

# Extremal Graph Theory for Book-Embeddings

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## Abstract

A *book-embedding* of a graph is a linear ordering of its vertices, and a partition of its edges into non-crossing sets (called *pages*). The *pagenumber* of a graph is the minimum number of pages for which it has a book-embedding.

After introducing concepts relevant to book-embeddings and their applications, we will describe the main results on the pagenumber for classes of graphs, particularly results on the pagenumber of planar graphs. We will briefly cover extremal graph theory before discussing extremal results with regards to the pagenumber of graphs.

The original results consider *edge-maximal book-embeddings*: book-embeddings to which no edge can be added without increasing the pagenumber. We prove that the minimum number of edges of such graphs with three pages and  $n$  vertices is  $\lceil \frac{7n}{2} \rceil - 8$ . Generalising to  $k$  pages, we prove an upper bound of  $\frac{(k+4)n}{2} - \frac{3k}{2}$  and a lower bound of  $\sqrt{\frac{k}{2}}n$  on the minimum number of edges.

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# Chapter 1

## Introduction

A *book-embedding* of a graph consists of a linear ordering of its vertices, with the edges partitioned into sets such that, under the given layout of the vertices, edges which are in the same set can be drawn without crossing on half-planes bounded by the line of the vertex embedding. Each of the non-crossing sets of edges is a *page* of the book-embedding, and the *pagenumber* of a graph is the minimum number of pages in any of its book-embeddings.

The concepts of book-embeddings and pagenumber were first introduced by Ollmann in 1973 [37], and have since been shown to have many interesting applications in fields such as circuit design and complexity theory. Most research on book-embeddings consists of finding a book-embedding for a particular class of graphs.

Book-embeddings are a particular layout for graphs, and several different types of problems arise in research in this field. One approach is to characterise the graphs of a given pagenumber. For example, graphs with pagenumber at most two are sub-hamiltonian planar. Another approach is to look for properties of book-embeddings for families of graphs, for example all planar graphs can be embedded in four pages. Finding optimal book-embeddings for various classes of graphs is a common type of problem that appears in this field.

With regards to the applications that exist for book-embeddings, there tend to be two main types of approach. Some applications enforce a predetermined ordering on the layout of the vertices, while there are other types of applications in which determining the optimal ordering of the vertices is part of the problem. Even in the case where the ordering is pre-specified, it is NP-complete to determine whether a graph has a  $k$ -page book-embedding for  $k \geq 2$  [18]. For some particular classes of graphs, there are efficient ways of getting optimal, and near optimal, book-embeddings.

There are several different variations on the terms used when discussing book-embeddings, which arise from the various approaches to the subject. For example, the equivalent term *stack layout* arises from the interpretation of book-embeddings

as representations of PUSH and POP operations on a stack data structure. Many of the applications can be seen to relate to this particular interpretation of book-embeddings, which is why most uses for book-embeddings arise from topics in computer science.

## Overview of Topics

The focus of this thesis will be on presenting ideas with regards to book-embeddings and extremal graph theory that build towards the main results on maximal book-embeddings and the minimum number of edges in these graphs.

In Chapter 2 the concepts of book-embeddings and pagewidth will be introduced, along with related concepts relevant to research on this topic. Some applications will also be discussed to provide motivation for research in this field and to give some context for where these concepts might arise.

Chapter 3 will look into some of the basic theorems that have arisen from this field. Some results on book-embeddings for particular classes of graphs will be discussed, graphs with small pagewidth will be characterised, and some optimal book-embeddings will be described.

The subject of book-embeddings for planar graph will be addressed in chapter 4, where the history of results for this topic will be briefly covered, and Buss and Shor's [6] constructive proof for nine-page book-embeddings of planar graphs will be covered in detail. While this result has since been superseded, it is still of particular importance in that it was the first to show that planar graphs have bounded pagewidth, disproving the conjecture by Bernhart and Kainen [4] that no such bound exists.

Chapter 5 will make a brief detour to introduce the field of extremal graph theory, providing some context for later chapters. Turán's theorem is considered to have initiated research in this field, and will be introduced along with some other results from extremal graph theory to provide an intuition for the types of results that will be considered in later chapters with regards to book-embeddings. As extremal graph theory is a large and varied field, the focus will be restricted to results which consider the maximum and minimum number of edges in graphs relative to other graph invariants, such as chromatic number. From there, some extremal results specific to the concept of book-embeddings will be described in Chapter 6. In particular, results which relate pagewidth to other graph invariants will be considered.

In Chapter 7 the concept of maximal book-embeddings will be introduced, and consideration will be given to the size of such graphs. In particular, some original results on the minimum number of edges in a maximal book-embedding will be

proven. We show that the minimum size of a maximal book-embedding with  $n$  vertices on  $k$  pages is at most  $\frac{(k+4)n}{2} - \frac{3k}{2}$  and at least  $\sqrt{\frac{k}{2}}n$ . For the case where  $k = 3$ , the minimum number of edges in a maximal book-embedding is shown to be  $\lceil \frac{7n}{2} \rceil - 8$  for  $n \geq 3$ .



# Chapter 2

## Preliminaries

### 2.1 Concepts and Definition

A *graph* is defined as an ordered pair  $G = (V, E)$ , consisting of a vertex set  $V$  along with an edge set  $E$  whose elements are two-element subsets of the vertex set  $V$ . The terms *order* and *size* are used to refer to the number of elements in the vertex and edge set, respectively. A graph  $G = (V, E)$  is said to be a *subgraph* of another graph  $G' = (V', E')$  if  $V \subset V'$  and  $E \subset E'$ .

The *degree* of a vertex  $v$ , denoted  $\deg(v)$ , is the number of edges which are incident to  $v$ . The *minimum degree* of a graph  $G$  is denoted by  $\delta(G)$ , and is the minimum degree of all vertices in  $V(G)$ . Similarly, the *maximum degree* of a graph  $G$ , denoted by  $\Delta(G)$ , is the maximum of the degrees of all the vertices in  $V(G)$ .

An *embedding* of a graph  $G$  on a surface  $S$  is defined as a drawing of  $G$  with the vertices and edges drawn on the surface  $S$  such that no edges of  $G$  intersect, except at a common endpoint. The main type of graph embedding which will be considered is the class of graphs embeddable in the plane.

**Definition 1.** A *planar graph* is a graph that can be embedded in the plane without crossings.

**Definition 2.** A *maximal planar graph* is a planar graph  $G$  such that adding any new edge to  $G$  will result in a non-planar graph.

A *face* of a planar graph is one of the regions defined by its edges. It can be seen that all faces in a maximal planar graph are triangles, as otherwise there would be at least one edge crossing that face which could be added without violating the condition of planarity.

**Definition 3.** An *outerplanar graph* is a graph that can be embedded in the plane with all vertices on the outer face, such that all edges are pairwise non-crossing.

A common way of visualising outerplanar graphs is with the vertices embedded in a circular layout, ordered such that no edges cross. All outerplanar graphs are planar, however the converse is not true. For example, while  $K_4$  can be embedded in the plane without any edges crossing, it is not outerplanar.

Graph colouring problems involve assigning labels, referred to as colours, to some elements of the graph, subject to certain constraints. There are two standard types of colourings of graphs, in which either the vertices or the edges might be considered as the object to be coloured. An *edge colouring* of a graph is an assignment of colours to the edges of a graph such that no edges incident to the same vertex are assigned the same colour. A *vertex colouring* of a graph is an assignment of colours to the vertices of a graph such that no vertex is assigned the same colour as any of its neighbours.

**Definition 4.** The *chromatic number* of a graph  $G$ , denoted  $\chi(G)$ , is the minimum number of colours required for a vertex colouring of  $G$ .

When referring to a class of graphs, the chromatic number of that class is the maximum chromatic number of any graph in that class. One of the major results in the topic of graph colouring is that the chromatic number for the class of planar graphs is four [2].

A *k-page book-embedding* can be considered as a graph with vertices embedded on a circle, and edges drawn inside the circle and coloured with  $k$  colours such that no two edges of the same colour cross. Other representations involve a restricted embedding of the graph in a topological structure referred to as a book.

**Definition 5.** A *book* is a topological structure consisting of half planes joined along a common boundary line, called the *spine*.

**Definition 6.** A *page* of a book is one of its half-planes.

When embedding a graph in a book, we impose restrictions on the layout of edges on the same page.

**Definition 7.** Given a book-embedding with vertices laid out in the order  $v_1, v_2, \dots, v_n$ , edges are said to *cross* if they have labels  $v_a v_b$  and  $v_i v_j$  such that  $a < i < b < j$ .

**Definition 8.** Given a book-embedding with vertices laid out in the order  $v_1, v_2, \dots, v_n$ , edges are said to be *nested* if they have labels  $v_a v_b$  and  $v_i v_j$  such that  $a < i < j < b$ .

A  $k$ -page book-embedding can also be considered as an ordering of the vertices along with a partition of the edges into  $k$  sets such that, given the vertices ordered along a line in the specified order, no edges which are in the same part cross. There are several conceptually distinct, yet theoretically equivalent, ways of defining book-embeddings of graphs, which can relate to different ways of visualising the graph

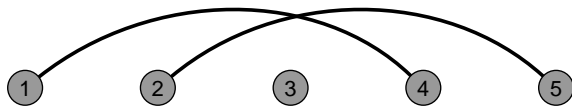


Figure 2.1: Edges of a book-embedding which are crossing

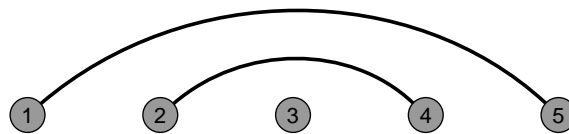


Figure 2.2: Edges of a book-embedding which are nested

drawing.

