 2$ vw-paths of length 4 and two of length 3.

Note also that Theorem 5.2 generalises for directed graphs G with diameter at most 2. That is, for all vertices v and w there is a directed path from v to w of length at most 2. Let v and w be distinct vertices in G. Let $d := \min\{\mathrm{outdeg}(v), \mathrm{indeg}(w)\}$. Then a proof almost identical to that of Theorem 5.2 proves that there are d edge-disjoint directed paths of length at most 4 from v to w.

Theorem 5.2 implies the following corollary for visibility graphs.

Corollary 5.3. *Let G be a non-collinear visibility graph. Then the edge-connectivity of G equals its minimum degree. Moreover, for distinct vertices v and w, there are* $\min\{\deg(v), \deg(w)\}$ *edge-disjoint vw-paths of length at most* 4*, including at least one of length at most* 2*.*

We now show that not only is the edge connectivity as high as possible, but it is realised by paths with at most one bend.

Theorem 5.4. *Let G be a visibility graph with n vertices, at most ℓ of which are collinear. Then G is $\lceil \frac{n-1}{\ell-1} \rceil$-edge-connected, which is best possible. Moreover, between each pair of vertices, there are $\lceil \frac{n-1}{\ell-1} \rceil$ edge-disjoint 1-bend paths.*

Proof. Let v and w be distinct vertices of G. Let V^* be the set of vertices of G not on the line vw. Let $m := |V^*|$. Thus $m \geqslant n - \ell$.

Let $\mathcal{L}$ be the pencil of lines through v and the vertices in V^*. Let $\mathcal{M}$ be the pencil of lines through w and the vertices in V^*. Let H be the bipartite graph with vertex set $\mathcal{L} \cup \mathcal{M}$, where $L \in \mathcal{L}$ is adjacent to $M \in \mathcal{M}$ if and only if $L \cap M$ is a vertex in V^*.

Thus H has m edges, and maximum degree at most $\ell - 1$. Hence, by König's theorem [51], H is $(\ell - 1)$-edge-colourable. Thus H contains a matching of at least $\frac{m}{\ell-1}$ edges. This matching corresponds to a set S of at least $\frac{m}{\ell-1}$ vertices in V^*, no two of which are collinear with v or w.

For each vertex $x \in S$, take the path in the visibility graph from v straight to x and then straight to w. These paths are edge-disjoint. Adding the path

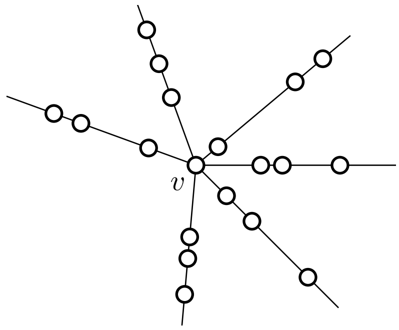


Figure 5.2: If each ray from v through $V(G)$ contains ℓ vertices, the degree of v is $\frac{n-1}{\ell-1}$.

straight from v to w, we get at least $\frac{m}{\ell-1}+1$ paths, which is at least $\frac{n-1}{\ell-1}$. Figure 5.2 shows that this bound is best possible. □

We now prove that minimum sized edge cuts in non-collinear visibility graphs are only found around a vertex. To do this, we first characterise the diameter 2 graphs for which it does not hold.

Proposition 5.5. *Let G be a graph with diameter at most 2 and minimum degree $\delta \geqslant 2$. Then G has an edge cut of size δ that is not the set of edges incident to a single vertex if and only if $V(G)$ can be partitioned into $A \cup B \cup C$ such that:*

- *$G[A] \cong K_\delta$ and $|B \cup C| \geqslant \delta$,*
- *each vertex in A has exactly one neighbour in B and no neighbours in C,*
- *each vertex in B has at least one neighbour in A, and*
- *each vertex in B is adjacent to each vertex in C.*

Proof. If G has the listed properties then the edges between A and B form a cut of size δ that is not the set of edges incident to a single vertex.

Conversely, suppose an edge cut of size δ separates the vertices of G into two sets X and Y with $|X| > 1$ and $|Y| > 1$. Each vertex of X is incident to at least $\delta - (|X| - 1)$ edges of the cut. It follows that $\delta \geqslant |X|(\delta - (|X| - 1))$. Consequently, $|X|(|X| - 1) \geqslant \delta(|X| - 1)$ and thus $|X| \geqslant \delta$. Analogously, $|Y| \geqslant \delta$. Since G has diameter 2, there are no vertices $x \in X$ and $y \in Y$, such that all the neighbours of x are in X, and all the neighbours of y are in Y. Thus we may assume without loss of generality that all vertices in X have a neighbour in Y. Since there are only δ edges between X and Y, $|X| = \delta$ and each vertex in X has exactly one neighbour in Y. The minimum degree condition implies that all edges among X are present. Let $A := X$, $B := \bigcup_{x \in X}\{N(x) \setminus X\}$ and $C := V(G) \setminus (A \cup B)$. Each vertex $c \in C$ must be joined to all vertices in B, otherwise there would be a vertex in A at distance greater than 2 from c. $\square$

We now prove that diameter 2 graphs such as those described in Proposition 5.5 cannot be visibility graphs.

Theorem 5.6. *Every minimum edge-cut in a non-collinear visibility graph is the set of edges incident to some vertex.*

Proof. Let G be a non-collinear visibility graph. Suppose for the sake of contradiction that G has an edge cut of size $\delta(G)$ that is not the set of edges incident to a single vertex. Since G is non-collinear, $\delta \geqslant 2$. By Proposition 5.5, $V(G)$ can be partitioned into $A \cup B \cup C$ with $|A| = \delta$, $|B \cup C| \geqslant \delta$, and δ edges between A and B. Furthermore, the vertices in A can pairwise see each other and each vertex in A has precisely one neighbour in B.

Choose any $a \in A$ and draw the pencil of δ rays from a to all other vertices of the graph. All rays except one contain a point in $A \setminus \{a\}$. Say two rays are *neighbours* if they bound a sector of angle less than π with no other ray inside it. Observe that every ray has at least one neighbour.

First suppose a is in the interior of the convex hull of $V(G)$, as in Figure 5.3(a). Then every ray has two neighbours, so each point in $B \cup C$ can see at least one point of $A \setminus \{a\}$ on a neighbouring ray. Hence C is empty and $|B| \geqslant \delta$. Along with the edge from a to its neighbour in B we have at

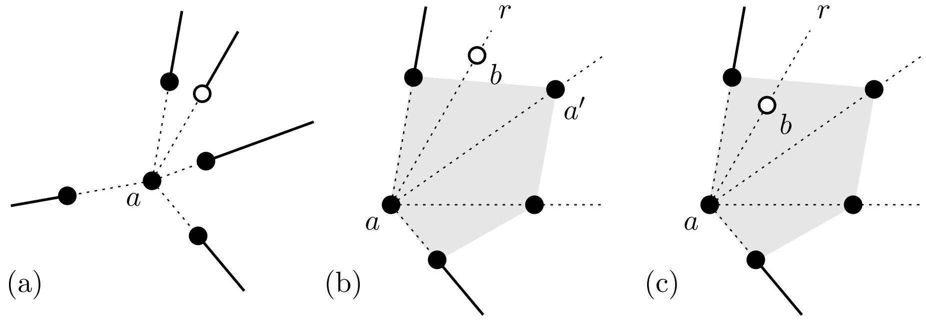


Figure 5.3: In each case the remaining points of $B \cup C$ must lie on the solid segments of the rays.

least $\delta + 1$ edges between A and B, a contradiction.

If we cannot choose a in the interior of $\mathrm{conv}(V(G))$, then A is in strictly convex position because no three points of A are collinear. Let the rays from a containing another point from A be called A-*rays*. The A-rays are all extensions of diagonals or edges of $\mathrm{conv}(A)$. There is one more ray r that contains only points of $B \cup C$. In fact, r has only one point b on it, since all of r is visible from the point in A on a neighbouring ray. Furthermore, the rays which extend diagonals of $\mathrm{conv}(A)$ contain no points of $B \cup C$ since A lies in the boundary of $\mathrm{conv}(V(G))$. Hence the rest of $B \cup C$ must lie in the two rays which extend the sides of $\mathrm{conv}(A)$. If these rays both have a neighbouring A-ray, then we can argue as before and find $\delta + 1$ edges between A and B. We are left with the case where some A-ray has r as its only neighbour. If b lies outside $\mathrm{conv}(A)$ and $\delta > 2$ (Figure 5.3(b)), then we can change our choice of a to a point a' on a ray neighbouring r, and then we are back to the previous case. (If $\delta = 2$ then the other point of A will see b so there can be no more points of $B \cup C$). Otherwise b is the only point in the interior of $\mathrm{conv}(A)$ (Figure 5.3(c)), and is therefore the only point in B since it sees all of A. In this case C must be empty since b blocks a point $c \in C$ from at most one point in A. Thus $|B \cup C| = 1$, a contradiction. $\square$

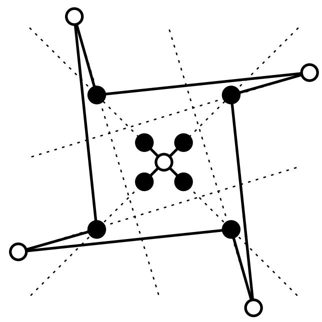


Figure 5.4: Two properly coloured non-crossing geometric graphs with no black-white edge between them.

5.2 A key lemma

We call a plane graph drawn with straight edges a *non-crossing geometric graph*. The following interesting fact about non-crossing geometric graphs will prove useful. It says that, except for some degenerate cases, two properly coloured non-crossing geometric graphs that are separated by a line can be joined by an edge such that the union is a properly coloured non-crossing geometric graph. Note that this is false if the two graphs are not separated by a line, as demonstrated by the example in Figure 5.4. A similar result, but with the vertices assumed to be in general position, was proved by Hurtado et al. [46].

Lemma 5.7. *Let G_1 and G_2 be two properly coloured non-crossing geometric graphs with at least one edge each. Suppose their convex hulls are disjoint and that $V(G_1) \cup V(G_2)$ is not collinear. Then there exists an edge $e \in V(G_1) \times V(G_2)$ such that $G_1 \cup G_2 \cup \{e\}$ is a properly coloured non-crossing geometric graph.*

Proof. Let h be a line separating G_1 and G_2. Assume that h is vertical with G_1 to the left. Let $G := G_1 \cup G_2$.

Call a pair of vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ a *visible pair* if the line segment between them does not intersect any vertices or edges of G. We aim to find a visible pair with different colours, so assume for the sake of contradiction that every visible pair is monochromatic.

We may assume that G_1 and G_2 are edge maximal with respect to the colouring, since the removal of an edge only makes it easier to find a bichromatic visible pair.

Suppose the result holds when there are no isolated vertices in G. Then, if there are isolated vertices, we can ignore them and find a bichromatic visible pair (v_1, v_2) in the remaining graph. If the edge v_1v_2 contains some of the isolated vertices, then it has a sub-segment joining two vertices of different colours. If these vertices lie on the same side of h then the graphs were not edge maximal after all. If they are on different sides, then they are a bichromatic visible pair. Thus we may assume that there are no isolated vertices in G.

Let l be the line containing a visible pair (v_1, v_2), then the *height* of the pair is the point at which l intersects h. Call the pair *type-1* if v_1 and v_2 both have a neighbour strictly under the line l (Figure 5.5(a)). Call the pair *type-2* if there are edges v_1w_1 in G_1 and v_2w_2 in G_2 such that the line g containing v_1w_1 intersects v_2w_2 (call this point x), w_2 lies strictly under g, and the closed triangle v_1v_2x contains no other vertex (Figure 5.5(b)). Here x may equal v_2, in which case $g = l$. A visible pair is also type-2 in the equivalent case with the subscripts interchanged.

A particular visible pair may be neither type-1 nor type-2, but we may assume there exists a type-1 or type-2 pair. To see this, consider the highest visible pair (v_1, v_2) and assume it is neither type-1 nor type-2 (see Figure 5.5(c)). Note that v_1v_2 is an edge of the convex hull of G. Since all of G lies on or below the line l containing v_1v_2, both vertices must have degree 1 and their neighbours w_1 and w_2 must lie on l. For $i = 1, 2$, let x_i be a vertex of G_i not on l that minimizes the angle $\angle v_iw_ix_i$. Since $V(G)$ is not collinear, at least one of x_i exists. By symmetry, we may assume that either only x_1 exists, or both x_1 and x_2 exist and $\text{dist}(x_1, l) \leqslant \text{dist}(x_2, l)$. In either case, (x_1, v_2) and (x_1, w_2) are visible pairs and at least one of them is bichromatic.

So now assuming there exists a type-1 or type-2 visible pair, let (u_1, u_2) be the lowest such pair:

Case (i) The pair (u_1, u_2) is type-1 (see Figure 5.5(d)). Let u_1w_1 be the first edge of G_1 incident to u_1 in a clockwise direction, starting at u_1u_2. Let u_2w_2 be the first edge of G_2 incident to u_2 in a counterclockwise direction, starting at u_2u_1. Let x be the point on the segment u_1w_1 closest to w_1 such that the open triangle u_1u_2x is disjoint from G. Similarly, let y be the point on the segment u_2w_2 closest to w_2 such that the open triangle u_1u_2y is disjoint from G.

Without loss of generality, the intersection of u_1y and u_2x is to the left of h, or on h. Therefore the segment xu_2 is disjoint from G_2. Let $v \in V(G_1)$ be the vertex on xu_2 closest to u_2. Thus (v, u_2) is a visible pair of height less than (u_1, u_2). We may assume that $v \neq w_1$, otherwise (v, u_2) would be bichromatic. The point w_2 is under the line vu_2 and v has no neighbour above the line vu_2. Hence v either has a neighbour under the line vu_2 and (v, u_2) is type-1, or v has a neighbour on the line vu_2 and (v, u_2) is type-2. This contradicts the assumption that (u_1, u_2) was the lowest pair of either type.

Case (ii) The pair (u_1, u_2) is type-2 with neighbours w_1 and w_2, such that the line u_2w_2 intersects the edge u_1w_1 at some point x (see Figure 5.5(e)). Let y be the point on the segment u_1w_1 closest to w_1 such that the open triangle u_1u_2y is disjoint from G. Note y is below x by the definition of type-2.