**Theorem 9.** *The following definitions of book-embeddings are equivalent;*

1. *A book-embedding is a graph with vertices embedded in a straight line, with edges assigned colours such that no edges of the same colour cross, and no edges cross the line in which the vertices are embedded (see Figure 2.3).*
2. *A book-embedding is a graph with vertices embedded in a circle, and edges (also called chords) drawn across the circle and coloured in such a way that no crossing edges are of the same colour (see Figure 2.4).*
3. *A book-embedding is a topological embedding of a graph, with vertices embedded in a straight line (called the spine of the book), and edges residing on half-planes (pages) such that no edges on the same plane cross (see Figure 2.5).*

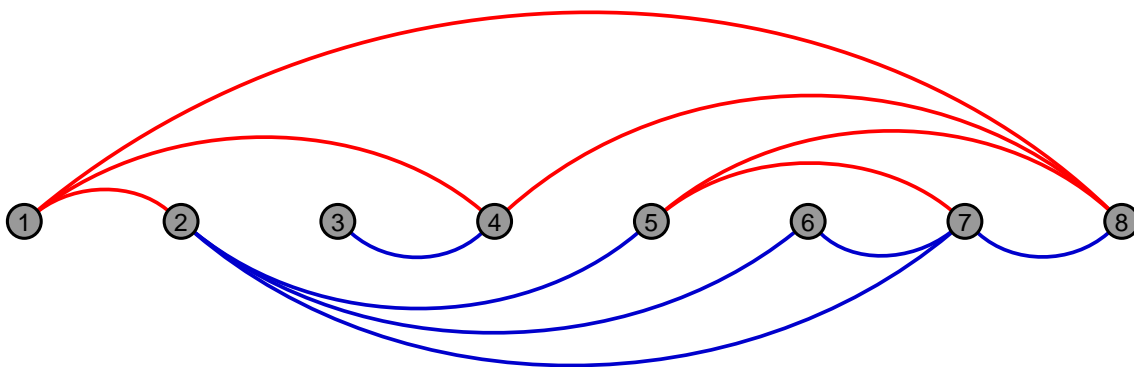


Figure 2.3: A two page book-embedding of a graph represented with vertices in linear formation

*Proof.* We first show how a book-embedding in a linear layout can be drawn in a circular layout. We then show how a circular layout can be drawn in a topological layout, and then how a topological layout can be drawn as a linear book-embedding.

1. If we have a book-embedding in the linear layout of definition one, with all edges drawn above the line of vertices, we can get the equivalent circular layout

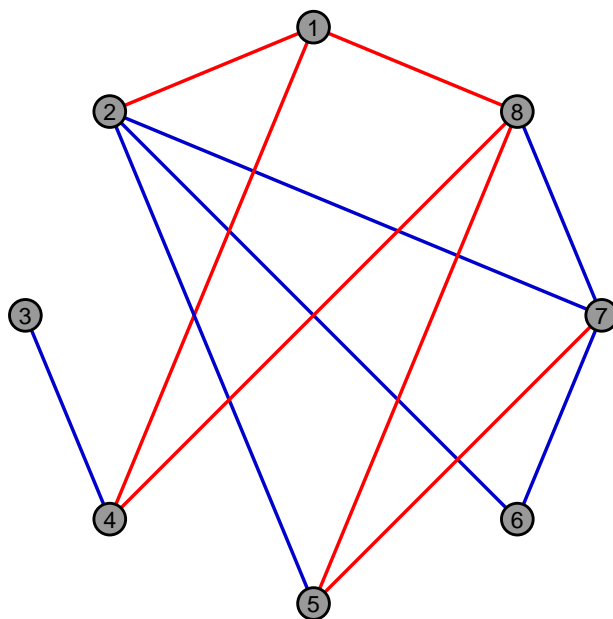


Figure 2.4: A two page book-embedding of the same graph, represented as a circular embedding

as follows; first consider a conceptual line to be drawn through the vertices along the line in which they are embedded, then bring the end vertices on the line around to get the vertices in a circular layout. It is clear that any edges that were not crossing in the initial layout will not cross in the new layout, hence the pages of the book-embedding are preserved.

2. If we have a circular layout of a book-embedding, we can get an equivalent topological embedding by taking one of the edges on the outside of the circular embedding, and bringing the adjacent vertices down and apart in such a way that all the vertices are in a line. Consider this line to be the spine of the book

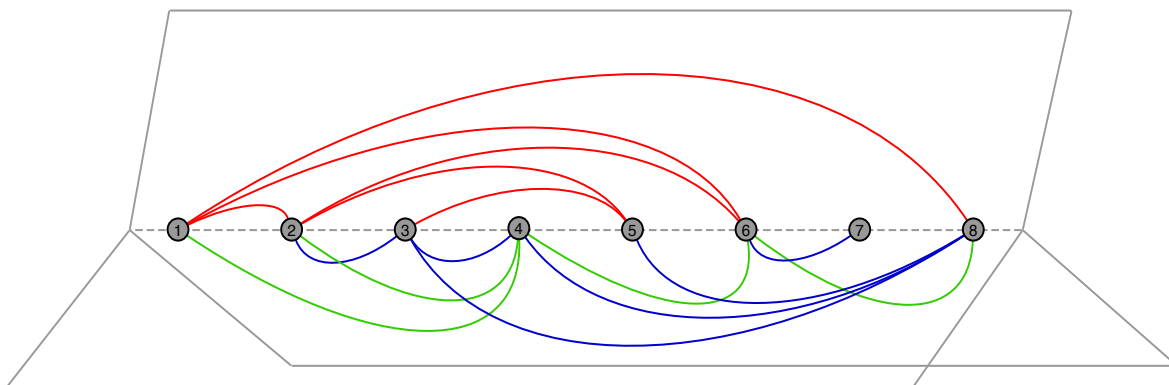


Figure 2.5: A three page book-embedding represented as an embedding in a topological book

the graph will be embedded in. For each of the colours used in the initial layout, take all the edges in that colour and embed them in a single page of the book, using a new page for each colour. This gives a topological representation of the initial book-embedding which maintains the relative ordering of the vertices and partitioning of the edges.

3. From definition three, with the edges embedded in half-planes, and the vertices in pages of a topological book, we can easily get the linear layout of definition one by assigning each of the pages to a distinct colour, so all edges on the same page are in the same colour set. Then all the pages can be embedded in the plane above the spine to give a linear layout of the book-embedding. Since edges on the same page don't cross, there are no crossing edges of the same colour.

Using these relations we can go from a book-embedding in any of the three formats to any of the other formats, showing equivalence.

□

When considering book-embeddings in a circular layout, it can be useful to distinguish between the following two types of edges.

**Definition 10.** The edges of a book-embedding that are between consecutive vertices on the external face are called *external edges*.

The edges of a book-embedding which are not external edges are said to be *internal edges*.

The different ways of representing book-embeddings reflect different perspectives from which book-embeddings can be considered. Some properties of book-embeddings are more easily visualised, and some applications more readily modelled, by using particular layouts for book-embeddings.

For example, linear layouts of book-embeddings are often more convenient when considering book-embeddings as a model for stack operations. Another way of referring to book-embeddings is as *stack layouts* of graphs, as they can be considered as representing processes on a stack data structure. When traversing the vertices of the book-embedding in order along the spine, edges embedded in the same page are encountered in a Last-In-First-Out order, which is associated with stacks.

For example, consider the one page book-embedding shown in Figure 2.6. The linear ordering of the vertices in the book-embedding is  $\pi = 1, 2, 3, 4, 5, 6$ .

Starting at vertex 1 and stepping through the vertices, we we push and pop the edges in the following order:

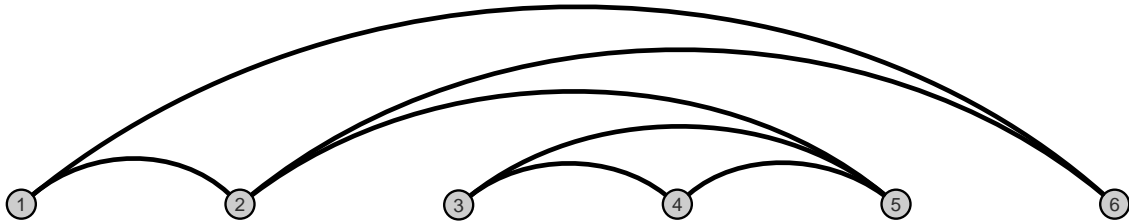


Figure 2.6: The ordering of the vertices from left to right can be interpreted in terms of stack operations

vertex 1:	$PUSH(1, 6), PUSH(1, 2)$
vertex 2:	$POP(1, 2), PUSH(2, 6), PUSH(2, 5)$
vertex 3:	$PUSH(3, 5), PUSH(3, 4)$
vertex 4:	$POP(3, 4), PUSH(4, 5)$
vertex 5:	$POP(4, 5), POP(3, 5), POP(2, 5)$
vertex 6:	$POP(2, 6), POP(1, 6)$

We can see from this that the Last-In First-Out behaviour of the stack is maintained by the ordering of the edges in the book-embedding.

Related is the concept of a queue layout, arising from the conceptually similar queue data structure.

**Definition 11.** A  $k$ -queue layout of a graph consists of a total ordering of the vertices, along with a partition of the edge set into  $k$  sets of pairwise non-nested elements [10].

While the pages of a book-embedding can be considered in terms of stack operations, for a queue layout of a graph the pages relate instead to queue operations.

Most questions about book-embeddings relate to the minimum number of pages required for book-embeddings of a particular class of graphs. There are many ways for any non-trivial graph to be embedded in a book, which arises from the number of possible orderings which can be imposed on the vertices. Clearly the number of pages required for a book-embedding varies according to the order in which the vertices are embedded along the spine.

**Definition 12.** The *pagenumber* of a graph  $G$ , denoted  $pn(G)$ , is the minimum number of pages in which it can be embedded without crossings.

The pagenumber of a graph can also be referred to as its *book-thickness*, *stack number* or *fixed outer-thickness*. It is a ubiquitous concept when discussing book-embeddings, as most results in this area focus on minimising the number of pages

used. It can also be of interest to minimise the pagewidth in a book-embedding of a graph. The *cutwidth* of a page in a book-embedding is the maximum number of edges which cross some line perpendicular to the spine of the book.

**Definition 13.** The *pagewidth* of a book-embedding of a graph is the maximum width of its pages.

Sometimes when considering book-embeddings of graphs we will be interested in minimising the pagewidth of the layout as well as the pagenumber.

Any graph can trivially be given a book-embedding with pagewidth one by embedding each of its edges in different pages, so while we can consider the pagenumber of a graph, the pagewidth is a property of book-embeddings. An alternative term which may be used for the pagewidth of a book-embedding is *stackwidth*.

For a given ordering of the vertices of a graph  $G$ , the *width* of an edge is the number of edges along the spine between the endpoints. More formally, for a permutation  $\pi = (v_1, \dots, v_n)$  of the vertices of  $G$ , an edge  $(v_i, v_j) \in E(G)$  has width of  $|i - j|$  in  $\pi$ . The bandwidth of a permutation  $\pi$  on graph  $G$  is the maximum width of all the edges in  $E(G)$  under the vertex ordering specified by permutation  $\pi$ .

**Definition 14.** The *bandwidth* of a graph  $G$  is the minimum bandwidth over all permutations of  $V(G)$ .

If a graph has bandwidth  $k$  it is referred to as a bandwidth- $k$  graph. The class of bandwidth- $k$  graphs has pagenumber  $k - 1$ , although the ordering of the vertices which optimises the pagenumber is not the same as the one which specifies the bandwidth [40].

For a queue layout of a graph, the equivalent concept in place of pagenumber is referred to as the *queue number*.

**Definition 15.** The *queue number* of a graph  $G$ , denoted  $qn(G)$ , is the minimum number  $k$  such that  $G$  has a  $k$ -queue layout [10].

There are many results on queue layouts for classes of graphs with minimal queue number. Applications of queue layouts are fairly similar to those of book-embeddings, as many of the application relate to the underlying connection between these graph drawings and the relevant data structures.

## 2.2 Motivations and Applications

Book-embeddings of graphs relate to many interesting question with regards to other topics in graph theory. Most of the results in this area of research involve considering the least sufficient pagenumber for a given class of graphs, and several of these classes

have many interesting real-world applications for which the pagewidth might be of interest. For example some important interconnection networks have been shown to have 3-page book-embeddings [17].

Another use of book-embeddings for graphs relates to compact encodings of graphs. For example, Jacobson [27] described a succinct data structure for representing graphs of bounded pagewidth, which gives a good space complexity representation for planar graphs as they have pagewidth of four.

There are several direct applications motivating the study of book-embeddings, mostly pertaining to questions which arise from the fields of computer science and electrical engineering, in particular the areas of circuit design and complexity theory. Most of these examples consider systems that can be naturally modelled by graphs in which the features of the book-embedding, usually the pagewidth, have some convenient analogy with the application. In many of the applications, the association between book-embeddings and stacks becomes apparent.

### 2.2.1 Optimised VLSI Design

One application of book-embeddings is that of optimised layouts for VLSI circuits. The design of Very Large-Scale Integration (VLSI) systems involves integrating a large number (billions) of electronic components, such as transistors and wires, into a single chip. Optimised design techniques become a priority in order to place such a large number of transistors on a single integrated circuit. VLSI circuits are a ubiquitous technology, appearing in most levels of modern technology, from computers to cars to phones, so the value of this application cannot be understated.

VLSI circuits can be visualised as a graph, where the processor elements are represented as vertices, with the edges representing the interconnections between these elements. Several of the graph theoretic questions which arise from problems regarding VLSI circuits relate to minimising some cost function by laying out the vertices of the graph on a line [45].

One approach to VLSI design involves simplifying the problem of laying out the processing elements by placing them on a line, with the interconnecting wires running along parallel tracks.

The cutwidth of a graph corresponds with the minimum number of tracks required for routing the interconnecting wires in the initial VLSI circuit. So the problem of finding an ordering of the processor elements which minimises the required number of tracks, and which would therefore minimise the necessary area for the wiring, is related to the cutwidth of circuits representative graph [45].



## 2.2.2 Layout of Fault Tolerant Processor Arrays

Another motivation for the study of book-embeddings which arises from the field of electrical engineering is the problem of designing fault tolerant VLSI processor arrays. The DIOGENES approach was proposed by Rosenberg [39] as a method for designing fault tolerant arrays of processing elements. The logic behind this approach is as follows. The processing elements, some of which might be faulty, are laid out in a (conceptual) line, corresponding to the spine of a book-embedding. Parallel to the line of processors are bundles of wires with embedded switches. The switches built into the chip allow the bundles of wires to function as a stack, hence each of these bundles can be considered as a page of a book-embedding.

For any given processor, a connection to a processor on its right is PUSHed on a stack, whereas connections to processors on its left would be POPped from a stack. Therefore, each connection to a good processor requires one stack operation at that processor. If a processor is bad, then no stack operations occur at that processor. Whether a processor is good or bad can be considered as a binary variable, so a single control signal can cause the PUSHing and POPping of many connections. By switching in only good processors, fault tolerance can be achieved.

The required array of the processors is modelled by a connection graph, where the vertices represent the processors, and edges correspond with the desired connections between these processors. The DIOGENES design layout problem considers the number of stacks, and the number of connections carried by each of the stacks, that is necessary for implementing the array of processors. In other words, it considers the pagenumber (or stack number) and pagewidth (or stackwidth) of the connection graph.

As noted before, a single page of a book-embedding can be considered as a representation of a stack, and hence the required number of stacks is exactly the number of pages necessary for a book-embedding of the connection graph, and the width of these stacks is the width of the pages in the book-embedding.

## 2.2.3 Sorting With Parallel Stacks

Book-embeddings also have also been discussed with regards to the question of realising fixed permutations of  $N$  distinct elements with disjoint, parallel stacks [14].

This question was described by Tarjan [41] as “The Switchyard Problem”, and was considered in terms of the layout of a railroad switchyard. This description of the question considers a train pulling  $N$  cars travelling from one end of a switchyard to the other, and looks at the possible rearrangements that can be made to the cars before the train reaches the end of the switchyard. The switchyard can be

visualised as a directed acyclic graph (dag) possessing a unique source and sink. Each vertex represents a side track of the railway, with the edges representing connections between them. The vertices are assumed to have indefinite storage space, and can be considered as either stacks or queues. It is the version of this problem which considers the vertices as stacks that gives rise to the book-embedding connection, as would be expected given the natural connection of book-embeddings with the stack data structure. The version of this model that considers side tracks to operate as queues rather than stacks would relate instead to a queue layout graph.

This problem has also been considered by Even and Itai [14] in terms of colouring a permutation graph. For a permutation  $\pi$  of  $\{1, \dots, n\}$ , its permutation graph has vertex set  $\{1, \dots, n\}$  and an edge  $(i, j)$  whenever  $i < j$  and  $\pi^{-1}(i) < \pi^{-1}(j)$  or  $i > j$  and  $\pi^{-1}(i) > \pi^{-1}(j)$ . Even and Itai show that minimising the number of stacks required is equivalent to colouring a circle graph, and is therefore NP-complete [18].

This problem can also be considered in terms of finding the possible permutations of  $\{1, \dots, N\}$  that can be obtained using non-communicating stacks [7]. Each of the numbers is initially pushed, in increasing order, onto any one of the stacks. After all of the numbers have been pushed onto the stacks, the permutation is formed by popping the numbers from the stacks.

Suppose that for some permutation  $\pi$  of  $\{1, \dots, N\}$ , we wish to know the minimum number of stacks required in this formulation to realise permutation  $\pi$ . To solve this, we use a graph theoretic model of the problem.

This can be achieved by visualising the process as a bipartite graph, with the partitions corresponding to the initial and final permutations of the numbers. This is modelled as follows. For our permutation  $\pi$ , we construct a graph  $G_\pi$  with vertices  $\{a_1, \dots, a_N, b_1, \dots, b_N\}$ , with edges between vertices  $a_i$  and  $b_{\pi(i)}$  representing the mapping of  $i$  under the permutation  $\pi$ .

For our problem of finding the minimum number of stacks required for realising some permutation  $\pi$ , we now have a neat relation with book-embeddings, as the problem now corresponds to finding the pagewidth of the permutation graph. This is because the pages in a book-embedding of  $G_\pi$  correspond with numbers which are inversely ordered under the permutation  $\pi$ , which means that when they are PUSHed onto a stack, the order in which they will be POPped from the stack is the proper ordering under the new permutation given by  $\pi$ .

Even and Itai [14] showed that minimising the number of stacks required to realise such a permutations is NP-complete by showing that this problem is equivalent to colouring a circle graph.

The related model of this problem using queues in place of stacks would use the queue number of the permutation graph as the minimum number of queues required for realising permutation  $\pi$ .

## 2.2.4 Turing Machine Graphs

A Turing machine computation of  $t$  steps can be modelled by constructing a corresponding  $t$ -vertex graph [7]. The construction is as follows; each step of the computation is represented by a vertex of the graph, and two vertices  $t_1$  and  $t_2$  are adjacent in the graph when one of the machines tapeheads has visited the same tape square at both of the timesteps corresponding to the vertices  $t_1$  and  $t_2$ , and not at any intervening time. From this construction it follows that the dependency graph of every  $k$ -tape Turing machine graph can be embedded in a book with  $2k$  pages.

Since we can associate the computations of a Turing machine with graphs of a bounded pagewidth, it is possible that results characterising graphs which can be embedded in a given number of pages might have interesting interpretations with regards to results in complexity theory.

For example, a proof that  $k$ -page graphs have a small (subquadratic) bandwidth would imply that a one-tape non-deterministic Turing machine could simulate a two-tape machine in subquadratic time [16].

## 2.2.5 Modelling RNA Structures

While most applications of book-embeddings are with regards to topics in computer science and electrical engineering, there have been some attempts to use them in biology, particularly for modelling secondary structures of RNA sequences. In this application, the order in which the vertices are embedded is predetermined by the structures nucleotide sequence.

The *primary structure* of a RNA sequence is an ordering of monomers, or nucleotides, and can be considered equivalent to the spine of a book-embedding. The *secondary structure* is a set of hydrogen bonds between pairs of monomers, and can be represented by edges between the equivalent vertices. A *knot* in such a structure describes when two or more edges are mutually crossing, and is referred to as a *pseudo-knot* when it can be drawn on two pages without crossings. Many RNA molecules feature pseudo-knots, which poses some difficulties as they don't fit under conventional models of secondary structures [19].

In such a model, the pagewidth of the graph represents the minimum number of disjoint secondary structures that the initial structure can be decomposed into, where each page represents a secondary structure of the RNA structure [8]. From this, we can see that unknotted secondary structures can be embedded in one page, and knotted structures require at least two.

# Chapter 3

## Book-Embeddings and Pagenumber

### 3.1 Introduction to Graph Layout

Book-embeddings are a specific example of a graph layout, where the vertices are embedded in either a line or a circle, depending on the approach used.

Chung *et.al.* [7] distinguish two approaches for considering the problem of constructing book-embeddings of graphs. The first, and simplest, case is where the ordering in which the vertices are embedded along the spine is pre-defined. This formulation appears in such applications as single row routing, and sorting with parallel stacks. The second case is when finding the optimal ordering of vertices is a part of the problem itself, such as in the construction of fault tolerant processor arrays. The number of permutations of the vertices make this a much more complicated problem than the first case. In this section, some results of this second kind are given for certain types of graphs.