First assume that G_2 intersects yu_2. Let v_2 be the closest vertex to u_2 on yu_2. Thus (u_1, v_2) is a visible pair of height less than (u_1, u_2). Let z be a neighbour of v_2. If z is under the line u_1v_2 then (u_1, v_2) is type-1 since w_1 is also under this line. Note z cannot lie above the line yu_2 since u_1 and u_2 see each other and the open triangle u_1u_2y is empty. Furthermore, if z lies on yu_2 then $z = u_2$ and (u_1, v_2) is bichromatic. Thus if z is not under the line u_1v_2, the line v_2z must intersect the edge u_1w_1 at a point above y, so (u_1, v_2) is a type-2 pair. Hence the pair (u_1, v_2) is type-1 or type-2, a

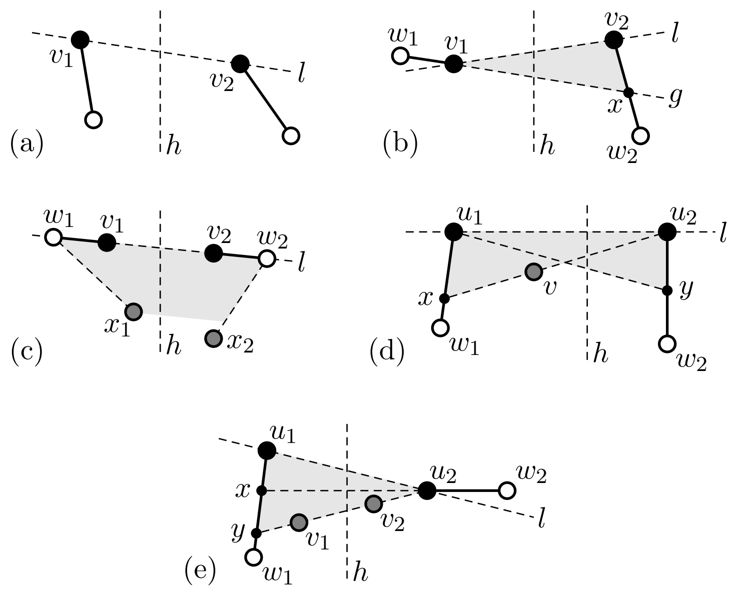


Figure 5.5: Proof of Lemma 5.7. The shaded areas are empty. (a) A type-1 visible pair. (b) A type-2 visible pair. (c) The highest visible pair. (d) The lowest pair is type-1. (e) The lowest pair is type-2.

contradiction.

Now assume that yu_2 does not intersect G_2, and therefore does intersect G_1, and let $v_1 \in V(G_1)$ be the vertex on yu_2 closest to u_2. Thus (v_1, u_2) is a visible pair of height less than (u_1, u_2). We may assume that $v_1 \neq w_1$, otherwise (v_1, u_2) would be bichromatic. Since w_2 is under the line v_1u_2, if v_1 has a neighbour under the line v_1u_2 then (v_1, u_2) is a type-1 pair. Otherwise the only neighbour of v_1 is on the line v_1u_2 which makes (v_1, u_2) a type-2 pair. Hence the pair (v_1, u_2) is type-1 or type-2, a contradiction. $\square$

5.3 Vertex connectivity

As is common practice, we will often refer to vertex-connectivity simply as connectivity. Connectivity of visibility graphs is not as straightforward as edge-connectivity since there are visibility graphs with connectivity strictly less than the minimum degree (see Figure 5.1). Our aim in this section is to show that the connectivity of a visibility graph is at least half the minimum degree (Theorem 5.11). This follows from Theorem 5.10 below, which says that bivisibility graphs contain large non-crossing subgraphs. In the proof of Theorem 5.10 we will need a version of the Ham Sandwich Theorem for point sets in the plane, and also Lemma 5.9.

Theorem 5.8. *(Ham Sandwich. See [58].) Let A and B be finite sets of points in the plane. Then there exists a line h such that each closed half-plane determined by h contains at least half of the points in A and at least half of the points in B.*

Lemma 5.9. *Let A be a set of points lying on a line l. Let B be a set of points, none of them lying on l. Let $|A| \geqslant |B|$. Then there is a non-crossing spanning tree in the bivisibility graph of A and B.*

Proof. We proceed by induction on $|B|$. If $|B| = 1$ then the point in B sees every point in A, and we are done. Now assume $1 < |B| \leqslant |A|$.

First suppose that all of B lies to one side of l and consider the convex hull C of $A \cup B$. An end point a of A is a corner of C and there is a point b of B visible to it in the boundary of C. There exists a line h that separates $\{a, b\}$ from the rest of $A \cup B$. Applying induction and Lemma 5.7 we find a non-crossing spanning tree among $A \cup B \setminus \{a, b\}$ and an edge across h to the edge ab, giving a non-crossing spanning tree of $\mathcal{B}(A, B)$.

Now suppose that there are points of B on either side of l. Then we may apply the inductive hypothesis on each side to obtain two spanning trees. Their union is connected, and thus contains a spanning tree. □

Theorem 5.10. *Let A and B be disjoint sets of points in the plane with $|A| = |B| = n$ such that $A \cup B$ is not collinear. Then the bivisibility graph $\mathcal{B}(A, B)$ contains a non-crossing subgraph with at least $n + 1$ edges.*

Proof. We proceed by induction on n. The statement holds for $n = 1$, since no valid configuration exists. For $n = 2$, any triangulation of $A \cup B$ contains at least five edges. At most one edge has both endpoints in A, and similarly for B. Removing these edges, we obtain a non-crossing subgraph of $\mathcal{B}(A,B)$ with at least three edges. Now assume $n > 2$.

Case (i) First suppose that there exists a line l that contains at least n points of $A \cup B$. Let $A_0 := A \cap l$, $B_0 := B \cap l$, $A_1 := A \setminus l$ and $B_1 := B \setminus l$. Without loss of generality, $|A_0| \geqslant |B_0|$.

If $|A_0| > |B_0|$ then $|A_0| + |B_1| > |B_0| + |B_1| = n$. Since $|A_0| + |B_0| \geqslant n = |B_1| + |B_0|$ we have $|A_0| \geqslant |B_1|$, so we may apply Lemma 5.9 to A_0 and B_1. We obtain a non-crossing subgraph of $\mathcal{B}(A,B)$ with $|A_0| + |B_1| - 1 \geqslant n$ edges, and by adding an edge along l if needed, we are done.

Now assume $|A_0| = |B_0|$. We apply Lemma 5.9 to A_0 and B_1, obtaining a non-crossing subgraph with $n-1$ edges, to which we may add one edge along l. We still need one more edge. Suppose first that one open half-plane determined by l contains points of both A_1 and B_1. Let a and b be the furthest points of A_1 and B_1 from l in this half-plane. Since $|A_0| = |B_0|$ we may assume that a is at least as far from l as b. Then we may add an edge along the segment ab, because none of the edges from A_0 to B_1 cross it. It remains to consider the case where l separates A_1 from B_1. Then applying Lemma 5.9 on each side of l we find a non-crossing subgraph with $2n-1$ edges: $|A_0| + |B_1| - 1$ on one side, $|B_0| + |A_1| - 1$ on the other side, and one more along l.

Case (ii) Now assume that no line contains n points in $A \cup B$. By Theorem 5.8 there exists a line h such that each of the closed half-planes determined by h contains at least $\frac{n}{2}$ points from each of A and B. Assume that h is horizontal. Let A^+ be the points of A that lie above h along with any that lie on h that we choose to assign to A^+. Define A^-, B^+ and B^- in a similar fashion. Now assign the points on h to these sets so that each has exactly $\lceil \frac{n}{2} \rceil$ points. In particular, assign the required number of *leftmost* points of $h \cap A$ to A^+ and *rightmost* points of $h \cap A$ to A^-. Do the same for $h \cap B$ with left and right interchanged. If n is even then $A^+ \cup A^-$ and $B^+ \cup B^-$ are partitions of A and B. If n is odd then $|A^+ \cap A^-| = |B^+ \cap B^-| = 1$.

Since there is no line containing n points of $A \cup B$, the inductive hypothesis may be applied on either side of h. Thus there is a non-crossing subgraph with $\lceil \frac{n}{2} \rceil + 1$ edges on each side. The union of these subgraphs has at least $n+2$ edges, but some edges along h may overlap. Due to the way the points on h were assigned, one of the subgraphs has at most one edge along h. (If n is odd, this is the edge between the two points that get assigned to both sides.) Deleting this edge from the union yields a non-crossing subgraph of $\mathcal{B}(A, B)$ with at least $n + 1$ edges. □

Theorem 5.11. *Every non-collinear visibility graph with minimum degree δ has connectivity at least $\frac{\delta}{2} + 1$.*

Proof. Suppose $\{A, B, C\}$ is a partition of the vertex set of a non-collinear visibility graph such that C separates A and B, and $|A| \leqslant |B|$. By considering a point in A we see that $\delta \leqslant |A| + |C| - 1$. By removing points from B until $|A| = |B|$ whilst ensuring that $A \cup B$ is not collinear, we may apply Theorem 5.10 and Observation 5.1 to get $|C| \geqslant |A| + 1$. Combining these inequalities yields $|C| \geqslant \frac{\delta}{2} + 1$. □

The following observations are corollaries of Theorem 5.11, though they can also be proven directly by elementary arguments.

Proposition 5.12. *The following are equivalent for a visibility graph G: (1) G is not collinear, (2) $\kappa(G) \geqslant 2$, (3) $\lambda(G) \geqslant 2$ and (4) $\delta(G) \geqslant 2$.*

Proposition 5.13. *The following are equivalent for a visibility graph G: (1) $\kappa(G) \geqslant 3$, (2) $\lambda(G) \geqslant 3$ and (3) $\delta(G) \geqslant 3$.*

5.4 Vertex connectivity with bounded collinearities

For the visibility graphs of point sets with n points and at most ℓ collinear, connectivity is at least $\frac{n-1}{\ell-1}$, just as for edge-connectivity. Bivisibility graphs will play a central role in the proof of this result. For point sets A and B an *AB-line* is a line containing points from both sets.

Theorem 5.14. *Let $A \cup B$ be a non-trivial partition of a set of n points with at most ℓ on any AB-line. Then the bivisibility graph $\mathcal{B}(A, B)$ contains a non-crossing forest with at least $\frac{n-1}{\ell-1}$ edges. In particular, if $\ell = 2$ then the forest is a spanning tree.*

Proof. The idea of the proof is to cover the points of $A \cup B$ with a large set of disjoint line segments each containing an edge of $G := \mathcal{B}(A, B)$. Start with a point $v \in A$. Consider all open ended rays starting at v and containing a point of B. Each such ray contains at least one edge of G and at most $\ell - 1$ points of $(A \cup B) \setminus v$. For each ray r, choose a point $w \in B \cap r$. Draw all maximal line segments with an open end at w and a closed end at a point of A in the interior of the sector clockwise from r. Figure 5.6 shows an example. If one sector S has central angle larger than π then some points of A may not be covered. In this case we bisect S, and draw segments from each of its bounding rays into the corresponding half of S (assign points on the bisecting line to one sector arbitrarily). Like the rays, these line segments all contain at least one edge of G and at most $\ell - 1$ points of $(A \cup B) \setminus \{v, w\}$. Together with the rays, they are pairwise disjoint and cover all of $(A \cup B) \setminus v$. Hence the edges of G contained in them form a non-crossing forest with at least $\frac{n-1}{\ell-1}$ edges. Note that if $\ell = 2$ we have a forest with $n - 1$ edges, hence a spanning tree. $\square$

Note that the $\ell = 2$ case of Theorem 5.14 is well known [47].

Corollary 5.15. *Let G be the visibility graph of a set of n points with at most ℓ collinear. Then G has connectivity at least $\frac{n-1}{\ell-1}$, which is best possible.*

Proof. Let $\{A, B, C\}$ be a partition of $V(G)$ such that C separates A and B. Consider the bivisibility graph of $A \cup B$. Applying Observation 5.1 and Theorem 5.14 (with $n' = n - |C|$ and $\ell' = \ell - 1$) yields $|C| \geqslant \frac{n-|C|-1}{\ell-2}$, which implies $|C| \geqslant \frac{n-1}{\ell-1}$. As in the case of edge-connectivity, the example in Figure 5.2 shows that this bound is best possible. $\square$

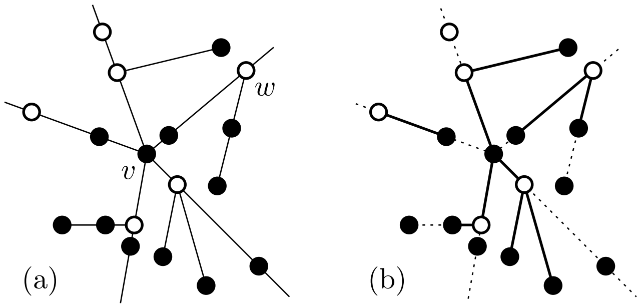


Figure 5.6: Covering $A \cup B$ with rays and segments (a), each of which contains an edge of the bivisibility graph (b).

In the case of visibility graphs with at most three collinear vertices, it is straightforward to improve the bound in Theorem 5.11.

Proposition 5.16. *Let G be a visibility graph with minimum degree δ and at most three collinear vertices. Then G has connectivity at least $\frac{2\delta+1}{3}$.*

Proof. Let $\{A, B, C\}$ be a partition of $V(G)$ such that C separates A and B. Thus each AB-line contains only two vertices in $A \cup B$. Applying Theorem 5.14 (with $\ell = 2$) and Observation 5.1 to $\mathcal{B}(A, B)$ gives $|C| \geqslant |A| + |B| - 1$. For $v \in A$ and $w \in B$ note that $\delta \leqslant \deg(v) \leqslant |A| + |C| - 1$ and $\delta \leqslant \deg(w) \leqslant |B| + |C| - 1$. Combining these inequalities gives $|C| \geqslant \frac{2\delta+1}{3}$. □

In the case of visibility graphs with at most four collinear vertices, the same improvement is found as a corollary of the following theorem about bivisibility graphs. Lemma 5.7 is an important tool in the proof.

Theorem 5.17. *Let A and B be disjoint point sets in the plane with $|A| = |B| = n$ such that $A \cup B$ has at most three points on any AB-line. Then the bivisibility graph $\mathcal{B}(A, B)$ contains a non-crossing spanning tree.*

Proof. We proceed by induction on n. The statement is true for $n = 1$. Apply Theorem 5.8 to find a line h such that each closed half-plane defined by h has at least $\frac{n}{2}$ points from each of A and B. Assume that h is horizontal.

The idea of the proof is to apply induction on each side of h to get two spanning trees, and then find an edge joining them together. In most cases the joining edge will be found by applying Lemma 5.7.

We will construct a set A^+ containing the points of A that lie above h along with any that lie on h that we choose to assign to A^+. We will also construct A^-, B^+ and B^- in a similar fashion. By the properties of h, there exists an assignment[1] of each point in $h \cap (A \cup B)$ to one of these sets such that $|A^+| = |B^+| = \lceil \frac{n}{2} \rceil$ and $|A^-| = |B^-| = \lfloor \frac{n}{2} \rfloor$.

Consider the sequence s_h of signs ($+$ or $-$) given by the chosen assignment of points on h from left to right. If s_h is all the same sign, or alternates only once from one sign to the other, then it is possible to perturb h to h' so that $A^+ \cup B^+$ lies strictly above h' and $A^- \cup B^-$ lies strictly below h'. Thus we may apply induction on each side to obtain non-crossing spanning trees in $\mathcal{B}(A^+, B^+)$ and $\mathcal{B}(A^-, B^-)$. Then apply Lemma 5.7 to find an edge between these two spanning trees, creating a non-crossing spanning tree of $\mathcal{B}(A, B)$.