Many classes of graphs have been shown to be able to be embedded in a bounded number of pages, with some of these bounds known to be strict. Graphs with a pagenumber of zero, one and two can be shown to have particular properties, as for these values there exist class equivalence relations. For example, the set of graphs with pagenumber at most one are exactly the class of outerplanar graphs.

### 3.2 Graphs with Small Pagenumber

Graphs which can be embedded in few pages can be shown to have particular properties. In particular, for values of  $k$  strictly less than three, the set of graphs which can be embedded in  $k$  pages can be shown to be exactly equivalent to another class of graphs.

**Theorem 16.** [7] A graph  $G$  has a book-embedding using a single page if and only if  $G$  is an outerplanar graph.

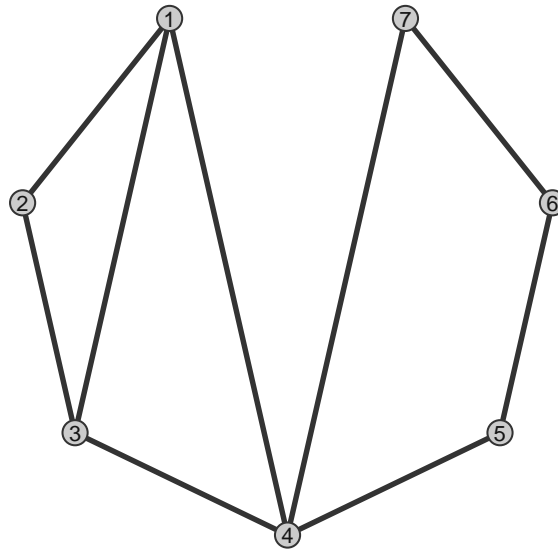


Figure 3.1: An outerplanar graph  $G$

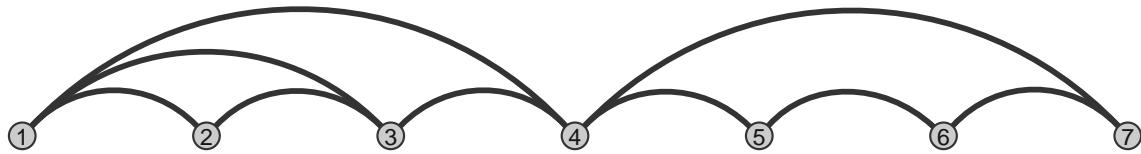


Figure 3.2: Outerplanar graph  $G$  embedded in one page

*Proof.* A graph  $G$  is referred to as outerplanar precisely when its vertices can be placed in a circular formation such that its edges are internal to the circle and do not cross any other edges. Let  $G$  be some arbitrary outerplanar graph, drawn in the specified formation, for example the graph shown in 3.1. We can obtain a one-page book-embedding of this graph by cutting the circle between any two vertices,  $v_1$  and  $v_2$ . The vertices of the graph  $G$  can now be straightened out to form a line from  $v_1$  to  $v_2$  in which the ordering of the vertices between them is preserved. The edge  $v_1, v_2$  can be added back into the graph without crossing any other edges. We see the result of this for our example in Figure 3.2. So any arbitrary outerplanar graph can be embedded on a single page.

Conversely, a one-page book-embedding of  $G$  can be converted into the above formation for an outerplanar graph. This is achieved by considering a conceptual line to be passed through the vertices in the order of their embedding, where we label the first vertex as  $v_1$  and the end vertex  $v_n$ . The vertices  $v_1$  and  $v_n$  can then

be brought around to form a circle preserving both the order of the vertices and the non-crossing property of the edges, thus demonstrating its outerplanarity.

An example of an outerplanar graph and its equivalent one-page book-embedding can be seen in Figures 3.1 and 3.2.  $\square$

Similarly, graphs with a queuenumber of one can also be characterised as a class of planar graphs. Heath and Rosenberg characterised the graphs with queuenumber of one as a class of planar graph with a particular type of planar embedding, which are referred to as *arched leveled-planar graphs* [24]. However, while identifying whether a graph has pagenumber one can be completed in linear time, the equivalent problem for identifying graphs with queuenumber one is NP-complete [23]. But when the vertex ordering is pre-specified, finding the pagenumber of the graph is NP-complete, while finding the queuenumber can be done in polynomial time [14].

The next relation demonstrates that graphs with two page book-embeddings are a sub-class of planar graphs. First, it is useful to give the following definitions.

**Definition 17.** A *Hamiltonian cycle* of a graph  $G$  is a cycle in  $G$  that contains each vertex in  $V(G)$  exactly once.

A graph is described as *Hamiltonian* if it contains a Hamiltonian cycle. A graph is *subhamiltonian planar* if it is a subgraph of a planar Hamiltonian graph. The problem of identifying Hamiltonian cycles in graphs is a well known NP-complete problem [29].

**Theorem 18.** [4] *A graph  $G$  has pagenumber at most two if and only if  $G$  is a subgraph of a hamiltonian planar graph.*

*Proof.* It is quite simple to show that all two page embeddings are subhamiltonian planar. Starting with any arbitrary two page book-embedding  $G$  with  $n$  vertices, it is possible to draw  $G$  on the plane with one of its pages above the spine, and the other page below the spine. Clearly this is a planar drawing of  $G$ . Consider the labelling of the vertices in the order of the spine to be  $v_1, v_2, \dots, v_n$ . For any two consecutive vertices  $v_i, v_{(i+1 \bmod n)}$ , either the edge  $(v_i, v_{(i+1 \bmod n)})$  is already in  $G$ , or it can be added to  $G$  without crossing any other edge, since no edges in  $G$  cross the spine. Calling this new graph  $G'$ , it is clearly true that  $G'$  is Hamiltonian, as it contains the Hamiltonian cycle  $v_1, v_2, \dots, v_n, v_1$ . Since the initial graph  $G$  was planar, and none of the edges added in making  $G'$  cross any other edges,  $G'$  is also planar. So the initial graph  $G$  is a subgraph of Hamiltonian planar graph  $G'$ , and hence any arbitrary two page book-embedding is a subhamiltonian planar graph.

From any subhamiltonian planar graph  $G$ , we can add edges to  $G$  to obtain another planar graph  $G'$  with a Hamiltonian cycle. The Hamiltonian cycle in  $G'$  partitions the edges of  $G'$  into three sets, which we will refer to as the edges along the

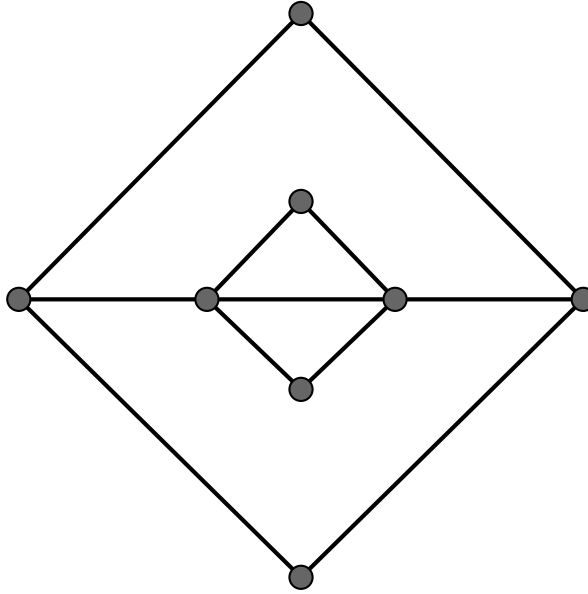


Figure 3.3: A planar graph  $G$

cycle, the edges on the external face of the cycle, and the edges on the internal face of the cycle. The edges along the cycle are exactly the edges in the Hamiltonian cycle of  $G'$ . The *internal face* of the cycle is the area which is enclosed by the Hamiltonian cycle, and the *external face* is the area which is not enclosed by the cycle.

The edges in the Hamiltonian cycle can be considered as being along the spine of the book-embedding, imposing the required order on the vertices.

Ignoring the edges on the external face, the remaining graph is clearly outerplanar, as all the vertices are on the outer face, and none of the edges cross. This is also true if the internal edges are ignored, as all the vertices will then be on the internal face of the remaining graph.

So by embedding the vertices along the spine in order of the Hamiltonian cycle of  $G'$ , we can then embed the edges internal to the cycle on one page, the edges external to the cycle on another page, and the edges along the cycle can be embedded on either of these pages as they are along the spine, and no edges cross the spine. Since  $G$  was a subgraph of  $G'$ , by removing the edges which were added in making  $G'$  we have a two page book-embedding of  $G$ . So any subhamiltonian planar graph can be embedded in two pages.

Since all two page book-embeddings are subhamiltonian planar, and all subhamiltonian planar graphs can be embedded in two pages, the equivalence of these two classes follows.  $\square$

So determining whether a given graph  $G$  has a two page book-embedding is equivalent to determining whether it is subhamiltonian planar. Therefore determin-

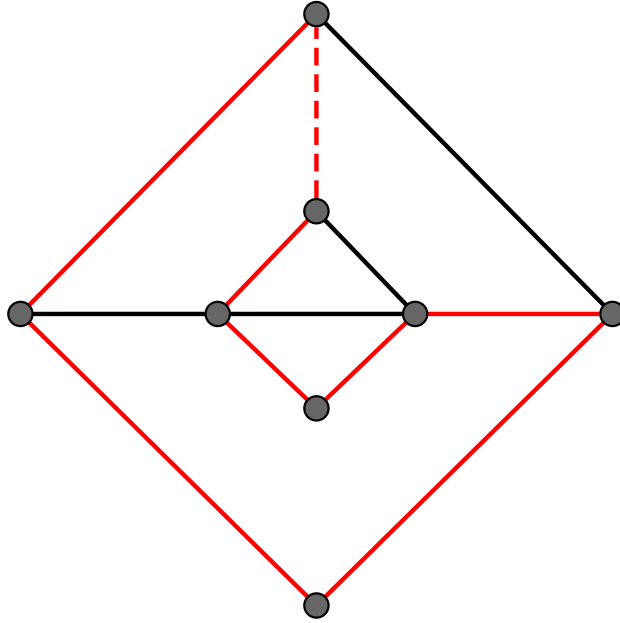


Figure 3.4: A Hamiltonian graph  $G'$  with  $G$  as a subgraph

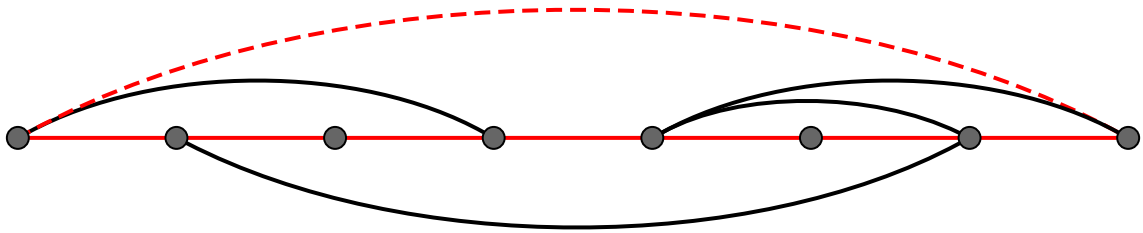


Figure 3.5: Graph  $G'$  laid out on two pages along its Hamiltonian cycle

ing whether a given planar graph has a two page book-embedding is a NP-complete problem [7], since the Hamiltonian circuit problem for maximal planar graphs is known to be NP-complete [44].

The above result provides a trivial lower bound on the pagenumber of non-planar graphs.

**Corollary 19.** *For any non-planar graph  $G$ , there is no book-embedding of  $G$  with fewer than three pages.*

It also should be noted that, while outerplanar graphs are equivalent to graphs with pagenumber equal to one, and sub-hamiltonian planar graphs are equivalent to graphs of pagenumber less than two, there exists no such pagenumber equivalence relation for the class of planar graphs. Since there exist non-Hamiltonian planar graphs, such as in Figure 3.7, it is obvious from Theorem 18 that two pages are not sufficient for book-embeddings of planar graphs. Furthermore, there exist non-planar graphs that can be embedded in three pages, for example  $K_5$ . So the



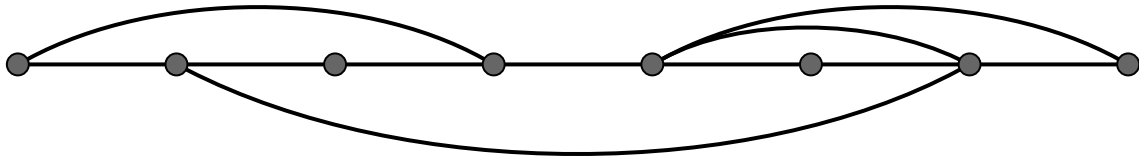


Figure 3.6: A two page book-embedding of a subhamiltonian planar graph

class of planar graphs can not be defined by bounded pagenumber, although from Theorem 35, all planar graphs have pagenumber less than four.

The question of finding a sufficient pagenumber for book-embeddings of planar graphs is one which has an interesting history, as such a bound was initially considered not to exist. Since then the best bound for book-embeddings of planar graphs has been iteratively improved until the current bound of four pages was given by Yannakakis [46]. The topic of book-embeddings of planar graphs will be discussed in more detail in the next chapter.

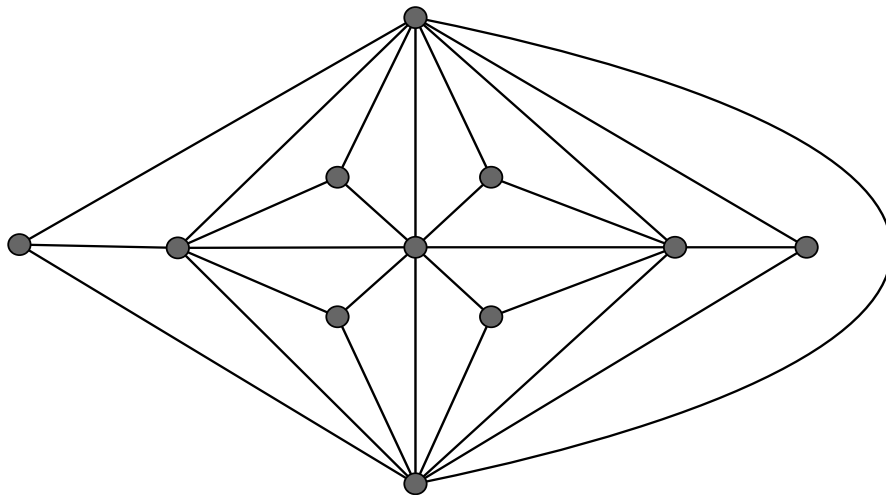


Figure 3.7: A non-Hamiltonian maximal planar graph

### 3.3 Book-Embeddings of Complete Graphs

It is useful to consider the pagenumber for book-embeddings of complete graphs, in order to find a rough upper bound on the pagenumber for any graph given its order.

**Definition 20.** A graph  $G$  is *complete* when, for any pair of distinct vertices  $v, u \in V(G)$ , there is an edge  $(v, u) \in E(G)$ .

The complete graph on  $n$  vertices is denoted  $K_n$ .

**Theorem 21.** [4] For  $n \geq 4$ , the pagenumber of the complete graph  $K_n$  is  $\lceil \frac{n}{2} \rceil$ .

*Proof.* There are  $\frac{n(n-1)}{2}$  edges in a complete graph on  $n$  vertices. A book-embedding of such a graph would have  $n$  external edges, which gives the number of internal edges as  $\frac{n(n-3)}{2}$ . Since there can be at most  $n - 3$  internal edges on any page, for there to be a  $k$ -page book-embedding we require that  $k(n-3) \geq \frac{n(n-3)}{2}$ , which means that the minimum number of pages in which we could embed this many edges is  $\lceil \frac{n}{2} \rceil$ .

Since we know that that pagenumber is at least  $\frac{n}{2}$ , we now need to show that there exist book-embeddings of  $K_n$  with this many pages. For even values of  $n$  we can write  $n = 2k$ , so we need to show that  $K_{2k}$  has a book-embedding with  $k$  pages.

We can consider the vertices as being in a circular layout, with the vertex ordering  $(v_0, v_1, \dots, v_{n-1})$ . For each of  $0 \leq i \leq k - 1$  we can define the sets:

$$E_i = \left\{ v_a v_b : \left\lceil \frac{1}{2}(a + b) \right\rceil \equiv i \pmod{\frac{n}{2}} \right\}$$

We can see that no pair of edges in the same set cross, as that would give two edges  $v_a v_b$  and  $v_i v_j$  such that  $a < i < b < j$ , so  $a + b \leq i + j - 2$  which means that these two edges can't be in the same set by the construction. Since  $k = \frac{n}{2}$  we can see that each of these  $k$  sets are distinct.

Each of these  $k$  disjoint partitions of the edges of  $K_{2k}$  contains  $n - 3$  internal edges which can be embedded without crossing on a single page. By adding the external edges to any of these pages we get a  $k$ -page book-embedding of  $K_{2k}$ .

Since  $K_{2k-1}$  is a subgraph of  $K_{2k}$  the equivalent result for odd values of  $n$  follows from the even case. □

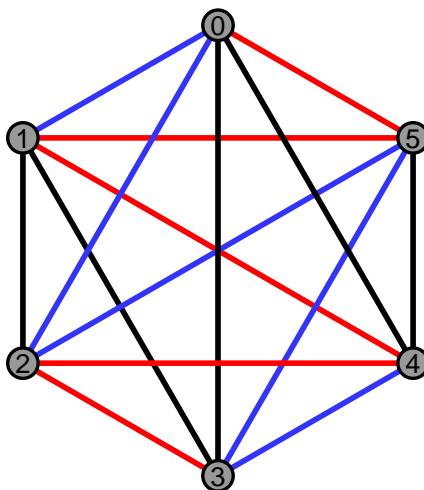


Figure 3.8: 3-page book-embedding of  $K_6$

We can see this illustrated for the case of  $K_6$  in Figure 3.8. If we consider the

edge set for  $i = 0$  we can see that we get the edges  $E_0 = \{v_0v_5, v_5v_1, v_1v_4, v_4v_2, v_2v_3\}$ , which is represented by the edges coloured red.

So given any graph on  $n$  vertices, we know that the maximum number of pages that could be required for it to be embedded in a book is  $\lceil \frac{n}{2} \rceil$ .

A similar result has been found regarding the lower bound on the pagewidth of complete bipartite graphs.

**Definition 22.** A graph  $G$  is *bipartite* if the vertex set  $V(G)$  can be partitioned into two sets such that there are no edges between vertices in the same set.

An alternate and equivalent characterisation of bipartite graphs is as graphs which contain no odd cycles [30].

**Definition 23.** A graph  $G$  is a *complete bipartite graph* if the vertex set  $V(G)$  can be partitioned into sets  $V_1(G)$  and  $V_2(G)$  such that there are no edges between vertices in the same set, and for every pair of vertices  $u \in V_1(G), v \in V_2(G)$  there exists an edge  $(u, v) \in E(G)$ .

The complete bipartite graph with one part containing  $n$  vertices, and the other part containing  $m$  vertices, is denoted  $K_{n,m}$ . It should be noted that  $K_{n,m}$  and  $K_{m,n}$  are isomorphic.

**Theorem 24.** [4] *The pagewidth of the complete bipartite graph  $K_{n,m}$  is at least  $\min(\lceil \frac{n}{2} \rceil, \lceil \frac{m}{2} \rceil)$ .*

Some work has been done in obtaining upper bounds on the pagewidth of complete bipartite graphs [35] [12].

### 3.4 Book-Embeddings for Other Graphs

For many different classes of graphs, there are known results for optimal book-embeddings and bounds on the maximum pagewidth. In this section, some of these results will be introduced.

Since all trees are outerplanar, it follows that they allow a book-embedding with a single page. So for embeddings of trees it is more informative to consider how to construct one page book-embeddings with small pagewidth. For the case of binary trees, there is a construction which is optimal for both the pagewidth and pagewidth.

**Theorem 25.** [7] *All binary trees can be embedded in one page with pagewidth  $\log_2 n$ .*

Embedding binary trees in a single page is quite simple; starting from the root of the tree, traverse the tree using a depth-first search algorithm to obtain a pre-ordering of the vertices, which specifies the layout of the vertices along the spine [7].

The pagewidth of this embedding is given by the height of the tree, which is  $\log_2 n$ . This construction gives a book-embedding of binary trees for which the pagewidth is optimal [5].