Otherwise, s_h alternates at least twice (so there are at least three points on h). This need never happen if there are only points from one set on h, since the points required above h can be taken from the left and those required below h from the right. Without loss of generality, the only remaining case to consider is that h contains one point from A and two from B. If the two points from B are consecutive on h, then without loss of generality $s_h = (+, -, +)$ and the points of B are on the left. In this case the signs of the points from B may be swapped so s_h becomes $(-, +, +)$. If the point from A lies between the other two points, it is possible that s_h must alternate twice. In this case, use induction to find spanning trees in $\mathcal{B}(A^+, B^+)$ and $\mathcal{B}(A^-, B^-)$. These spanning trees have no edges along h, so we may add an edge along h to connect them, as shown in Figure 5.7. $\square$

[1] We need only consider one of the sets, say A. Say there are x points above h, y points on h and z points below h. Then $x + y \geqslant \lceil n/2 \rceil \geqslant \lfloor n/2 \rfloor \geqslant x$ so we can ensure $|A^+| = \lceil n/2 \rceil$. A^- is the complement and therefore has $\lfloor n/2 \rfloor$ points.

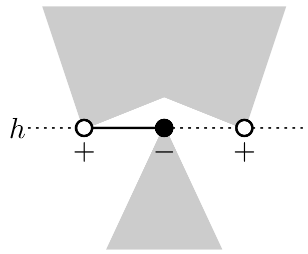


Figure 5.7: The only case in which h may not be perturbed to separate the points assigned above h from those assigned below.

Theorem 5.18. *Let G be a visibility graph with minimum degree δ and at most four collinear vertices. Then G has connectivity at least $\frac{2\delta+1}{3}$.*

Proof. Let $\{A, B, C\}$ be a partition of $V(G)$ such that C separates A and B and $|A| \leqslant |B|$. By considering a point in A we can see that $\delta \leqslant |A|+|C|-1$. If necessary remove points from B so that $|A| = |B|$. Applying Theorem 5.17 and Observation 5.1 yields $|C| \geqslant 2|A| - 1$. Combining these inequalities yields $|C| \geqslant \frac{2\delta+1}{3}$. □

It turns out that Proposition 5.16 and Theorem 5.18 are best possible. There are visibility graphs with at most three collinear vertices and connectivity $\frac{2\delta+1}{3}$. The construction was discovered by Roger Alperin, Joe Buhler, Adam Chalcraft and Joel Rosenberg in response to a problem posed by Noam Elkies. Elkies communicated their solution to Todd Trimble who published it on his blog [96]. Here we provide a brief description of the construction, but skip over most background details. Note that the original problem and construction were not described in terms of visibility graphs, so we have translated them into our terminology.

The construction uses real points on an elliptic curve. For our purposes a *real elliptic curve* $\mathcal{C}$ is a curve in the real projective plane (which we model as the Euclidean plane with an extra 'line at infinity') defined by an equation of the form $y^2 = x^3 + \alpha x + \beta$. The constants α and β are chosen so that the discriminant $\Delta = -16(4\alpha^3 + 27\beta^2)$ is non-zero, which ensures that the curve is non-singular. We define a group operation '+' on the points of $\mathcal{C}$ by declaring that $a+b+c=0$ if the line through a and b also intersects $\mathcal{C}$ at c, that is, if a, b and c are collinear. The identity element 0 corresponds to the point at infinity in the $\pm y$-direction, so that for instance $a+b+0=0$ if the line through a and b is parallel to the y-axis. Furthermore, $a+a+b=0$ if the tangent line at a also intersects $\mathcal{C}$ at b. It can be shown that this operation defines an abelian group structure on the points of $\mathcal{C}$ (see a standard text such as [83]).

We will use two facts about real elliptic curves and the group structure on them. Firstly, no line intersects an elliptic curve in more than three points. Secondly, the group acts continuously: adding a point e which is close to 0 (i.e. very far out towards infinity) to another point a results in a point close to a (in terms of distance along $\mathcal{C}$).

Proposition 5.19. (Alperin, Buhler, Chalcraft and Rosenberg) *For infinitely many integers δ, there is a visibility graph with at most three vertices collinear, minimum degree δ, and connectivity $\frac{2\delta+1}{3}$.*

Proof. Begin by choosing three non-zero collinear points a, b and c on a real elliptic curve $\mathcal{C}$, such that c lies between a and b. Then choose a point e very close to 0. Now define

$$\begin{aligned} A :=& \{a + ie : 0 \leqslant i \leqslant m-1\} \\ B :=& \{b + je : 0 \leqslant j \leqslant m-1\} \\ C :=& \{-(a+b+ke) : 0 \leqslant k \leqslant 2m-2\}. \end{aligned}$$

Let G be the visibility graph of $A \cup B \cup C$. Since the points are all on $\mathcal{C}$, G has at most three vertices collinear. Observe that the points $a+ie$ and $b+je$ are collinear with the point $-(a+b+(i+j)e)$. Since e was chosen to be very close to 0, by continuity the set A is contained in a small neighbourhood of a, and similarly for B and C. Therefore, the point from C is the middle point in each collinear triple, and so C is a vertex cut in G, separating A and B.

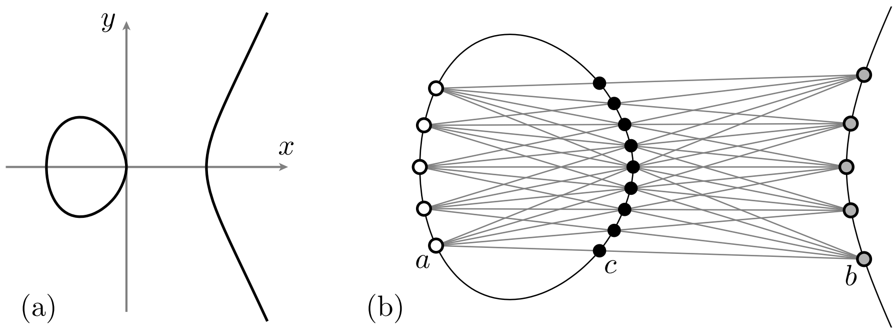


Figure 5.8: (a) The elliptic curve $y^2 = x^3 - x$. (b) The black points separate the white points from the grey points.

By choosing a, b and c away from any points of inflection, we can guarantee that there are no further collinear triples among the sets A, B or C. Thus a point in A sees all other points in $A \cup C$, a point in B sees all other points in $B \cup C$, and a point in C sees all other points. Therefore the minimum degree of G is $\delta = 3m - 2$, attained by the vertices in $A \cup B$. Hence (also using Proposition 5.16) the connectivity of G is $|C| = 2m - 1 = \frac{2\delta+1}{3}$. $\square$

In Figure 5.8 we have chosen $\mathcal{C}$ to be the curve $y^2 = x^3 - x$ and the points a, b and c on the x-axis. We have taken advantage of the symmetry about the x-axis to choose $A = \{a \pm ie\}$ (and similarly for B and C), which is slightly different to the construction outlined in Proposition 5.19.

We close our discussion of the connectivity of visibility graphs with the following conjecture.

Conjecture 5.20. *Every visibility graph with minimum degree δ has connectivity at least $\frac{2\delta+1}{3}$.*

Chapter 6

Bivisibility graphs

Bivisibility graphs turned out to be a useful tool in the study of visibility graphs in Chapter 5. In this chapter we collect further results about bivisibility graphs. Recall that, given disjoint points sets A and B in the plane, the *bivisibility graph* $\mathcal{B}(A, B)$ of A and B has vertex set $A \cup B$, where a point $v \in A$ and a point $w \in B$ are adjacent if and only if they are visible with respect to $A \cup B$.

6.1 Connectedness of bivisibility graphs

Visibility graphs are always connected, but bivisibility graphs may have isolated vertices. However, we now prove that non-collinear bivisibility graphs have at most one component that is not an isolated vertex.

Lemma 6.1. *Let A and B be disjoint point sets such that $A \cup B$ is not collinear. Let T be a (closed geometric) triangle with vertices $a \in A$, $b \in B$ and $c \in A \cup B$. Then a or b has a neighbour in $\mathcal{B}(A, B)$ lying in $T \setminus ab$.*

Proof. There is at least one point of $A \cup B$ in T not lying on the line ab (namely, c). The one closest to ab sees both a and b, and is therefore adjacent to one of them. $\square$

Theorem 6.2. *Let A and B be disjoint point sets such that $A \cup B$ is not collinear. Then $\mathcal{B}(A, B)$ has at most one component that is not an isolated vertex.*

Proof. Assume for the sake of contradiction that $\mathcal{B}(A, B)$ has two components both with one or more edges. Choose a pair of edges ab and $a'b'$, one from each component, such that the area of $C := \text{conv}(a, b, a', b')$ is minimal. If ab and $a'b'$ lie on one line, then they are joined by a path through the closest point to that line, a contradiction. If they do not lie on a line, then both ends of at least one of the edges are corners of C. Assume this edge is ab and let v be another corner of C (v is either a' or b'). Then by Lemma 6.1, a or b has a neighbour w in $\triangle abv \setminus ab$. Without loss of generality, w is a neighbour of a. If $w = v$, then ab and $a'b'$ are in the same component, a contradiction. If $w \neq v$, then the convex hull of $\{a', b', a, w\}$ is contained in C but does not contain b. This contradicts the assumption that C had minimal area. $\square$

Corollary 6.3. *A non-collinear bivisibility graph is connected if and only if it has no isolated vertices.*

6.2 Number of edges and complete bipartite subgraphs

In Chapter 5 we saw that non-collinear bivisibility graphs on n red and n blue points contain non-crossing subgraphs with at least $n+1$ edges (Theorem 5.10). A *bichromatic line* is a line containing at least one point of each colour. Since each bichromatic line contains at least one edge of the bivisibility graph, lower bounds on the number of bichromatic lines give lower bounds on the number of edges.

Theran [95] proved the following extension of Beck's Theorem (3.9) to bichromatic point sets.

Theorem 6.4 (Theran)**.** *Let P be a set of n red and n blue points in the* *plane with at most ℓ collinear. Then P determines at least $cn(2n-\ell)$ bichromatic lines, for some constant $c > 0$.*

However, this theorem is also an immediate corollary of earlier results of Pach and Pinchasi [65].

Theorem 6.5 (Pach and Pinchasi)**.** *Let P be a set of n red and n blue points, not all collinear. Then (1) P determines at least $n/2$ bichromatic lines with at most two red and at most two blue points, and (2) the number of bichromatic lines with at most six points is at least $1/10$ times the total number of lines determined by P.*

Part (2) implies, using our constant of $1/93$ for Beck's Theorem (3.15), that $c \geqslant 1/465$ in Theorem 6.4. Pach and Pinchasi [65] also proved the following result for unequal sized colour classes.

Theorem 6.6 (Pach and Pinchasi)**.** *Let P be a set of n red and cn blue points, not all collinear, and with $c \geqslant 1$. Then the number of bichromatic lines determined by P with at most $8c$ points is at least $1/25c^2$ times the total number of lines determined by P.*

This can again be combined with Theorem 3.15.

Corollary 6.7. *Let P be a set of n red and cn blue points, with at most ℓ collinear, and with $c \geqslant 1$. Then the number of bichromatic lines determined by P is at least $\frac{1+c}{2325c^2}n((1+c)n-\ell)$.*

Purdy and Smith [80] extended the work of Pach and Pinchasi. A bichromatic line is called *equichromatic* if the difference between the number of red points and the number of blue points is at most 1.

Theorem 6.8 (Purdy and Smith)**.** *Let P be a set of n red points and $n-k$ blue points, not all collinear, and let L be the total number of lines determined by P. Then the number of equichromatic lines determined by P is at least $\frac{1}{4}(L+2n+3-k(k+1))$.*

Corollary 6.9. *Let P be a set of n red and $n-k$ blue points in the plane with at most ℓ collinear. Then P determines at least*

$$\frac{1}{4}\left(\frac{2n-k}{93}(2n-k-\ell)+2n+3-k(k+1)\right)$$

bichromatic lines.

When $k=0$ this gives the best constant for Theorem 6.4.

Corollary 6.10. *Let P be a set of n red and n blue points in the plane with at most ℓ collinear. Then P determines at least $\frac{1}{186}n(2n-\ell)$ bichromatic lines.*

As well as giving a lower bound on the number of edges in a bivisibility graph, this implies the following general lower bound on the maximum degree.

Corollary 6.11. *Let A be a set of n red points and B a set of n blue points in the plane, such that $A \cup B$ is not collinear. Then the bivisibility graph $\mathcal{B}(A,B)$ has maximum degree at least $n/94$.*

Proof. Let ℓ be the size of the largest line. Then by Corollary 6.10 the maximum degree is at least $\frac{1}{186}(2n-\ell)$. Suppose at least half the points on the largest line are blue. Then the closest red point to the line has degree at least $\ell/2$. The minimum of these two functions is $n/94$. $\square$

Another consequence of Corollary 6.10 (and Theorem 6.4) is a version of the Big-Line-Big-Clique Conjecture (2.7) for bivisibility graphs. As mentioned in Chapter 2, standard extremal graph theory results are insufficient to prove the Big-Line-Big-Clique Conjecture by themselves. On the other hand, in the case of bipartite graphs, there are useful known results. The problem of determining the maximum number of edges in a bipartite graph that does not contain a given complete bipartite subgraph is known as the Zarankiewicz Problem [102]. The following upper bound was given by Kővári, Sós and Turán [53].

Theorem 6.12 (Kővári, Sós and Turán)**.** *Fix an integer t, and let G be a bipartite graph with n vertices in each part. If G has no $K_{t,t}$ subgraph then the number of edges in G is $O(n^{2-1/t})$.*

Corollary 6.13. *For all integers $t, \ell \geqslant 2$, there exists an integer N such that, if $n \geqslant N$, then every bivisibility graph on n red and n blue points contains a $K_{t,t}$ subgraph or ℓ collinear points.*

Proof. Let G be a bivisibility graph on n red and n blue points with no ℓ collinear. Suppose G has no $K_{t,t}$ subgraph. Then by Theorem 6.12, G has $o(n^2)$ edges, but by Corollary 6.10 the number of edges in G is $\Omega(n^2)$, which is a contradiction for sufficiently large n. □

Of course, since bivisibility graphs are subgraphs of visibility graphs we also have the following.

Corollary 6.14. *For all integers $t, \ell \geqslant 2$, there exists an integer n such that every visibility graph on n or more points contains a $K_{t,t}$ subgraph or ℓ collinear points.*

6.3 Kleitman–Pinchasi Conjecture

We may also ask for linear lower bounds on the number of bichromatic lines that do not depend on the maximum number of collinear points ℓ. In 2003, Kleitman and Pinchasi [50] studied this question under the assumption that neither colour class is collinear. They made the following conjecture.

Conjecture 6.15 (Kleitman–Pinchasi Conjecture)**.** *Let P be a set of n red, and n or $n-1$ blue points in the plane. If neither colour class is collinear, then P determines at least $|P|-1$ bichromatic lines.*

If true, this conjecture would be tight for the arrangement of $n-1$ red and $n-1$ blue points on a line, along with one red and one blue point off the line, and collinear with some point on the line. As discussed in Chapter 2, one motivation for this conjecture is that it would imply Theorem 2.9 of de Bruijn and Erdős [15], which states that every non-collinear set of n points in the plane determines at least n lines.

Kleitman and Pinchasi [50] came very close to proving Conjecture 6.15, establishing the following theorem.

Theorem 6.16 (Kleitman and Pinchasi)**.** *Let P be a set of n red, and n or $n-1$ blue points in the plane. If neither colour class is collinear, then P determines at least $|P|-3$ bichromatic lines.*

Purdy and Smith [80] proved Conjecture 6.15 for $n \geqslant 79$ using their Theorem 6.8 and a result of Kelly and Moser [49]. In this section we improve on the methods of Kleitman and Pinchasi and show, firstly, that Conjecture 6.15 is true for $n \geqslant 10$, and secondly, that for all n the number of bichromatic lines is at least $|P|-2$.

Kleitman and Pinchasi use proof by induction. They establish an inductive step that works for $n \geqslant 20$ for both Theorem 6.16 and Conjecture 6.15. To establish the inductive base case for $n \leqslant 19$ and finish the proof, they apply computer based linear programming methods along with the following lemma [50, Claim 2.1].

Lemma 6.17 (Kleitman and Pinchasi)**.** *Let P be a set of n red, and n or $n-1$ blue points in the plane. If P determines a line with n points or more, then there are at least $|P|-1$ bichromatic lines.*

Let $s_{i,j}$ be the number of lines determined by P with exactly i red points and j blue points. The linear program aims to minimise the number of bichromatic lines under the following constraints. For simplicity, they are stated only for the case of n red and n blue points. The case of $n-1$ blue points is very similar.

- $\sum \binom{i}{2} s_{i,j} = \binom{n}{2}$ (Counting red pairs)
- $\sum \binom{j}{2} s_{i,j} = \binom{n}{2}$ (Counting blue pairs)

- $\sum ijs_{i,j} = n^2$ (Counting bichromatic pairs)
- $\sum(i+j-3)s_{i,j} \leqslant -3$ (Melchior's Inequality (3.10))
- If $i+j \geqslant n$ then $s_{i,j} = 0$ (Lemma 6.17)
- $s_{i,j} \in \mathbb{N}_0$.

We improve the linear program in several ways. First, we add Hirzebruch's Inequality (3.5) to the list of constraints. Recall that Hirzebruch's Inequality holds as long as at most $|P|-3$ points are collinear. Lemma 6.17 ensures this is so. Second, we improve Lemma 6.17 to make more coefficients zero. Third, we introduce further constraints that are tight for general position colour classes.

Lemma 6.18. *Suppose P is a set of n red and n (or $n-1$) blue points, and suppose there is a line L with r red and b blue points. Let $b' = \min\{n-r, b\}$ and $r' = \min\{n-b, r\}$ (or $r' = \min\{n-1-b, r\}$). Then the number of bichromatic lines is at least*

$$\sum_{i=0}^{b'-1} b-i + \sum_{i=0}^{r'-1} r-i = (bb' - b'^2/2 + b'/2) + (rr' - r'^2/2 + r'/2) \ .$$

Moreover, if $b+r < n$, then $b' = b$, $r' = r$ and the number of bichromatic lines is at least $(b^2+b+r^2+r)/2$.

Proof. The bichromatic lines with a red point on L are distinct from those with a blue point. To count those with a red point, take any r' blue points not on L. Order these blue points p_1, p_2, p_3, etc.. There are b lines from p_1 to the blue points on L. For p_2 there are also b such lines, but p_1 may lie on one of them (but not more). So there are $b-1$ lines that were not yet counted. Similarly, for p_3 there are at least $b-2$ lines that are not counted previously, and for p_i there are $b-i+1$. □

Observation 6.19. *Suppose there are n blue points. Each red point can lie on at most $\lfloor n/2 \rfloor$ lines determined by two or more blue points.*

Thus we have the following further constraints for the linear program (for the case of n red and n blue points):

Table 6.1: The minimum number of bichromatic lines, minus $(|P|-1)$.

n	Even	Odd	n	Even	Odd
4	1		12	3	3
5	0	-1	13	5	4
6	-2	-1	14	7	6
7	0	-1	15	9	8
8	0	-1	16	11	10
9	0	-1	17	13	12
10	1	0	18	16	14
11	2	1	19	19	17

- $s_{1,1}+s_{0,2}+s_{2,0}+\frac{3}{4}(s_{0,3}+s_{1,2}+s_{2,1}+s_{3,0}) \geqslant 2n+\sum_{i+j\geqslant 5}(2i+2j-9)s_{i,j}$ (Hirzebruch's Inequality (3.5))
- If $i^2+i+j^2+j \geqslant 4n-2$ then $s_{i,j}=0$ (Lemma 6.18)
- $\sum_{j\geqslant 2} i s_{i,j} \leqslant n\lfloor n/2 \rfloor$ (Observation 6.19)
- $\sum_{i\geqslant 2} j s_{i,j} \leqslant n\lfloor n/2 \rfloor$ (Observation 6.19).

Table 6.1 shows the difference between the minimum number of bichromatic lines as given by this linear program[1], and the target bound of $|P|-1$. Results are given for both the case of n red and n blue (even) and the case of n red and $n-1$ blue (odd). A non-negative value in Table 6.1 indicates that Conjecture 6.15 is true for that case, so in particular it is true for each case with $n \geqslant 10$. This can be combined with the inductive step of Kleitman and Pinchasi.

Theorem 6.20. *Let P be a set of n red, and n or $n-1$ blue points in the plane, where $n \geqslant 10$. If neither colour class is collinear, then P determines at least $|P|-1$ bichromatic lines.*

The only case in Table 6.1 in which the number of bichromatic lines may be $|P|-3$ is that of six red and six blue points. In this case the linear

[1] The input files for the linear programming software, as well as a program used to generate the files are available from the author's web page www.ms.unimelb.edu.au/~mspayne/.

program has a solution with $s_{2,2} = 9$, $s_{0,2} = 6$ and $s_{2,0} = 6$, giving just 9 bichromatic lines. We will show that this is not geometrically realisable. We will work in the projective plane and make use of the following well known fact. It is simply the statement that one projective basis can be transformed to another.

Proposition 6.21. *Let V and W be real projective planes. Given $v_1, \ldots, v_4 \in V$ in general position and $w_1, \ldots, w_4 \in W$ in general position, there exists a unique collineation (a bijection that preserves collinearities) from V to W that maps each v_i to w_i.*

Proposition 6.22. *It is not possible to arrange six red points and six blue points in the plane so that $s_{2,2} = 9$.*

Proof. Suppose for contradiction that there are nine lines with two red and two blue points. This gives 36 bichromatic pairs, so there can be no more bichromatic lines. This implies that every point is on three such lines. Label the points $r_1, \ldots, r_6$ and $b_1, \ldots, b_6$. Suppose $\{r_1, r_2, b_5, b_6\}$ lie on a line L. Since r_1 is collinear with two pairs in $\{b_1, b_2, b_3, b_4\}$, this set is in general position. Hence by Proposition 6.21 we may assume that they are the vertices of a square, with coordinates $(-1, 1), (1, 1), (-1, -1)$ and $(1, -1)$ respectively, as shown in Figure 6.1. Since r_1 and r_2 are also each collinear with two other blue pairs, we may also assume, without loss of generality, that $r_1 = \overline{b_1 b_2} \cap \overline{b_3 b_4} = (\infty, 0)$[2] and $r_2 = \overline{b_1 b_4} \cap \overline{b_2 b_3} = (0, 0)$.

There is another red point on the line $\overline{b_1 b_2}$ (with equation $y = 1$), say r_3, and a further red point on $\overline{b_3 b_4}$ (with equation $y = -1$), say r_4. The position of either r_3 or r_4 determines the set $\{b_5, b_6\}$. That is, $\{b_5, b_6\} = \{L \cap \overline{r_3 b_3}, L \cap \overline{r_3 b_4}\} = \{L \cap \overline{r_4 b_1}, L \cap \overline{r_4 b_2}\}$. Since the configuration described thus far is symmetric about the line $y = 0$, it follows that if $r_3 = (a, 1)$ for some real number a, then $r_4 = (a, -1)$.

At this stage there are six bichromatic lines with only one red point: $\overline{b_1 b_4}$, $\overline{b_4 b_6}, \overline{b_6 b_2}, \overline{b_2 b_3}, \overline{b_3 b_5}$ and $\overline{b_5 b_1}$. There are two red points left to determine, r_5 and r_6, and each must lie on three of these lines. Note that the bichromatic

[2] That is, the point at infinity in the direction of the x-axis.

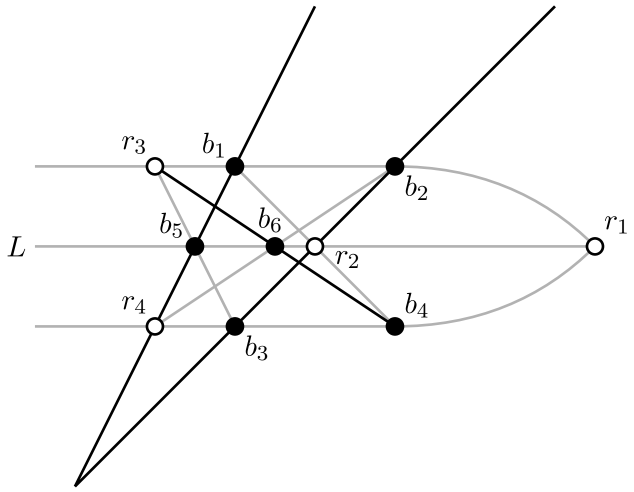


Figure 6.1: Construction for Proposition 6.22.

lines form a cycle on the blue points in the order listed. Neighbours in the cycle share a blue point, so cannot share a red point, and so r_5 and r_6 lie on alternating lines in the cycle. Thus we may assume r_5 lies on $\overline{b_2b_3}, \overline{b_4b_6}$ and $\overline{b_5b_1}$.

Since $\overline{b_2b_3}$ is the line $x = y$, we can say that $r_5 = (c, c)$ for some real number c[3]. Since r_5 lies on $\overline{b_5b_1} = \overline{r_4b_1}$, we have

$$(c, c) = \lambda(a, -1) + (1 - \lambda)(-1, 1)$$

for some parameter λ. Eliminating λ from these two equations yields

$$ac = a - 1 - 3c \ .$$

Similarly, since r_5 lies on $\overline{b_4b_6} = \overline{r_3b_4}$, we have

$$(c, c) = \gamma(a, 1) + (1 - \gamma)(1, -1)$$

for some parameter γ. Eliminating γ from these two equations yields

$$ac = 3c - a - 1 \ .$$

[3]The point r_5 could also be at infinity on $\overline{b_2b_3}$. This case is easily excluded by inspection since both $\overline{b_4b_6}$ and $\overline{b_5b_1}$ would need to be parallel to $\overline{b_2b_3}$. There is no value of a that achieves this.

Equating both expressions for ac yields $a = 3c$, and substituting this into the above equation yields $3c^2 = -1$. This contradiction concludes the proof. $\square$

Proposition 6.22 implies that $s_{2,2} \leqslant 8$ in the case of six red and six blue points. Adding this as an extra constraint in our linear program results in the minimum number of bichromatic lines increasing to 10. Thus we are one step closer to the complete Kleitman–Pinchasi conjecture for all n.

Theorem 6.23. *Let P be a set of n red, and n or $n-1$ blue points in the plane. If neither colour class is collinear, then P determines at least $|P|-2$ bichromatic lines.*

Finally, we note that without the geometric restrictions, the system of bichromatic lines in the proof of Proposition 6.22 can be completed as a combinatorial structure. In addition to the lines already constructed (as shown in Figure 6.1) we may include r_5 in the quadruples $\{r_2, r_5, b_2, b_3\}$, $\{r_3, r_5, b_4, b_6\}$ and $\{r_1, r_5, b_1, b_5\}$ and r_6 in the quadruples $\{r_4, r_6, b_2, b_6\}$, $\{r_3, r_6, b_3, b_5\}$ and $\{r_2, r_6, b_1, b_4\}$. This means that the Kleitman-Pinchasi Conjecture is not true in a combinatorial setting such as that of Theorem 2.11 of Meshulam [61]. It is an interesting question whether this combinatorial construction can be generalised to an infinite family of counterexamples.

Chapter 7

Empty pentagons

This chapter addresses the problem of finding empty pentagons in point sets with bounded collinearities. A subset X of a point set P is an *empty k-gon* if X is a strictly convex k-gon and $P \cap \mathrm{conv}(X) = X$. The status of the general problem of finding empty k-gons was discussed in Chapter 2.4. Recall that Abel et al. [1] showed that every finite set of at least $\mathrm{ES}\left(\frac{(2\ell-1)^{\ell}-1}{2\ell-2}\right)$ points in the plane contains an empty pentagon or ℓ collinear points, where $\mathrm{ES}(n)$ is the Erdős–Szekeres function (see Theorem 2.12). The function $\mathrm{ES}(k)$ is known to grow exponentially [33, 34], so this bound is doubly exponential in ℓ. See [11, 23, 81] for more on point sets with no empty pentagon. We prove the following theorem without applying the Erdős–Szekeres Theorem.

Theorem 7.1. *Let P be a finite set of points in the plane. If P contains at least $328\ell^2$ points, then P contains an empty pentagon or ℓ collinear points.*

This quadratic bound is optimal up to a constant factor since the $(\ell-1) \times (\ell-1)$ square grid has $(\ell-1)^2$ points and contains neither an empty pentagon nor ℓ collinear points. Also note that another way of interpreting Theorem 7.1 is to say that any set of n points with no empty pentagon contains $\Omega(\sqrt{n})$ collinear points.

The point set P is assumed to be finite, and indeed Theorem 7.1 does not hold for infinite sets. A countably infinite point set in general position with no empty pentagons can be constructed recursively from any finite set in general position by repeatedly placing points inside every empty pentagon, avoiding collinearities. On the other hand, Theorem 7.1 easily generalises to *locally finite* point sets, point sets which contain only finitely many points in any bounded region. The result of Abel et al. [1] already implies that an infinite locally finite set with no empty pentagon contains ℓ collinear points for every positive integer ℓ.

The remainder of this section introduces terminology that is used throughout the chapter. The *convex layers* $L_1, \ldots, L_r$ of P are defined recursively as follows: L_i is the subset of P lying in the boundary of the convex hull of $P \setminus \bigcup_{j=1}^{i-1} L_j$, and L_r is the innermost layer, so $P = \bigcup_{i=1}^{r} L_i$ and $L_i \neq \emptyset$ for $i = 1, \ldots, r$. Note that each layer is in weakly convex position.