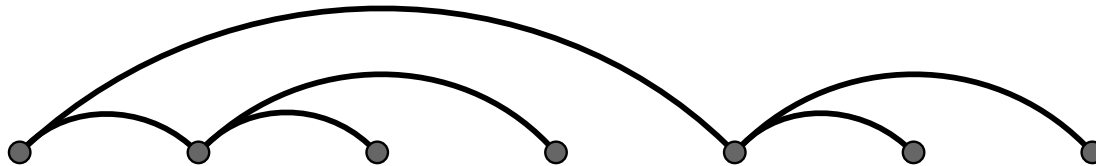


Figure 3.9: One page book-embedding of a binary tree

While in the case of binary trees there are book-embeddings which optimise both pagenumber and pagewidth, for many other classes of graphs layouts with optimal pagenumber have relatively large pagewidth. Many results on book-embeddings consider some trade-off between the pagenumber and pagewidth of book-embeddings. For example, while it was shown in Theorem 16 that outerplanar graphs are equivalent to one page book-embeddings, we can also consider book-embeddings of outerplanar graphs with near-optimal pagenumber and small pagewidth.

**Theorem 26.** [21] *Every  $n$ -vertex outerplanar graph  $G$  with a maximum degree of  $\Delta(G) = d$  admits a book-embedding with two pages and pagewidth  $O(d \log(n))$ .*

In general, adding edges to a graph increases the number of pages in which it allows a book-embedding. An obvious question with regards to the pagenumber of graphs would be the rate of growth with regards to the number of edges.

**Theorem 27.** [32] *All graphs with  $E$  edges have pagenumber of  $O(\sqrt{E})$ .*

Since the pagenumber of a graph is the minimum number of half-planes in which its edges can be embedded, we might consider how this relates other surfaces in which a graph allows an embedding. A graph  $G$  is embeddable in a surface  $\Sigma$  if there is a drawing of  $G$  on  $\Sigma$  such that no edges of  $G$  cross. A surface with genus  $g$  is a sphere with  $g$  handles.

**Definition 28.** The *genus* of a graph  $G$  is the minimum genus of an orientable surface into which  $G$  can be embedded without any edges crossing.

From this we can see that planar graphs have genus zero, as they are able to be embedded on a sphere without any edge crossings. So now we can consider how the genus of a graph relates to the pagenumber.

**Theorem 29.** [31] *Any graph of genus  $g$  has a book-embedding with  $O(\sqrt{g})$  pages.*

This improves upon the earlier result of Heath and Istrail showing that graphs of genus  $g$  allow book-embeddings with  $O(g)$  pages [22].

We can see that this bound is tight, since complete graphs on  $n$  vertices are known to have genus  $\Theta(n^2)$  [38] and were shown earlier to have pagenumber  $\Theta(n)$ .

## 3.5 Topological Book-Embeddings

The standard definition of book-embeddings considered the vertices to be embedded along the spine, with each edge embedded on a single page in such a way that no edges on the same page cross. If the restriction that an edge be embedded on a single page were to be lifted, it would result in a different type of book-embedding, defined as follows.

**Definition 30.** A *topological book-embedding* of a graph  $G$  consists of the vertices of  $G$  embedded along the spine of a book, with each edge embedded on any number of the pages in such a way that no two edges cross.

Topological book-embeddings have also been referred to as *homeomorphic book-embeddings* [28]. When discussing topological book-embeddings, the standard concept of book-embeddings in which edges do not cross the spine have been referred to as *combinatorial book-embeddings* [11].

There has been much research on the topic of topological book-embeddings. Of interest are some classifications of graphs which allow topological book-embeddings on one or two pages, which give results which are comparable to the equivalent cases on book-embeddings.

**Theorem 31.** [11] *A graph  $G$  is outerplanar if and only if it has a topological book-embedding with one page.*

When restricted to one page, topological book-embeddings are equivalent to book-embeddings, as the allowance that edges be embedded on more than one page has no effect for this case. For topological book-embeddings, unlike with general book-embeddings, there is a nice relation for planar graphs and the number of pages.

**Theorem 32.** [11] *A graph  $G$  is planar if and only if it has a topological book-embedding with two pages.*

So while for general book-embeddings, graphs with pagenumber two are sub-hamiltonian planar, allowing edges to cross the spine generalises this to all planar graphs. This is because two pages are equivalent to the plane, so allowing edges to cross the spine means that a two-page topological book-embedding is an embedding of the graph on the plane with vertices embedded in a line.

While we can construct graphs which require an arbitrarily large number of pages for a book-embedding, with topological book-embeddings there is a bounded number of pages which is sufficient for embedding any graph.

**Theorem 33** ([4],[3]). *For every graph  $G$  there is a topological book-embedding of  $G$  which uses three pages.*

*Proof.* In order to construct a three page topological book-embedding for some arbitrary graph  $G$ , we can use any ordering of the vertices along the spine. To embed an edge of the graph, we add *pseudonodes* at each endpoint of the edge, which are artificial nodes representing the first position at which the edge crosses the spine, relative to each of the vertices it connects.

Consider some vertex  $v$ . For every edge adjacent to  $v$  we add pseudonodes along the spine next to  $v$  and use a single page to embed edges from  $v$  to the pseudonodes. Doing this for every vertex in  $G$ , we get  $2m$  pseudonodes in total, where  $m$  is the number of edges in  $G$ . For any edge in  $G$  we have two corresponding pseudonodes, and to embed an edge in the book we draw an edge between the two pseudonodes using the remaining two pages.

For example, if we consider the graph  $K_5$  with its vertices embedded in a line. Since each vertex is adjacent to all the other four vertices, we get four pseudonodes for each vertex. In Figure 3.10 the pseudonodes and the single-page embedding of the connecting edges is shown.



Figure 3.10: The vertices of  $K_4$  with associated pseudonodes

Since each page is a half-plane and we allow crossing the spine, two pages are equivalent to the plane, and we have a linear arrangement of vertices which we need to pair up using non-intersecting edges. When matching up any pair of vertices, the edge does not divide the plane, so it will always be possible to draw a line between the remaining pairs of vertices.

For example, in Figure 3.11 we can see how the pseudonodes for  $K_5$  can be connected using two pages and crossing the spine. The embedding of an edge between two edges can be seen by starting from one of the end vertices and taking the path through the associated pseudonodes. If we ignore the pseudonodes, this gives the embedding of the edge of the initial graph.

So for any graph  $G$  we can add pseudonodes at each of the vertices corresponding with the adjacent edges. Using a single page, we can connect each of the vertices to its pseudonodes. With the two remaining pages we can connect up the pseudonodes associated with the endpoints of each edge of the graph, and from this we obtain a three page topological book-embedding of  $G$  with no crossings.

So for any graph there is topological book-embedding using only three pages.  $\square$

So if a graph is outerplanar it has a one page topological book-embedding, if

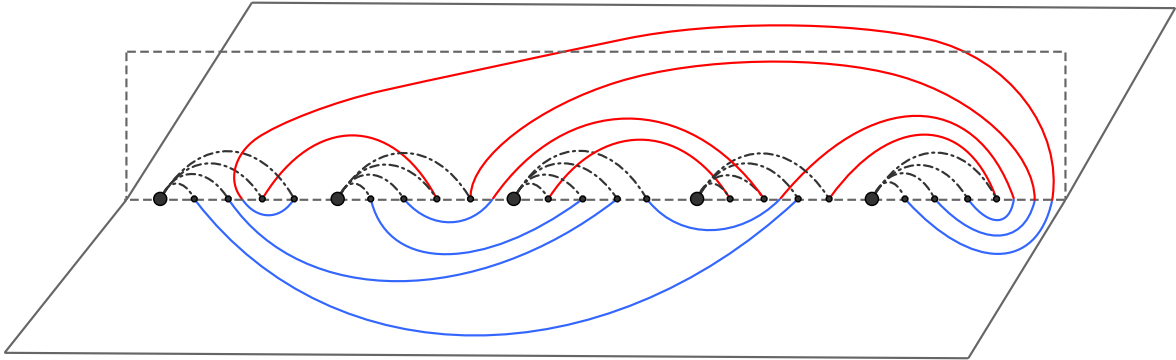


Figure 3.11: A topological book-embedding of  $K_5$

it is planar it can be embedded in two pages, and all other graphs allow three page topological book-embeddings. Most research on book-embeddings considers the number of pages required for embedding particular graphs, however we have seen that determining the number of pages for topological book-embeddings reduces to determining whether a graph is outerplanar, planar or non-planar.

Since topological book-embeddings allow edges to cross the spine, it is a feature of some research on this topic to consider topological book-embeddings which minimise the number of crossings of the spine.

**Theorem 34.** [34] *For any graph  $G$  with  $n$  vertices, there is a three page topological book-embedding of  $G$  in which each edge crosses the spine no more than  $n$  times.*

# Chapter 4

## Book-Embeddings of Planar Graphs

### 4.1 Introduction and History

One obvious question that arises from the topic of book-embeddings of graphs is finding an upper bound on the pagewidth for planar graphs. This question was first considered by Bernhart and Kainen, with the conjecture that no such bound exists, and that there exists planar graphs with arbitrarily large pagewidth [4].

This was later refuted, when Buss and Shor [6] proved that all planar graphs could be embedded in nine pages. This result was improved to seven pages by Heath [20], and further improved by Istrail [26] to six pages.

The latest result on this question was a proof by Yannakakis showing that four pages are sufficient for book-embeddings of planar graphs [46]. He further claimed that there exist planar graphs which can't be embedded in three pages. However, the details of this proof have never been published, and there are currently no known examples of planar graphs that require four pages for a book-embedding.

### 4.2 Embedding Planar Graphs in Books

In this section, the proof for the sufficiency of nine pages for book-embeddings of planar graphs, and consequently that a bound on the pagewidth exists, will be given. While this result was the first to show that it is possible to embed any planar graph in a bounded number of pages, the bound itself has since been improved upon. The current best bound is as follows.

**Theorem 35.** [46] *For any planar graph  $G$  the pagewidth of  $G$  is at most four.*

In proving this result, Yannakakis gives a linear-time algorithm for obtaining a four-page book-embedding of any planar graph. The existence of a bounded pa-



genumber sufficient for book-embeddings of planar graphs has many interesting consequences, as it implies that results in terms of a graphs pagenumber can potentially be used to provide bounds for the class of planar graphs.

For example, Jacobson [27] provides a succinct data structure for planar graphs by providing a more general construction for graphs of bounded pagenumber.

## Nine Page Book-Embeddings

The proof, as given by Buss and Shor [6], on the sufficiency of nine pages for book-embeddings of planar graphs uses the idea that subhamiltonian planar graphs can be embedded in two pages.

The construction in this proof is based on two main ideas. First is the result from Theorem 18, from which we know that the pagenumber of subhamiltonian planar graphs is two. Second is the idea that the vertices of a graph can be divided into successive levels in such a way that every edge connects vertices which are either on the same or adjacent levels. The appropriate nesting of levels guarantees that edges belonging to non-adjacent levels can be assigned independent pages in a book-embedding.