Points of P will also be referred to as *vertices* and line segments connecting two points of P as *edges*. The *edges of a layer* are the edges between consecutive points in the boundary of the convex hull of that layer. Edges of layers will always be specified in clockwise order. A single letter such as e is often used to denote an edge. For an edge e, let $l(e)$ denote the line containing e. Some edges will be used to determine half-planes. The open half-planes determined by $l(e)$ will be denoted e^+ and e^-, where the $+$ and $-$ sides will be determined later. Similarly, the closed half-planes determined by $l(e)$ will be denoted $e^{\oplus}$ and $e^{\ominus}$.

Gerken [39] introduced the notion of k-sectors. If $p_1p_2p_3p_4$ is a strictly convex quadrilateral (that is, a strictly convex 4-gon), then the 4-*sector* $S(p_1, p_2, p_3, p_4)$ is the set of all points q such that $qp_1p_2p_3p_4$ is a strictly convex pentagon. Note that the order of the arguments is significant. $S(p_1, p_2, p_3, p_4)$ is the intersection of three open half-planes, and may be bounded or unbounded, as shown in Figure 7.1. The closure of a 4-sector will be denoted by square brackets, $S[p_1, p_2, p_3, p_4]$. If P contains no empty pentagon and $p_1p_2p_3p_4$ is an empty quadrilateral in P, then $P \cap S(p_1, p_2, p_3, p_4) = \emptyset$. Otherwise, since P is finite, there exists a point $x \in P \cap S(p_1, p_2, p_3, p_4)$ closest to the line $l(p_1p_4)$, and $xp_1p_2p_3p_4$ is an empty pentagon.

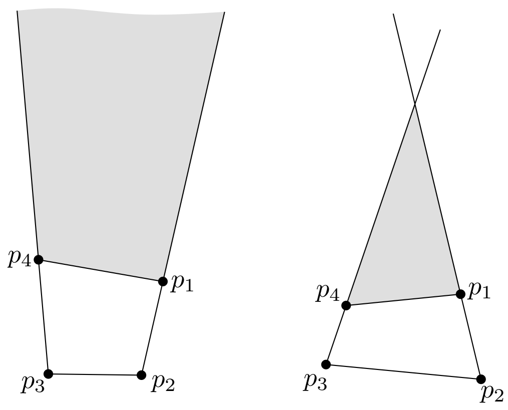


Figure 7.1: The shaded regions represent the 4-sector $S(p_1, p_2, p_3, p_4)$, which may be bounded or unbounded.

7.1 Large subsets in weakly convex position

The first major step in proving Theorem 7.1 is to establish the following theorem concerning point sets with large subsets in weakly convex position.

Theorem 7.2. *If a point set P contains 8ℓ points in weakly convex position, then P contains an empty pentagon or ℓ collinear points.*

This result was also obtained independently by Cibulka and Kynčl (private communication, Pavel Valtr, 2012). Theorem 7.2 immediately implies that every point set with $\mathrm{ES}(8\ell)$ points contains an empty pentagon or ℓ collinear points, which is already a substantial improvement on the result of Abel et al. [1] mentioned above. The rest of this section is dedicated to proving it.

Throughout this section, let P be a set of points in the plane that contains 8ℓ points in weakly convex position but contains no ℓ collinear points. Suppose for the sake of contradiction that P contains no empty pentagon. Let A be an inclusion-minimal weakly convex 8ℓ-gon in P. That is, there is no weakly convex 8ℓ-gon A' such that $\mathrm{conv}(A') \subsetneq \mathrm{conv}(A)$. An empty pentagon in $P \cap \mathrm{conv}(A)$ is an empty pentagon in P, so it can be assumed that $P \subseteq \mathrm{conv}(A)$, so A is the first convex layer of P. Let B be the second convex layer of P. For an edge e of A or B, let e^+ be the open half-plane

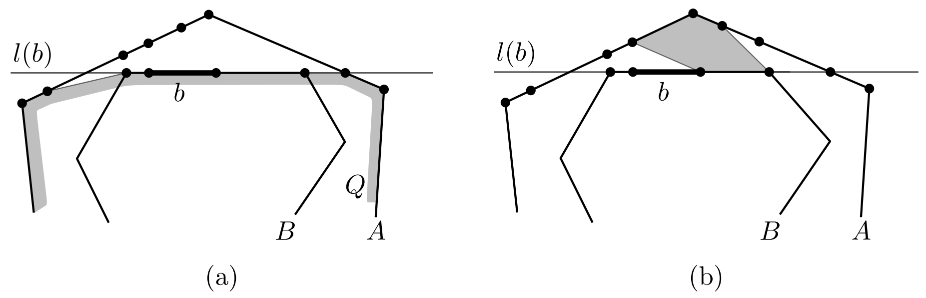


Figure 7.2: (a) If $|A \cap b^+| \leqslant |B \cap l(b)|$, then A is not minimal. (b) If b^+ contained three non-collinear points of A, there would be an empty pentagon.

determined by $l(e)$ that does not contain any point in B.

Observation 7.3. *For each edge b of B, $|A \cap b^+| > |B \cap l(b)|$. Similarly, if $b_1, b_2, \ldots, b_j$ are edges of B, then*

$$\left| A \cap \bigcup_{i=1}^{j} b_i^+ \right| > \left| B \cap \bigcup_{i=1}^{j} l(b_i) \right| .$$

Proof. If $|A \cap b^+| \leqslant |B \cap l(b)|$ then removing the vertices $A \cap b^+$ from A and replacing them by $B \cap l(b)$ gives a weakly convex m-gon Q such that $m \geqslant |A|$ and $\operatorname{conv}(Q) \subsetneq \operatorname{conv}(A)$, contradicting the minimality of A; see Figure 7.2(a). The second claim follows from the minimality of A in a similar way. □

Observation 7.4. *For each edge b of B, the vertices of $A \cap b^+$ are collinear.*

Proof. By Observation 7.3, there are at least 3 points in $A \cap b^+$. If $A \cap b^+$ is not collinear, then there is an empty pentagon; see Figure 7.2(b). □

The following lemma implies that B has at least 4ℓ vertices.

Lemma 7.5. $2|B| \geqslant |A|$.

Proof. Since $|A| \geqslant 8\ell$, A has at least nine corners. Thus $B \neq \emptyset$. If B is collinear then let h be the line containing B. There are at most two corners of A on h, so there are at least four corners of A strictly to one side of h. The interior of the convex hull of these four corners together with any point in B is empty. This implies that there is an empty pentagon in P, a contradiction.

Therefore B has at least three corners, and at least three sides, where a *side* of B is the set of edges between consecutive corners. Let $b_1, \ldots, b_k$ be edges of B, one in each side of B. By Observation 7.4, each of the sets $A \cap b_i^+$ is collinear for $i = 1, \ldots, k$. Thus $|A| \leqslant \Sigma_{i=1}^k |A \cap b_i^+| < k\ell$, and so $k \geqslant 9$. In other words, B has at least nine corners, so there is at least one point $z \in P$ in the interior of $\mathrm{conv}(B)$. Suppose that for some edge xy of A the closed triangle $\Delta[x, y, z]$ contains no point of B. Then there is an edge $x'y'$ of B that crosses this triangle. The 4-sector $S(x', x, y, y')$ contains z, contradicting the fact that P contains no empty pentagon. Thus every such closed triangle contains a point of B. Since each point of B is in at most two such closed triangles, $2|B| \geqslant |A|$. $\square$

The following lemma implies that for a set of points X, the first edge b in B in clockwise order such that $X \subseteq b^+$ is well defined, as long as there is at least one such edge.

Lemma 7.6. *For any set of points $X \neq \emptyset$, let E_X be the set of edges b in B such that $X \subseteq b^+$. Then the edges in E_X are consecutive in B, and not every edge of B is in E_X.*

Proof. If $X \cap \mathrm{conv}(B) \neq \emptyset$ then $E_X = \emptyset$. Take a point $x \in X$, so $x \notin \mathrm{conv}(B)$. Let y be a point in the interior of $\mathrm{conv}(B)$ that is not collinear with any two points of $B \cup \{x\}$. Then $l(xy)$ intersects precisely two edges b and $\tilde{b}$ of B, with $x \in b^+$ and $x \in \tilde{b}^-$. Thus, $X \not\subseteq \tilde{b}^+$, so E_X does not contain every edge of B.

If E_X contains only one edge then the lemma holds, so consider two edges b_1 and b_2 in E_X and suppose they are not consecutive. If $l(b_1) = l(b_2)$, then clearly the edges between b_1 and b_2 on $l(b_1)$ are also in E_X. Now suppose $l(b_1) \neq l(b_2)$. If $l(b_1)$ and $l(b_2)$ are parallel, then $b_1^+ \cap b_2^+ = \emptyset$, a contradiction.

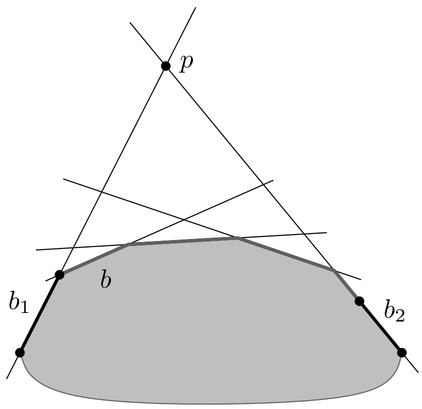


Figure 7.3: Lemma 7.6.

So $l(b_1)$ and $l(b_2)$ cross at a point p. Without loss of generality, p is above B with b_1 on the left and b_2 on the right, as shown in Figure 7.3. Let b be the next edge clockwise from b_1. Then clearly $p \in b^{\oplus}$, so $b_1^+ \cap b_2^+ \subseteq b^+$, and hence $b \in E_X$. Iterating this argument shows that every edge clockwise from b_1 until b_2 is in E_X. It follows that the edges in E_X are consecutive in B. $\square$

Let a be an edge of A such that $|A \cap l(a)| \geqslant 3$. Such an edge exists by Observations 7.3 and 7.4. Let $\{v_1, \ldots, v_k\}$ be $A \cap l(a)$ in clockwise order. Thus $k < \ell$.

Lemma 7.7. *There is an edge b of B such that either $\{v_1, v_2, v_3\} \subseteq b^+$ or $\{v_{k-2}, v_{k-1}, v_k\} \subseteq b^+$.*

Proof. Let b be an edge of B with $v_2 \in b^+$. Such an edge exists, since otherwise $v_2 \in \text{conv}(B)$. Observations 7.3 and 7.4 imply that $|A \cap b^+| \geqslant 3$ and $A \cap b^+$ is collinear. Thus if $v_1 \in b^+$, then $\{v_1, v_2, v_3\} \subseteq b^+$, as required. Otherwise $l(b)$ intersects $l(a)$ between v_1 and v_2, so $\{v_2, v_3, \ldots, v_k\} \subseteq b^+$ and $k \geqslant 4$, because if $k = 3$ then $|A \cap b^+| = 2$. $\square$

By Lemma 7.7, without loss of generality, there is an edge b of B such that $\{v_1, v_2, v_3\} \subseteq b^+$, and by Lemma 7.6 the edges with this property are consecutive in B. Let b_1 be the first one in clockwise order. For an

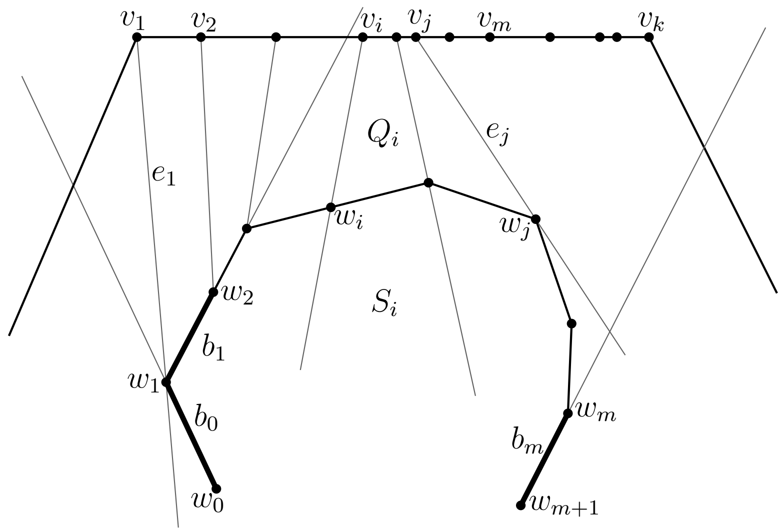


Figure 7.4: Definition of b_1 and the quadrilaterals Q_i.

illustration of the following definitions, see Figure 7.4. First observe that $|A \cap l(a) \cap \tilde{b}^+| \geqslant 3$ cannot hold for every edge $\tilde{b}$ of B, because otherwise $A \cap l(a) = A$ by Observation 7.4, and so $|A| < \ell$. Define the endpoints of b_1 to be w_1 and w_2 in clockwise order. Let $w_3, \ldots, w_{m+1}$ and $b_i := w_i w_{i+1}$ be subsequent vertices and edges of B in clockwise order, where $|A \cap l(a) \cap b_{m-1}^+| \geqslant 3$ but $|A \cap l(a) \cap b_m^+| \leqslant 1$. Then $m \leqslant |B \cap \bigcup_{i=1}^{m-1} l(b_i)| < |A \cap \bigcup_{i=1}^{m-1} b_i^+| \leqslant k$ by Observation 7.3. Now define $e_i := v_i w_i$ for $i = 1, \ldots, m$. Let e_i^- be the open half-plane determined by $l(e_i)$ that contains v_1, or that does not contain v_2 in the case of e_1.

Let j be minimal such that the closed half-plane $e_j^\ominus$ contains B. Clearly $j \neq 1$ since $w_2 \in e_1^+$. The following argument shows that j is well-defined. Call e_i *good* if w_i is the closest point of $l(e_i) \cap \operatorname{conv}(B)$ to v_i. First suppose that e_m is good, so in particular $v_m \in b_{m-1}^+$. Since m was chosen so that $|A \cap l(a) \cap b_{m-1}^+| \geqslant 3$ but $|A \cap l(a) \cap b_m^+| \leqslant 1$, and since $m < k$, it follows that $v_m \in b_m^\ominus$ also. This implies that $B \subseteq e_m^\ominus$, as illustrated in Figure 7.5(a), and so j is well-defined. Now suppose that e_m is not good. By the choice of b_1, both e_1 and e_2 are good, so let p be minimal such that e_p is not good. Thus $3 \leqslant p \leqslant m$. Then w_{p-2} is in e_{p-1}^- because e_{p-1} is good, and w_p is in e_{p-1}^- because e_p is not good, as shown in Figure 7.5(b). This implies that

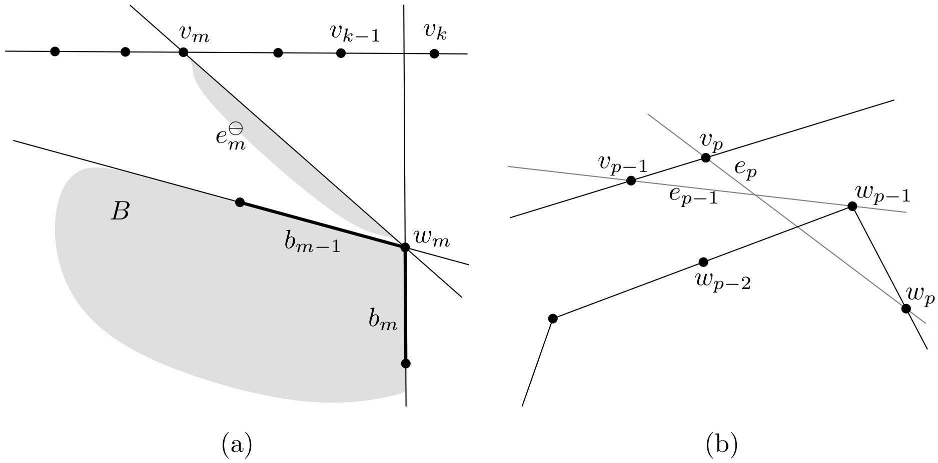


Figure 7.5: (a) If e_m is good then $B \subseteq e_m^\ominus$. (b) If e_{p-1} is good and e_p is not, then $B \subseteq e_{p-1}^\ominus$.

$B \subseteq e_{p-1}^\ominus$, so j is well-defined. Note that this also shows that e_i is good for all $i = 1, \ldots, j$.

Define the quadrilaterals $Q_i := w_i v_i v_{i+1} w_{i+1}$ for $i = 1, \ldots, j-1$. By the following argument, the quadrilaterals Q_i are strictly convex. Suppose on the contrary that Q_h is not strictly convex, and h is minimal. There are two possible order types for Q_h. The first possibility is that $v_h \in b_h^\ominus$ and so $B \subseteq e_h^\ominus$ (since e_h is good), contradicting the minimality of j; see Figure 7.6(a). The second possibility is that $v_{h+1} \in b_h^\ominus$ and so $A \cap \bigcup_{i=1}^{h} b_i^+ = \{v_1, \ldots, v_h\}$, which contradicts Observation 7.3 since $|B \cap \bigcup_{i=1}^{h} l(b_i)| \geqslant h+1$; see Figure 7.6(b).

Let $S_i := S[w_i, v_i, v_{i+1}, w_{i+1}]$ be the closed 4-sector of the quadrilateral Q_i for $i = 1, \ldots, j-1$. Note that $B \cap S_i = B \cap e_i^\oplus \cap e_{i+1}^\ominus$. Take a point $x \in B \cap e_1^\oplus$. Then $x \in e_j^\ominus$ since $B \subseteq e_j^\ominus$. Let h be minimal such that $x \in e_{h+1}^\ominus$. If $h = 0$ then $x \in l(e_1) \cap B \subseteq S_1$. Otherwise $x \notin e_h^\ominus$, so $x \in e_h^\oplus$, and so $x \in S_h$. Hence $B \cap e_1^\oplus \subseteq \bigcup_{i=1}^{j-1} S_i$.

The quadrilaterals Q_i are empty because they lie between the layers A and B. Therefore no S_i contains a point of B in its interior, and so all the points of $B \cap e_1^\oplus$ lie on the lines $l(e_1), \ldots, l(e_j)$. Since B is in weakly convex

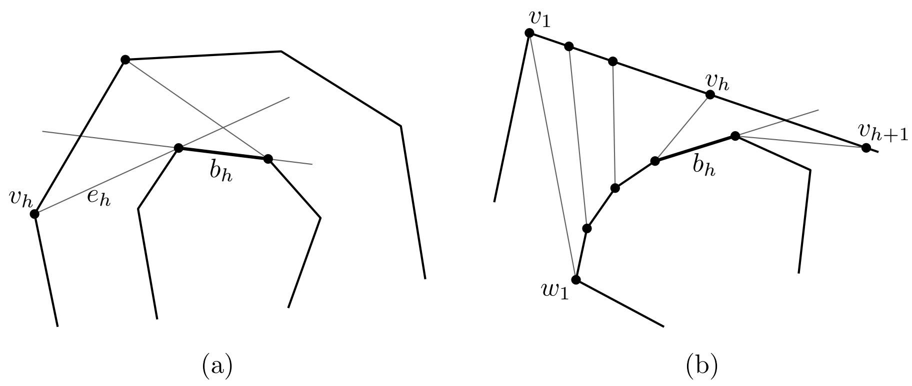


Figure 7.6: (a) If $v_h \in b_h^{\ominus}$ then $B \subseteq e_h^{\ominus}$. (b) If $v_{h+1} \in b_h^{\ominus}$ then $A \cap \bigcup_{i=1}^{h} b_i^+ = \{v_1, \ldots, v_h\}$.

position, $|B \cap l(e_i)| \leqslant 2$ for $i = 2, \ldots, j-1$. There can be at most $\ell - 2$ points of B on $l(e_1)$ and $l(e_j)$. In fact there are less points of B on $l(e_1)$ and $l(e_j)$, as the following argument shows. Note that w_m is a corner of B since $A \cap b_{m-1}^+ \neq A \cap b_m^+$. Therefore $B \cap l(e_j) \subseteq \{w_j, \ldots, w_m\}$, and so $|B \cap l(e_j)| \leqslant m - j + 1$. Since $j, m < \ell$, adding up the bounds for each $l(e_i)$ yields $|B \cap e_1^{\oplus}| \leqslant (\ell - 2) + 2(j-2) + (m - j + 1) < 3\ell$. Since $|B| \geqslant 4\ell$ by Lemma 7.5, this implies that $B \not\subseteq e_1^{\oplus}$, which implies that $|B \cap l(e_1)| \leqslant 2$. Hence $|B \cap e_1^{\oplus}| \leqslant 2(j-1) + (m - j + 1) < 2\ell$.

It remains to bound the size of the rest of B, that is, $|B \cap e_1^-|$. Define $v_0, v_{-1}, v_{-2}, \ldots$ and $w_0, w_{-1}, w_{-2}, \ldots$ to be the vertices of A and B proceeding anticlockwise from v_1 and w_1 respectively. Define $b_0 := w_0 w_1$. Since $B \not\subseteq e_1^{\oplus}$, it follows that $v_1 \in b_0^+$, as shown in Figure 7.7. Since b_1 is the first edge in clockwise order with $\{v_1, v_2, v_3\} \subseteq b_1^+$, neither v_2 nor v_3 is in b_0^+. Hence by Observation 7.3, $\{v_1, v_0, v_{-1}\} \subseteq b_0^+$. Also, by Observation 7.4, neither v_0 nor v_{-1} is in b_1^+, so b_0 is the first edge of B with $\{v_1, v_0, v_{-1}\} \subseteq b_0^+$ in anticlockwise order (recall that edges with this property are consecutive in B by Lemma 7.7). Therefore, the argument that started at b_1 and proceeded clockwise may be started at b_0 and proceed anticlockwise instead. In this situation, the edge e_1 will remain the same as before because the starting points v_1 and w_1 are unchanged. Thus the argument will cover $B \cap e_1^{\ominus}$ with 4-sectors and, analogously to before, show that $|B \cap e_1^{\ominus}| < 2\ell$. This implies

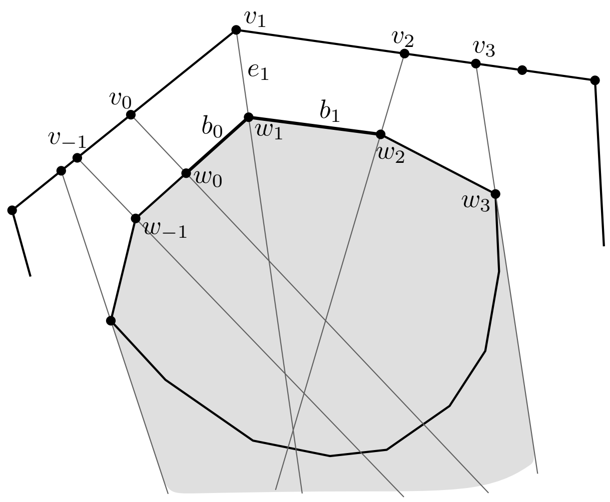


Figure 7.7: The convex hull of B is covered by the union of the closed sectors S_i.

that $|B| \leqslant |B \cap e_1^{\ominus}| + |B \cap e_1^{\oplus}| < 4\ell$, which contradicts the fact that $|B| \geqslant 4\ell$. This completes the proof of Theorem 7.2. $\square$

7.2 The empty edge lemma

The next lemma we need appears implicitly in the paper of Abel et al. [1]. The proof, which we include for completeness, is adapted directly from that paper, and the figures are reproduced with the kind permission of the authors. Abel et al. introduced the following definition. Fix a point z in the innermost layer of P. An edge xy in layer L_i of P is *empty* if the open triangle $\Delta(x, y, z)$ contains no points of L_{i+1}.

Lemma 7.8. *Let $L_1, \ldots, L_r$ be the convex layers of a point set P. If L_k contains an empty edge for some $k \in \{1, \ldots, r - \ell + 1\}$, then P contains an empty pentagon or ℓ collinear points.*

Proof. Suppose for contradiction that P contains no empty pentagon and no ℓ collinear points. Let z be a point in the innermost layer L_r of P. For now suppose xy is an empty edge of L_i for any $i \in \{1, \ldots, r - 2\}$. In this case,

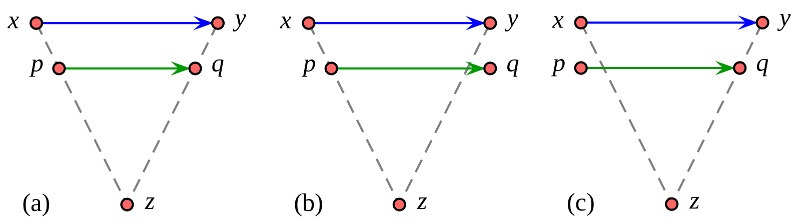


Figure 7.8: (a) Double-aligned. (b) Left-aligned. (c) Right-aligned.

the intersection of the boundary of $\text{conv}(L_{i+1})$ and $\Delta(x, y, z)$ is contained in an edge pq of L_{i+1}. Call pq the *follower* of xy. First some properties of followers are established.

Claim 1. If pq is the follower of xy, then $pxyq$ is an empty quadrilateral and pq is empty.

Proof. Let $Q := pxyq$. Since p and q are in the interior of $\text{conv}(L_i)$, both x and y are corners of Q. Both p and q are corners of Q, otherwise xy would not be empty. Thus Q is in strictly convex position. Q is empty by the definition of L_{i+1}.

Suppose that pq is not empty; that is, $\Delta(p, q, z) \cap L_{i+2} \neq \emptyset$. Then the 4-sector $S(p, x, y, q) \neq \emptyset$, so P contains an empty pentagon. This contradiction proves that pq is empty. □

As illustrated in Figure 7.8(a)–(c), the follower pq of xy is said to be:

- *double-aligned* if $p \in l(xz)$ and $q \in l(yz)$,
- *left-aligned* if $p \in l(xz)$ and $q \notin l(yz)$,
- *right-aligned* if $p \notin l(xz)$ and $q \in l(yz)$.

Claim 2. If pq is the follower of xy, then pq is either double-aligned or left-aligned or right-aligned.

Proof. Suppose that pq is neither double-aligned nor left-aligned nor right-aligned, as illustrated in Figure 7.9(a). By Claim 1, $pxyq$ is an empty quadri-

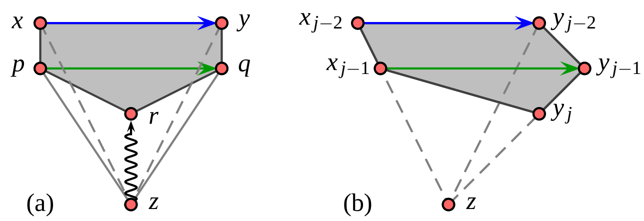


Figure 7.9: (a) Neither double-aligned nor left-aligned nor right-aligned. (b) The empty pentagon $x_{j-2}y_{j-2}y_{j-1}y_jx_{j-1}$.

lateral. But the 4-sector $S(p, x, y, q)$ contains the point z, so P contains an empty pentagon. □

Returning to the proof of Lemma 7.8, suppose x_1y_1 is an empty edge in L_k for some $k \in \{1, \dots, r - \ell + 1\}$. For $i = 2, 3, \dots, \ell - 1$, let x_iy_i be the follower of $x_{i-1}y_{i-1}$. By Claim 1 (at each iteration), x_iy_i is empty. For some $i \in \{2, \dots, \ell - 2\}$, the edge x_iy_i is not double-aligned, as otherwise $\{x_1, x_2, \dots, x_{\ell-2}, z\}$ are collinear and $\{y_1, y_2, \dots, y_{\ell-2}, z\}$ are collinear, which implies that $\{x_1, x_2, \dots, x_{\ell-1}, z\}$ are collinear or $\{y_1, y_2, \dots, y_{\ell-1}, z\}$ are collinear by Claim 2. Let i be the minimum integer in $\{2, \dots, \ell - 2\}$ such that x_iy_i is not double-aligned. Without loss of generality, x_iy_i is left-aligned. On the other hand, x_jy_j cannot be left-aligned for all $j \in \{i + 1, \dots, \ell - 1\}$, as otherwise $\{x_1, x_2, \dots, x_{\ell-1}, z\}$ are collinear. Let j be the minimum integer in $\{i + 1, \dots, \ell - 1\}$ such that x_jy_j is not left-aligned. Thus $x_{j-1}y_{j-1}$ is left-aligned and x_jy_j is not left-aligned. It follows that $x_{j-2}y_{j-2}y_{j-1}y_jx_{j-1}$ is an empty pentagon, as illustrated in Figure 7.9(b). This contradiction completes the proof. □

7.3 Proof of Theorem 7.1

Let P be a set of at least $328\ell^2$ points with no ℓ collinear points, and suppose for the sake of contradiction that P does not contain an empty pentagon. Let $L_1, \dots, L_r$ be the convex layers of P, with L_1 the outermost and L_r the innermost layer. Theorem 7.2 implies that $|L_i| < 8\ell$ for every i. The layers are divided into three groups as follows. The layers $L_{r-\ell+1}$ to L_r are the *inner layers*. Hence $|L_{r-\ell+1} \cup \cdots \cup L_r| < 8\ell^2$. The layers L_1 to L_a are the *outer layers*, where a is the minimum integer such that $|L_1 \cup \cdots \cup L_a| \geqslant 64\ell(\ell-1)$. This means that $|L_1 \cup \cdots \cup L_a| \leqslant 64\ell(\ell-1) + 8\ell < 64\ell^2$. The remaining layers L_{a+1} to $L_{r-\ell}$ are the *middle layers*.

The strategy of the proof is to analyse the structure of the middle layers and show that if there are too many middle layers, then the outer layers contain less points than the lower bound in the previous paragraph. This contradiction implies that there are not too many middle layers. Since the size of each layer is limited by Theorem 7.2, this yields an upper bound on the number of points in the middle layers. Adding this upper bound to those just established for the inner and outer layers will give a contradiction to the assumed size of P, completing the proof.