Before describing the proof of this theorem, it is necessary to provide some basic definitions, and to introduce a theorem of Whitney that is required.

**Definition 36.** A  $k$ -cycle is separating if its removal from the graph would result in the graph being separated into two or more distinct connected components.

Separating 3-cycles are referred to as *separating triangles*.

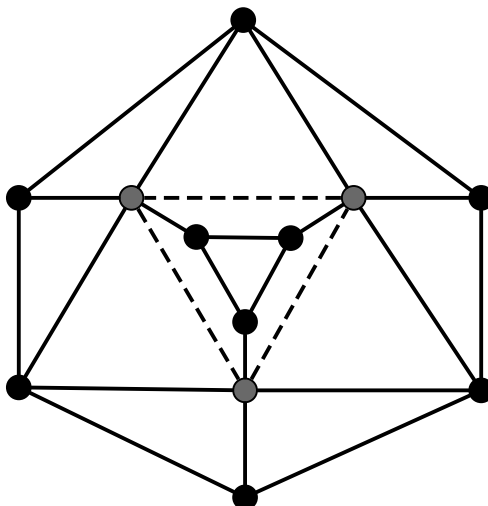


Figure 4.1: A separating triangle in a planar graph

**Definition 37.** A *triangulation* of a planar graph  $G$  is the graph obtained by adding edges to  $G$  that don't cross any other edge until the resulting graph has all triangular faces.

A triangulation of a planar graph results in a maximal planar graph. Note that a book-embedding for a triangulation of a planar graph gives an upper bound on the pagenumber for the original graph, so for this result it suffices to give a construction for maximal planar graphs.

For this construction we shall require the following theorem, which was proven by Hassler Whitney in 1931.

**Theorem 38** (Whitney's Theorem [43]). *A triangulated planar graph with no separating triangles has a Hamiltonian cycle.*

For the purpose of this proof we can consider Whitney's Theorem in the following form, which we will show to be equivalent.

**Lemma 39.** [6] *Consider a graph  $G$  which is embedded in the plane such that all faces of  $G$  are triangulated except for the outer face, and such that there are no separating triangles. Then there exists a two page book-embedding of  $G$  such that each triangular face of  $G$  is on a single page, with this last condition requiring that some edges be embedded on both pages.*

*Proof.* The proof of this lemma uses a three step process to augment graph  $G$  into graph  $G'$ , which meets the criteria of Whitney's Theorem. The first two steps are repeated as many times as possible, while the final step only occurs once.

1. While there exist some vertices  $p$ ,  $q$  and  $r$  which occur consecutively on the outer face, with  $q$  occurring again on the outer face, add edge  $(p, r)$  to the outer face.
2. While there exist some vertices  $p$ ,  $q$  and  $r$  occurring consecutively on the outer face, and there exists some other vertex  $s$  on the outer face for which there is an edge  $(q, s)$ , add a new vertex  $q'$  along with the edges  $(p, q')$ ,  $(q, q')$  and  $(r, q')$  in the outer face.
3. Add a new vertex  $t$  to the outer face, and connect it to all of the vertices on the outer face.

This process is clearly terminating, and the resulting graph contains neither duplicated edges nor separating triangles. So, by Whitney's Theorem,  $G'$  contains a Hamiltonian cycle, and therefore can be embedded on two pages. If the edges of the Hamiltonian cycle are embedded in both of the pages of the book, then the triangles of the graph are preserved. The desired two page book-embedding of the initial graph  $G$  can be obtained by removing the edges and vertices which were added.  $\square$

Consider some planar graph  $G$  which is embedded in the plane. The algorithm of Hopcroft and Tarjan can be used to obtain such an embedding [25]. For any non-triangular faces in  $G$ , add edges until the graph is triangulated.

Denote the set of vertices in  $G$  which are not enclosed in some separating triangle of  $G$  as  $V_0$ .

Assuming that sets  $V_0, V_1, \dots, V_n$  have been defined, let  $V^{n+1} = V(G) - \cup_{i=0}^n V_i$ , and let  $G^{n+1}$  be the subgraph of  $G$  induced by the vertex set  $V^{n+1}$ .

Let  $V_{n+1} \subseteq V^{n+1}$  be the set of vertices in  $G^{n+1}$  which are not enclosed by some separating triangle in some connected component of  $G^{n+1}$ .

Let  $G_n$  be the subgraph of  $G$  induced by the vertex set  $V_n$ . When referring to parts of our graph  $G$  we use the following terms:

- Each subgraph  $G_n$  is a *level* of  $G$ .
- A *section* of  $G$  is a connected component of some subgraph  $G_n$ .
- A section which is inside a section of the preceding level is a *subsection* of that section.

From this we can derive the following lemma, which can then be used in the proof of the sufficiency of nine pages for book-embeddings of planar graphs.

**Lemma 40.** [6] *For any section  $S$  of  $G$ , it is possible to order the vertices along the spine such that the edges of  $S$  and those from  $S$  to its subsections can be embedded in a book with six pages, where only three pages are used for edges to any one subsection. This book-embedding is not dependent on the ordering of the vertices of any subsection.*

*Proof.* By definition, each of the distinct levels of  $G$  has no separating triangles, so we can use the construction from Lemma 39 to embed  $S$  on two pages. Let  $S_1, S_2, \dots, S_m$  be the subsections of  $S$ . For each  $S_i$  there are three vertices of  $S$  in the separating triangle connected to  $S_i$ , which we denote as  $a_i, b_i$  and  $c_i$ , where  $a_i < b_i < c_i$  under the vertex ordering of  $S$ . Let  $a'_i$  be the successor of  $a_i$  in the vertex ordering of  $S$ . Then place  $S_i$  between vertices  $a_i$  and  $a'_i$ . If we have  $a_i = a_j$  for some  $i \neq j$ , then section  $S_i$  is placed before section  $S_j$  if and only if  $b_i > b_j$ .

Since the embedding used places all the triangles of  $S$  on the same page, the triangle  $a_i b_i c_i$  uses one page of the book. If the triangle  $a_i b_i c_i$  is on the first page of the embedding, then the edges  $(S_i, a_i)$ ,  $(S_i, b_i)$  and  $(S_i, c_i)$  are embedded on pages one, three and four, respectively. Otherwise, these edges are embedded on pages two, five and six. The edge  $(S_i, a_i)$  can obviously be placed on the same page as the triangle  $a_i b_i c_i$  as we placed  $S_i$  between vertices adjacent along the spine of the embedding of  $S$ . So the new edges on pages one and two cannot cross any edges of the same page, as they don't span any of the vertices in  $S$ .

If  $(S_i, a_i)$  was embedded on page one, then the edges placed on pages three and four which correspond to the different  $a_i$ 's run parallel to the edges  $(a_i, b_i)$  and  $(a_i, c_i)$  of  $S$ , which are embedded without conflict on page one. Similarly, for  $(S_i, a_i)$  on page two, the same argument holds for the edges placed on pages five and six. Edges which correspond to the same  $a$  don't conflict due to the ordering of the sections  $S_i$ , and they will only be placed on the same page if the corresponding triangles of  $S$  don't conflict.

So this six page book-embedding meets the required conditions.  $\square$

With all the preliminaries covered, a construction for nine page book-embeddings of planar graphs can now be described. In this construction, the boundaries between levels will be separating triangles.

**Theorem 41.** [6] *Any planar graph can be embedded in nine pages.*

*Proof.* Starting with level zero, we can apply Lemma 40 to the sections of each of the levels of the given planar graph. The six pages required by the lemma are always available for embedding the edges of each of these sections, as the edges which connect any section to the previous level only use three pages. Due to the nesting of the sections, there are no other possible conflicts in embedding the edges.  $\square$

# Chapter 5

## Extremal Graph Theory

### 5.1 Introduction to Extremal Graph Theory

Extremal graph theory is the study of maximal or minimal graphs which satisfy particular properties, usually with regards to graph invariants such as order, maximum vertex degree, genus or pagenumber. A standard extremal question would be to consider the smallest and largest size for a graph with some given property, in terms of the number of vertices. The properties in consideration can be in terms of some graph invariant, such as chromatic number or maximum vertex degree.

The inception of extremal graph theory is considered to be in 1941, with Turán's proof of his theorem on the size of  $k$ -clique-free graphs, although many results in this area have been proven earlier, including Mantel's Theorem, which gives a special case of Turán's result for  $k = 3$ . Both Mantel's Theorem, and Turán's generalisation thereof, will be proven in this chapter.

Some other standard results from extremal graph theory are also described, mostly focusing on the minimum and maximum number of edges for certain types of graphs. In particular, results pertaining to the chromatic number and connectivity of graphs will be described.

There are many possible extremal questions related to the topic of book-embeddings, mostly pertaining to the structure of graphs with a given pagenumber. We will consider questions from extremal graph theory which relate to the pagenumber of graphs in Chapter 6.

### 5.2 Turán's Theorem

First proven in 1941, Turán's theorem is considered to be the start of research in extremal graph theory. The general theorem relates to graphs which forbid subgraphs of the complete graph  $K_n$ ; the equivalent result specific to the special case of  $K_3$  is

referred to as Mantel's theorem, and was proven prior to Turán's result.

**Theorem 42** (Mantel's Theorem [1] [33]). *If  $G$  is a  $n$ -vertex triangle-free graph, then the maximum possible number of edges in  $G$  is  $\lfloor \frac{n^2}{4} \rfloor$ .*

*Proof.* Let  $\{i, j\}$  be some edge in a triangle free graph  $G$ . The other  $n - 2$  vertices can be adjacent to at most one of the vertices  $i, j$  without creating a triangle. So  $(\deg(i) - 1) + (\deg(j) - 1) \leq n - 2$ , which means that  $\deg(i) + \deg(j) \leq n$ .