For now, consider only the points in the middle layers L_{a+1} to $L_{r-\ell}$. For each point v in a middle layer L_i, define the left and right child of v as follows (see Figure 7.10(a)). Let x be the closest point to v in $\operatorname{conv}(L_{i+1}) \cap vz$ (where vz is the line segment from v to z). The *right child* of v is the point in L_{i+1} immediately clockwise from x. The *left child* of v is the point in L_{i+1} immediately anticlockwise from x. Note that although x may be in P, x is neither the left nor the right child of v.

A *right chain* is a sequence $v_1, \dots, v_t$ of points in $L_{a+1} \cup \cdots \cup L_{r-\ell}$ such that v_{i+1} is the right child of v_i. A *left chain* is defined in a similar fashion. A *subchain* is a chain contained in a larger chain, and a *maximal* chain is one that is not a proper subchain of another chain. A point cannot be the right child of two points u and v in L_i, otherwise the edge uv (or the edges in the segment uv if u and v are not adjacent) would be empty, contradicting Lemma 7.8. Similarly, a point cannot be the left child of two points. This implies that maximal right chains do not intersect one another, and similarly for maximal left chains. Furthermore, by construction each point in the middle layers has a left and a right child, so every maximal chain contains a point in $L_{r-\ell}$. Together these observations imply the following lemma.

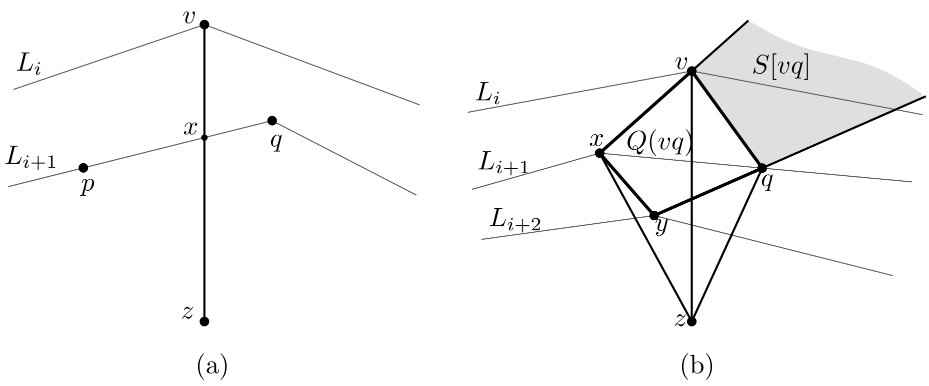


Figure 7.10: (a) The right child q and the left child p of v. (b) The quadrilateral $Q(vq)$ and the sector $S[vq]$.

Lemma 7.9. *Every point in the middle layers is in precisely one maximal right chain and one maximal left chain. The number of maximal right chains is $|L_{r-\ell}| \leqslant 8\ell - 1$, and similarly for maximal left chains.* □

The edges of a chain are the edges between consecutive vertices of the chain. A chain V is said to *wrap around* if every ray starting at z intersects the union of the edges of V at least twice. Since chains advance in the same direction around z with every step, this is equivalent to saying that V covers a total angle of at least 4π around z.

Lemma 7.10. *If the number of middle layers $r - \ell - a$ is at least 32ℓ, then there is a chain with at most 32ℓ vertices that wraps around.*

Proof. Let $V = (v_1, \ldots, v_t)$ be a right chain that starts at a point $v_1 \in L_{a+1}$. Since $r - \ell - a \geqslant 32\ell$, it can be assumed that $t = 32\ell$. By Lemma 7.9, each vertex v_i lies in some left chain, and there are at most $8\ell - 1$ maximal left chains, so some left chain intersects V at least five times. Let U be a left chain that intersects V in the points $p_1, \ldots, p_5$, where p_1 and p_5 are the first and last points of U respectively.

Recall that right chains advance clockwise around z with every step, and left chains anticlockwise. Therefore, the paths from p_i to p_{i+1} in U and V form a closed curve around z. So these paths cover an angle of 2π around z.

Hence U and V together cover a total angle of at least 8π around z. This implies that at least one of them covers a total angle of at least 4π, and thus wraps around. Both U and V have at most t vertices because they lie in the layers L_{a+1} to L_{a+t}. $\square$

If q is the right child of a vertex v in a middle layer L_i, then associate with vq the following quadrilateral, as illustrated in Figure 7.10(b). Let x be the point in L_{i+1} anticlockwise from q, so x either lies on vz or is the left child of v. Let y be a point in the open triangle $\Delta(x, q, z)$ closest to xq. Such a y exists in L_{i+2}, otherwise xq would be an empty edge. Then $Q(vq) := vxyq$ is the quadrilateral associated with vq. This quadrilateral is strictly convex by construction. The triangle $\Delta[x, q, y]$ is empty since x and q are neighbours in L_{i+1} and y is a closest point to xq. The triangle $\Delta[v, q, x]$ is empty because it can contain neither a point of L_i nor L_{i+1}. Thus $Q(vq)$ is an empty quadrilateral. Empty quadrilaterals determine 4-sectors that must be empty since there are no empty pentagons. Let $S[vq]$ be the closed 4-sector determined by $Q(vq)$, that is, $S[v, x, y, q]$ in the notation established previously.

Let $V = (v_1, \ldots, v_t)$ be a chain and let $e_i := v_iv_{i+1}$ be the edges of V. Let $e_i^{\oplus}$ be the closed half-plane defined by e_i that does not contain z. Consider a quadrilateral $Q(e_i) = v_ix_iy_iv_{i+1}$ and let c_i be the edge x_iv_i and let d_i be the opposite edge y_iv_{i+1}. Let $c_i^{\oplus}$ be $c_i^{\oplus}$ the closed half-plane defined by c_i that contains d_i, and let $d_i^{\oplus}$ be the closed half-plane defined by d_i that contains c_i. With these definitions, the 4-sector defined by $Q(e_i)$ is $S[e_i] = c_i^{\oplus} \cap d_i^{\oplus} \cap e_i^{\oplus}$.

Lemma 7.11. *If $V = (v_1, \ldots, v_t)$ wraps around, then the corresponding 4-sectors $S[e_i]$ cover the points of the outer layers L_1 to L_a.*

Proof. Let u be a point in $L_1 \cup \cdots \cup L_a$. Without loss of generality, suppose that V is a right chain, and that the line $l(uz)$ is vertical with u above z. Consider the ray h contained in $l(uz)$ that starts at z and does not contain u. Since V wraps around, it crosses h at least twice. Therefore there are two non-consecutive edges e_j and e_k of V that intersect h (with $j < k$), and there is an edge e_p between e_j and e_k that intersects the line segment zu.

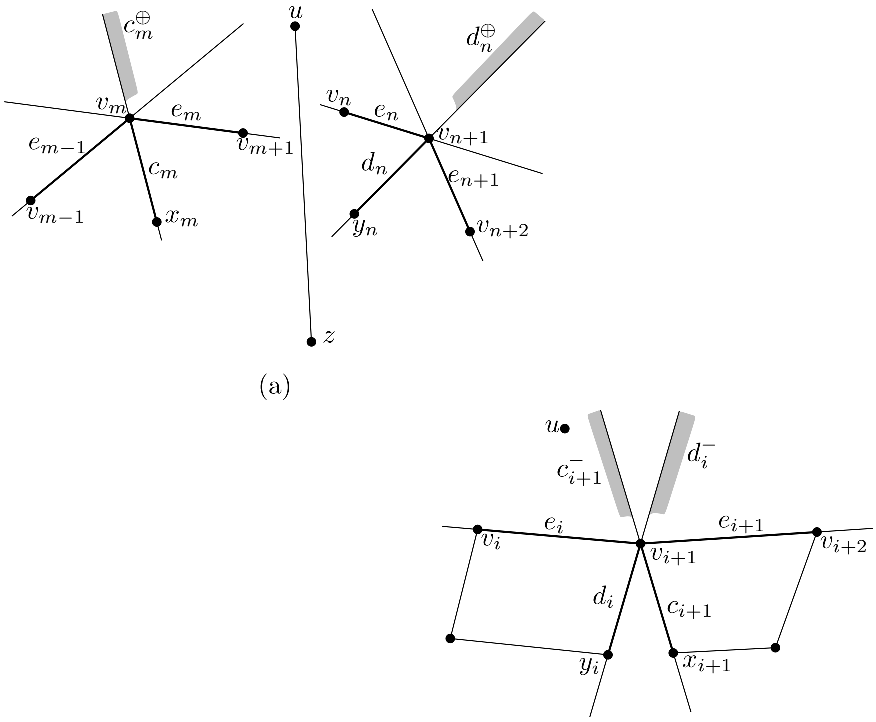


Figure 7.11: (a) $u \in c_m^\oplus$ and $u \in d_n^\oplus$. (b) u cannot be in both d_i^- and c_{i+1}^-.

Note that u lies in e_j^- and e_k^-, but u lies in e_p^+. Let $\tilde{V}$ be the maximal subchain of V that contains e_p and such that $u \in e^+$ for every edge e of $\tilde{V}$. Let e_m and e_n be the first and last edges of $\tilde{V}$. Since e_j and e_k are not in $\tilde{V}$ and $j < m \leqslant n < k$, the edges e_{m-1} and e_{n+1} are not in $\tilde{V}$. Thus $u \in e_{m-1}^\ominus \cap e_m^+$, as shown in Figure 7.11(a). Also, v_m lies to the left of $l(uz)$ since $j < m \leqslant p$. This implies that u and v_{m+1} are on the same side of $l(c_m)$, so $u \in c_m^\oplus$. Furthermore, $u \in e_n^+ \cap e_{n+1}^\ominus$, and v_{n+1} lies to the right of $l(uz)$ since $p \leqslant n < k$, as shown in Figure 7.11(a) also. This implies that $u \in d_n^\oplus$.

Since $u \in e_i^+ \cap e_{i+1}^+$ for $m \leqslant i \leqslant n-1$, the fact that y_i precedes x_{i+1} in L_{i+2} (or $y_i = x_{i+1}$) means that it is not possible for u to be in both d_i^- and c_{i+1}^-; see Figure 7.11(b). In order to prove that u is in some $S[e_i] = c_i^\oplus \cap d_i^\oplus \cap e_i^\oplus$, it suffices to show that $u \in c_i^\oplus \cap d_i^\oplus$ for some $i \in \{m, \ldots, n\}$. Let q be minimal such that $u \in d_q^\oplus$. Such a q exists because $u \in d_n^\oplus$. Then either $q = m$ or

$u \in d_{q-1}^{-}$, so in any case $u \in c_q^{\oplus}$. Therefore u lies in $S[e_q]$. $\square$

Lemma 7.10 says that if the number of middle layers $r-\ell-a$ is at least 32ℓ, then there is a chain $V=(v_1,\ldots,v_t)$ with $t=32\ell$ that wraps around. Since P contains no empty pentagons, Lemma 7.11 then implies that every point in the outer layers lies on one of the lines $l(c_i)$ or $l(d_i)$ that bound the sectors $S[e_i]$ corresponding to V. Thus the number of points in the outer layers is at most $2t(\ell-3)=64\ell(\ell-3)$. Recall however that a was chosen so that the outer layers contained at least $64\ell(\ell-1)$ points, so in fact the number of middle layers is less than 32ℓ. Therefore (by Theorem 7.2) the number of points in the middle layers is $|L_{a+1}\cup\cdots\cup L_{r-\ell}| < 32\ell\times 8\ell = 256\ell^2$. As noted at the beginning of the proof, $|L_1\cup\cdots\cup L_a| < 64\ell^2$, and also $|L_{r-\ell+1}\cup\cdots\cup L_r| < 8\ell^2$. Adding everything up gives $|P| = |L_1\cup\cdots\cup L_r| < 328\ell^2$. This contradicts the assumption that $|P| \geqslant 328\ell^2$, and so in fact P does contain an empty pentagon. This completes the proof of Theorem 7.1. $\square$

Bibliography

[1] Zachary Abel, Brad Ballinger, Prosenjit Bose, Sébastien Collette, Vida Dujmović, Ferran Hurtado, Scott Duke Kominers, Stefan Langerman, Attila Pór, and David R. Wood. Every large point set contains many collinear points or an empty pentagon. *Graphs Combin.*, 27(1):47–60, 2011. doi: 10.1007/s00373-010-0957-2.

[2] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Springer, 3rd edn., 2004.

[3] Miklós Ajtai, Vašek Chvátal, Monroe M. Newborn, and Endre Szemerédi. Crossing-free subgraphs. In *Theory and practice of combinatorics*, vol. 60 of *North-Holland Math. Stud.*, pp. 9–12. North-Holland, 1982.

[4] Jin Akiyama, Hiro Ito, Midori Kobayashi, and Gisaku Nakamura. Arrangements of n points whose incident-line-numbers are at most $n/2$. *Graphs Combin.*, 27(3):321–326, 2011. doi: 10.1007/s00373-011-1023-4.

[5] Noga Alon and Joel H. Spencer. *The probabilistic method*. John Wiley, 2nd edn., 2000.

[6] János Barát, Vida Dujmović, Gwenaël Joret, Michael S. Payne, Ludmila Scharf, Daria Schymura, Pavel Valtr, and David R. Wood. Empty pentagons in point sets with collinearities. 2012. `http://arxiv.org/abs/1207.3633`.

[7] József Beck. On the lattice property of the plane and some problems of Dirac, Motzkin and Erdős in combinatorial geometry. *Combinatorica*, 3(3-4):281–297, 1983. doi: 10.1007/BF02579184.

[8] Peter Brass. On point sets without k collinear points. In *Discrete geometry*, vol. 253 of *Monogr. Textbooks Pure Appl. Math.*, pp. 185–192. Dekker, New York, 2003. doi: 10.1201/9780203911211.ch12.

[9] Peter Brass, William Moser, and János Pach. *Research problems in discrete geometry.* Springer, New York, 2005.

[10] Vašek Chvátal. Remarks on a problem of Moser. *Canad. Math. Bull.*, 15:19–21, 1972.

[11] Josef Cibulka, Jan Kynčl, and Pavel Valtr. On planar point sets with the pentagon property. In *Proceedings of the Twenty-ninth Annual Symposium on Computational Geometry*, SoCG '13, pp. 81–90. ACM, New York, 2013. doi: 10.1145/2462356.2462406.

[12] Wikipedia contributors. Beck's theorem (geometry). *Wikipedia, the free encyclopedia*, 2010. `http://en.wikipedia.org/wiki/Beck's_theorem_(geometry)`.

[13] Wikipedia contributors. Crossing number (graph theory). *Wikipedia, the free encyclopedia*, 2013. `http://en.wikipedia.org/wiki/Crossing_number_(graph_theory)`.

[14] Peter Dankelmann and Lutz Volkmann. Degree sequence conditions for maximally edge-connected graphs depending on the clique number. *Discrete Math.*, 211(1-3):217–223, 2000. doi: 10.1016/S0012-365X(99)00279-4.

[15] Nicolaas G. de Bruijn and Paul Erdős. On a combinatorial problem. *Nederl. Akad. Wetensch., Proc.*, 51:1277–1279, 1948.

[16] Reinhard Diestel. *Graph theory*, vol. 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, 4th edn., 2010.

[17] Gabriel A. Dirac. Collinearity properties of sets of points. *Quart. J. Math., Oxford Ser. (2)*, 2:221–227, 1951. doi: 10.1093/qmath/2.1.221.

[18] Henry Dudeney. *Amusements in Mathematics.* Nelson, Edinburgh, 1917.

[19] Richard A. Duke, Hanno Lefmann, and Vojtěch Rödl. On uncrowded hypergraphs. *Random Structures Algorithms*, 6(2-3):209–212, 1995. doi: 10.1002/rsa.3240060208.

[20] Adrian Dumitrescu, János Pach, and Géza Tóth. A note on blocking visibility between points. *Geombinatorics*, 19(1):67–73, 2009. `http://www.cs.uwm.edu/faculty/ad/blocking.pdf`.

[21] György Elekes. On the number of sums and products. *Acta Arith.*, 81(4):365–367, 1997.

[22] György Elekes. A note on a problem of Erdős on right angles. *Discrete Math.*, 309(16):5253–5254, 2009. doi: 10.1016/j.disc.2009.02.037.

[23] David Eppstein. Happy endings for flip graphs. *J. Comput. Geom.*, 1(1):3–28, 2010.

[24] David Eppstein. Twenty proofs of Euler's formula: $V - E + F = 2$. `http://www.ics.uci.edu/~eppstein/junkyard/euler/`. Accessed December 2013.

[25] Paul Erdős. Appendix, in Klaus F. Roth, On a problem of Heilbronn. *J. London Math. Soc.*, 26:198–204, 1951. doi: 10.1112/jlms/s1-26.3.198.

[26] Paul Erdős. Some unsolved problems. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 6:221–254, 1961. `http://www.renyi.hu/~p_erdos/1961-22.pdf`.

[27] Paul Erdős. On some problems of elementary and combinatorial geometry. *Ann. Mat. Pura Appl. (4)*, 103:99–108, 1975.

[28] Paul Erdős. On some metric and combinatorial geometric problems. *Discrete Math.*, 60:147–153, 1986. doi: 10.1016/0012-365X(86)90009-9.

[29] Paul Erdős. Some old and new problems in combinatorial geometry. In *Applications of discrete mathematics*, pp. 32–37. SIAM, 1988.

[30] Paul Erdős and George Purdy. Some combinatorial problems in the plane. *J. Combin. Theory Ser. A*, 25(2):205–210, 1978. doi: 10.1016/0097-3165(78)90085-7.

[31] Paul Erdős and George Purdy. Extremal problems in combinatorial geometry. In *Handbook of combinatorics, Vol. 1, 2*, pp. 809–874. Elsevier, Amsterdam, 1995.

[32] Paul Erdős and George Purdy. Two combinatorial problems in the plane. *Discrete Comput. Geom.*, 13(3-4):441–443, 1995. doi: 10.1007/BF02574054.

[33] Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2:463–470, 1935. `http://www.numdam.org/item?id=CM_1935__2__463_0`.

[34] Paul Erdős and George Szekeres. On some extremum problems in elementary geometry. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 3–4:53–62, 1960/1961.

[35] Paul Erdős and Endre Szemerédi. On sums and products of integers. In *Studies in pure mathematics*, pp. 213–218. Birkhäuser, Basel, 1983.

[36] Zoltán Füredi. Maximal independent subsets in Steiner systems and in planar sets. *SIAM J. Discrete Math.*, 4(2):196–199, 1991. doi: 10.1137/0404019.

[37] Hillel Furstenberg and Yitzhak Katznelson. A density version of the Hales-Jewett theorem for $k = 3$. *Discrete Math.*, 75(1-3):227–241, 1989. doi: 10.1016/0012-365X(89)90089-7.

[38] Hillel Furstenberg and Yitzhak Katznelson. A density version of the Hales-Jewett theorem. *J. Anal. Math.*, 57:64–119, 1991. doi: 10.1016/S0167-5060(08)70577-6.

[39] Tobias Gerken. Empty convex hexagons in planar point sets. *Discrete Comput. Geom.*, 39(1-3):239–272, 2008. doi: 10.1007/s00454-007-9018-x.

[40] Timothy Gowers. A geometric Ramsey problem. 2011. `http://mathoverflow.net/questions/50928/a-geometric-ramsey-problem`. Accessed July 2012.

[41] Richard R. Hall, Terence H. Jackson, Anthony Sudbery, and K. Wild. Some advances in the no-three-in-line problem. *J. Combinatorial Theory Ser. A*, 18:336–341, 1975. doi: 10.1016/0097-3165(75)90043-6.

[42] Heiko Harborth. Konvexe Fünfecke in ebenen Punktmengen. *Elem. Math.*, 33(5):116–118, 1978.

[43] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[44] Friedrich Hirzebruch. Singularities of algebraic surfaces and characteristic numbers. In *The Lefschetz Centennial Conference, Part I*, vol. 58 of *Contemp. Math.*, pp. 141–155. Amer. Math. Soc., 1986. doi: 10.1090/conm/058.1/860410.

[45] Joseph D. Horton. Sets with no empty convex 7-gons. *Canad. Math. Bull.*, 26(4):482–484, 1983. doi: 10.4153/CMB-1983-077-8.

[46] Ferran Hurtado, Mikio Kano, David Rappaport, and Csaba D. Tóth. Encompassing colored planar straight line graphs. *Comput. Geom.*, 39(1):14–23, 2008. doi: 10.1016/j.comgeo.2007.05.006.

[47] Atsushi Kaneko and Mikio Kano. Discrete geometry on red and blue points in the plane—a survey. In *Discrete and computational geometry*, vol. 25 of *Algorithms Combin.*, pp. 551–570. Springer, Berlin, 2003.

[48] Jan Kára, Attila Pór, and David R. Wood. On the chromatic number of the visibility graph of a set of points in the plane. *Discrete Comput. Geom.*, 34(3):497–506, 2005. doi: 10.1007/s00454-005-1177-z.

[49] Leroy M. Kelly and William O. J. Moser. On the number of ordinary lines determined by n points. *Canad. J. Math.*, 10:210–219, 1958. doi: 10.4153/CJM-1958-024-6.

[50] Daniel J. Kleitman and Rom Pinchasi. A note on the number of bichromatic lines. 2003. `http://www2.math.technion.ac.il/~room/ps_files/KP_bichnum.pdf`.

[51] Dénes König. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Ann.*, 77(4):453–465, 1916. doi: 10.1007/BF01456961.

[52] Vitaly A. Koshelev. The Erdős-Szekeres problem. *Dokl. Akad. Nauk*, 415(6):734–736, 2007. doi: 10.1134/S106456240704031X.

[53] Thomas Kövari, Vera T. Sós, and Pál Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.

[54] Hanno Lefmann. Distributions of points in the unit square and large k-gons. *European J. Combin.*, 29(4):946–965, 2008. doi: 10.1016/j.ejc.2007.11.021.

[55] Hanno Lefmann. Extensions of the No-Three-In-Line Problem. 2012. `http://www.tu-chemnitz.de/informatik/ThIS/downloads/publications/lefmann_no_three_submitted.pdf`.

[56] F. Thomson Leighton. *Complexity Issues in VLSI*. MIT Press, 1983.

[57] Ben D. Lund, George B. Purdy, and Justin W. Smith. A pseudoline counterexample to the Strong Dirac Conjecture. 2012. `http://arxiv.org/abs/1202.3110`.

[58] Jiří Matoušek. *Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2003.

[59] Jiří Matoušek. Blocking visibility for points in general position. *Discrete Comput. Geom.*, 42(2):219–223, 2009. doi: 10.1007/s00454-009-9185-z.

[60] Eberhard Melchior. Über Vielseite der projektiven Ebene. *Deutsche Math.*, 5:461–475, 1941.

[61] Roy Meshulam. On multicoloured lines in a linear space. *J. Combin. Theory Ser. A*, 40(1):150–155, 1985. doi: 10.1016/0097-3165(85)90053-6.

[62] Leo Moser. Problem P.170. *Canad. Math. Bull.*, 13:268, 1970.

[63] Theodore S. Motzkin. Nonmixed connecting lines. *Notices Amer. Math. Soc.*, 14:837, 1967.

[64] Carlos M. Nicolás. The empty hexagon theorem. *Discrete Comput. Geom.*, 38(2):389–397, 2007. doi: 10.1007/s00454-007-1343-6.

[65] János Pach and Rom Pinchasi. Bichromatic lines with few points. *J. Combin. Theory Ser. A*, 90(2):326–335, 2000. doi: 10.1006/jcta.1999.3047.

[66] János Pach, Radoš Radoičić, Gábor Tardos, and Géza Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. *Discrete Comput. Geom.*, 36(4):527–552, 2006. doi: 10.1007/s00454-006-1264-9.

[67] János Pach and Micha Sharir. Repeated angles in the plane and related problems. *J. Combin. Theory Ser. A*, 59(1):12–22, 1992. doi: 10.1016/0097-3165(92)90094-B.

[68] János Pach and Géza Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997. doi: 10.1007/BF01215922.

[69] János Pach and Géza Tóth. Which crossing number is it anyway? *J. Combin. Theory Ser. B*, 80(2):225–246, 2000. doi: 10.1006/jctb.2000.1978.

[70] Michael S. Payne, Attila Pór, Pavel Valtr, and David R. Wood. On the connectivity of visibility graphs. *Discrete Comput. Geom.*, 48(3):669–681, 2012. doi: 10.1007/s00454-012-9446-0.

[71] Michael S. Payne and David R. Wood. On the general position subset selection problem. *SIAM J. Discrete Math.*, 27(4):1727–1733, 2013. doi: 10.1137/120897493.

[72] Michael S. Payne and David R. Wood. Progress on Dirac's Conjecture. 2013. `http://arxiv.org/abs/1207.3594`.

[73] Florian Pfender. Visibility graphs of point sets in the plane. *Discrete Comput. Geom.*, 39(1-3):455–459, 2008. doi: 10.1007/s00454-008-9056-z.

[74] Kevin T. Phelps and Vojtěch Rödl. Steiner triple systems with minimum independence number. *Ars Combin.*, 21:167–172, 1986.

[75] Ján Plesník. Critical graphs of given diameter. *Acta Fac. Rerum Natur. Univ. Comenian. Math.*, 30:71–93, 1975.

[76] Ján Plesník and Štefan Znám. On equality of edge-connectivity and minimum degree of a graph. *Arch. Math. (Brno)*, 25(1-2):19–25, 1989.

[77] Density H. J. Polymath. Density Hales-Jewett and Moser numbers. In *An irregular mind*, vol. 21 of *Bolyai Soc. Math. Stud.*, pp. 689–753. János Bolyai Math. Soc., Budapest, 2010. doi: 10.1007/978-3-642-14444-8_22.

[78] Attila Pór and David R. Wood. On visibility and blockers. *J. Computational Geometry*, 1(1):29–40, 2010.

[79] George Purdy. A proof of a consequence of Dirac's conjecture. *Geom. Dedicata*, 10(1-4):317–321, 1981. doi: 10.1007/BF01447430.

[80] George B. Purdy and Justin W. Smith. Bichromatic and equichromatic lines in $\mathbb{C}^2$ and $\mathbb{R}^2$. *Discrete Comput. Geom.*, 43(3):563–576, 2010. doi: 10.1007/s00454-009-9154-6.

[81] Stanley Rabinowitz. Consequences of the pentagon property. *Geombinatorics*, 14:208–220, 2005.

[82] Vojtěch Rödl and Edita Šiňajová. Note on independent sets in Steiner systems. *Random Structures Algorithms*, 5(1):183–190, 1994. doi: 10.1002/rsa.3240050117.

[83] Joseph H. Silverman. *The arithmetic of elliptic curves*, vol. 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986. doi: 10.1007/978-1-4757-1920-8.

[84] József Solymosi. On the number of sums and products. *Bull. London Math. Soc.*, 37(4):491–494, 2005. doi: 10.1112/S0024609305004261.

[85] Joel Spencer. Turán's theorem for k-graphs. *Discrete Math.*, 2:183–186, 1972. doi: 10.1016/0012-365X(72)90084-2.

[86] Benny Sudakov. Large K_r-free subgraphs in K_s-free graphs and some other Ramsey-type problems. *Random Structures Algorithms*, 26(3):253–265, 2005. doi: 10.1002/rsa.20035.

[87] James Joseph Sylvester. Question 2473. *Mathematical Questions and Solutions from the Educational Times*, 8:106–107, 1867.

[88] James Joseph Sylvester. Question 2572. *Mathematical Questions and Solutions from the Educational Times*, 45:127–128, 1886.

[89] James Joseph Sylvester. Question 3019. *Mathematical Questions and Solutions from the Educational Times*, 45:134, 1886.

[90] James Joseph Sylvester. Question 11851. *Mathematical Questions and Solutions from the Educational Times*, 59:98–99, 1893.

[91] László A. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.*, 6(3):353–358, 1997. doi: 10.1017/S0963548397002976.

[92] László A. Székely. A successful concept for measuring non-planarity of graphs: the crossing number. *Discrete Math.*, 276(1–3):331–352, 2004. doi: 10.1016/S0012-365X(03)00317-0.

[93] Endre Szemerédi and William T. Trotter, Jr. Extremal problems in discrete geometry. *Combinatorica*, 3(3-4):381–392, 1983. doi: 10.1007/BF02579194.

[94] Terence Tao and Van Vu. *Additive combinatorics*, vol. 105 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2006. doi: 10.1017/CBO9780511755149.

[95] Louis Theran. Lines induced by bichromatic point sets. 2011. http://arxiv.org/abs/1101.1488.

[96] Todd Trimble. Solution to problem of the week no.13: Highly coincidental. 2009. http://topologicalmusings.wordpress.com/2009/07/22/solution-to-pow-13-highly-coincidental/. Accessed March 2011.

[97] Paul Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[98] Pavel Valtr. On empty hexagons. In *Surveys on discrete and computational geometry*, vol. 453 of *Contemp. Math.*, pp. 433–441. Amer. Math. Soc., 2008. doi: 10.1090/conm/453/08809.

[99] Lutz Volkmann. Degree sequence conditions for equal edge-connectivity and minimum degree, depending on the clique number. *J. Graph Theory*, 42(3):234–245, 2003. doi: 10.1002/jgt.10087.

[100] David R. Wood. A note on colouring the plane grid. *Geombinatorics*, XIII(4):193–196, 2004.

[101] David R. Wood. Point sets with no empty pentagon. Accessed December 2013. Problem posted at Open Problem Garden, http://www.openproblemgarden.org/op/point_sets_with_no_empty_pentagon.

[102] Kazimierz Zarankiewicz. Problem P 101. *Colloq. Math.*, 3:19–30, 1954.