If we sum over all the edges of the graph, the number of times any vertex is counted is equal to its degree, which gives:

$$\sum_{\{i,j\} \in E(G)} (\deg(i) + \deg(j)) = \sum_{i=1}^n \deg(i)^2$$

Using this along with the previous inequality gives:

$$\begin{aligned} \sum_{i=1}^n \deg(i)^2 &= \sum_{\{i,j\} \in E(G)} (\deg(i) + \deg(j)) \\ &\leq \sum_{\{i,j\} \in E(G)} n = n|E(G)| \end{aligned}$$

If we multiply both sides of this by  $n$  we get:

$$n^2|E(G)| \geq \sum_{i=1}^n \deg(i)^2 n = \sum_{i=1}^n \deg(i)^2 \sum_{i=1}^n 1^2$$

Applying the Cauchy-Schwarz inequality, and using the observation that  $\sum_{i=1}^n \deg(i) = 2|E(G)|$ , gives:

$$\begin{aligned} n^2|E(G)| &\geq \sum_{i=1}^n \deg(i)^2 \sum_{i=1}^n 1^2 \\ &\geq \left( \sum_{i=1}^n \deg(i) \right)^2 = 4|E(G)|^2 \end{aligned}$$

From this follows the result that  $|E(G)| \leq \frac{n^2}{4}$ . □

The extremal graphs for Mantel's Theorem are complete bipartite graphs with  $n$  vertices and  $\frac{n}{2}$  in each partition, if  $n$  is even. We can see this has no triangles, as bipartite graphs are by definition. Each vertex is adjacent to all of the  $\frac{n}{2}$  vertices in the other partition, giving exactly  $\frac{n^2}{4}$  edges.

For example, the extremal case for  $n = 10$  is  $K_{5,5}$ , as shown in Figure 5.1, and it has exactly  $\frac{n^2}{4} = 25$  edges.

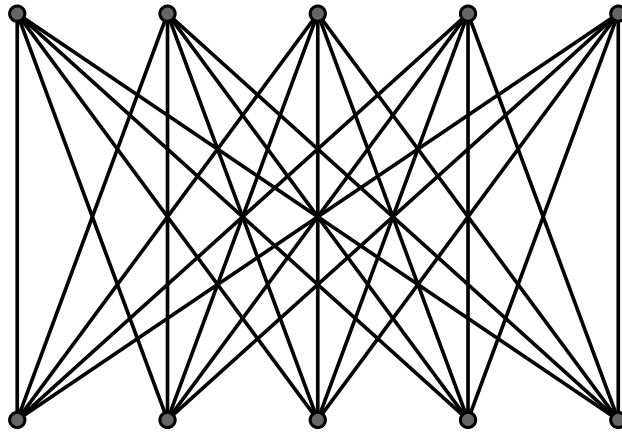


Figure 5.1: The complete bipartite graph  $K_{5,5}$

Turán's Theorem is an extension of Mantel's Theorem to graphs which don't contain a clique of a given size.

**Theorem 43** (Turán's Theorem [42]). *Let  $G$  be a graph on  $n$  vertices that contains no  $r$ -clique. Then the number of edges in  $G$  is at most  $\frac{(r-2)n^2}{2(r-1)}$ .*

*Proof.* The idea for this proof is to use induction on  $n$ . The statement is obviously true for small values of  $n$ . So we prove that it is true for graphs with  $n$  vertices given that it is true for all graphs with less than  $n$  vertices.

Let  $G$  be a graph with  $V(G) = \{1, 2, \dots, n\}$  such that  $G$  has no  $r$ -clique and has the maximal number of edges. It is clear that  $G$  must contain an  $(r-1)$ -clique as otherwise there would still be some edges that could be added.

Let  $A$  be one of the  $(r-1)$ -cliques in  $G$ , and let  $V(A)$  and  $E(A)$  denote the vertex and edge sets of  $A$  respectively. So the number of vertices in  $A$  is  $|V(A)| = r-1$  and the number of edges is  $E(A) = \binom{r-1}{2}$ .

From this we can define the set of vertices not in  $V(A)$ , which we denote as  $B = V(G) \setminus V(A)$ . Since  $B$  contains all vertices except the  $r-1$  in  $V(A)$ , we can see that  $|B| = n - r + 1$ .

Let  $E(B)$  be the number of edges on the vertices in  $B$ , and let  $E(A, B)$  denote the number of edges between  $V(A)$  and  $B$ . By induction, since  $|B| = n - r + 1$  and we assume that the theorem holds for graphs with fewer than  $n$  vertices, it can be seen that  $E(B) \leq \frac{r-2}{2(r-1)}(n - r + 1)^2$ . Since there are no  $r$ -cliques in  $G$ , every vertex  $v \in B$  is adjacent to at most  $r-2$  of the vertices in  $V(A)$ , and hence we obtain  $E(A, B) \leq (r-2)(n - r + 1)$ .

Adding all these together yields the result:

$$\begin{aligned} |E| &\leq \binom{r-1}{2} + \frac{r-2}{2(r-1)}(n-r+1)^2 + (r-2)(n-r+1) \\ &= \frac{r-2}{2(r-1)}n^2 \end{aligned}$$

Which gives the maximum number of edges in an  $r$ -clique-free graph with  $n$  vertices as  $\frac{r-2}{2(r-1)}n^2$ .  $\square$

The extremal graphs for Turán's Theorem are an extension of the extremal case for Mantel's Theorem.

**Definition 44.** A  $k$ -partite graph is a graph in which the vertices can be partitioned into  $k$  disjoint sets such that there are no edges between vertices in the same set.

We can see that the result given by Turán's Theorem is tight because complete  $r-1$ -partite graphs with  $\frac{n}{r-1}$  edges in each part can be seen to contain no  $r$ -clique, and have exactly  $\frac{r-2}{2(r-1)}n^2$  edges.

### 5.3 Other Extremal Results

While Turán's theorem is responsible for initiating the development of the field of extremal graph theory, this area of research has since grown into a major component of graph theory. Similar to Turán's theorem, many of the results in extremal graph theory consider the maximum or minimum number of edges, relative to the number of vertices, which can be in a graph with certain properties.

As examples, we will consider some well known results from this field relating to the chromatic number and connectivity of graphs, and the relation of these parameters to the number of edges.

**Definition 45.** A graph is said to be  $k$ -chromatic when it allows a vertex colouring with  $k$  colours, but no fewer.

From this we can give upper and lower bounds on the number of edges in a  $k$ -chromatic graph.

**Theorem 46.** *The minimum number of edges in a  $k$ -chromatic graph is  $\binom{k}{2}$ . In fact, for every integer  $k$  and for all integers  $n > k$  there exists a  $k$ -chromatic  $n$ -vertex graph with this many edges.*

*Proof.* If a graph  $G$  is  $k$ -chromatic, then we can partition the vertices into sets  $V_1, V_2, \dots, V_k$  such that there are no edges between vertices in the same set, and no such partitioning is possible with fewer than  $k$  partitions.



If there are two partitions  $V_i$  and  $V_j$  such that there is no edge between the vertices in these sets, then we could group these two sets together to reduce the number of sets used, which would mean that we could colour the graph with fewer than  $k$  colours. So for  $G$  to be  $k$ -chromatic, there must be an edge between every pair of the  $k$  sets, and to achieve this requires  $\binom{k}{2}$  edges. So for a graph to be  $k$ -chromatic, there must be at least  $\binom{k}{2}$  edges.

This bound is tight, as if we have  $n$  vertices, where  $n \geq k$ , then we can form a complete graph on  $k$  of these vertices and leave the remaining vertices disconnected. Each of the  $k$  vertices in the connected component must be in a distinct partition, and this graph has exactly  $\binom{k}{2}$  edges. The remaining  $n - k$  vertices which are disconnected can be assigned any of the  $k$  colours without restriction. This graph clearly can't be coloured with fewer than  $k$  colours, and allows a  $k$ -colouring, and is therefore a  $k$ -chromatic graph on  $n$  vertices with  $\binom{k}{2}$  edges.  $\square$

Since we have a lower bound on the number of edges in a  $k$ -chromatic graph, we might also wish to consider the upper bound.

**Theorem 47.** *The maximum number of edges in a  $k$ -chromatic  $n$ -vertex graph is  $\frac{n^2}{2} \left(1 - \frac{1}{k}\right)$ .*

*Proof.* Consider some  $k$ -chromatic graph  $G$  with  $n$  vertices. Since this graph is  $k$ -chromatic, we can separate the vertices into  $k$  sets according to their colouring. Vertices in the same colour set must not be adjacent, but there may exist a path between them via the vertices in the other sets.

Consider the  $i^{\text{th}}$  set of vertices, where the number of vertices in set  $i$  is  $n_i$ . The maximal case is when each of the vertices in any set  $i$  is connected to all the  $n - n_i$  vertices in the other sets, giving  $n_i(n - n_i)$  edges from each of the sets.

Since every edge is counted twice (once per end vertex) we get the relation:

$$\begin{aligned} |E(G)| &\leq \frac{1}{2} \sum_{i=1}^k n_i(n - n_i) \\ &\leq \frac{1}{2} k \frac{n}{k} \left(n - \frac{n}{k}\right) \\ &= \frac{n^2}{2} \left(1 - \frac{1}{k}\right) \end{aligned}$$

The last inequality is because the maximal case occurs when all the sets of vertices are equally of size  $\frac{n}{k}$ .

This bound is tight, because if  $n$  is a multiple of  $k$  we can construct a graph with  $\frac{n}{k}$  vertices of each colour, with every edge between vertices of a different colour present. So any given vertex would be adjacent to all of the  $n - \frac{n}{k} = n \left(1 - \frac{1}{k}\right)$  vertices assigned a different colour. This gives the number of edges in this graph as

$\frac{n^2}{2}(1 - \frac{1}{k})$ . So the extremal graph is complete  $k$ -partite graphs with  $\frac{n}{k}$  vertices in each part.

So the number of edges in a  $k$ -chromatic  $n$ -vertex graph is at most  $\frac{n^2}{2}(1 - \frac{1}{k})$ , and this is a tight bound.  $\square$

We can see that the extremal graph for this case is quite similar to that of Turán's theorem; in fact if we replace  $k$  with  $r - 1$  we find that we have the same upper bound on the number of edges. This is because  $k$ -chromatic graphs are  $k + 1$ -clique free.

Another graph invariant we can consider relates to the connectedness of a graph.

**Definition 48.** A graph is  $k$ -connected if there is no set of  $k - 1$  vertices whose removal would disconnect the graph.

As with the previous results on  $k$ -chromatic graphs, we can consider the minimum number of edges in graphs which are  $k$ -connected.

**Theorem 49.** For all  $k \geq 2$ , every  $k$ -connected  $n$ -vertex graph has at least  $\frac{kn}{2}$  edges, and for all values of  $k$  and  $n$  such that  $kn$  is even there exists a  $k$ -connected graph with this many edges.

*Proof.* For a graph to be  $k$ -connected, the minimum vertex degree must be at least  $k$ , because otherwise there would be some vertex with degree less than  $k$  which could be disconnected from the graph by the removal of its neighbours, violating the condition of  $k$ -connectivity.

So the minimum number of edges is at least when all vertices have degree  $k$ , giving  $|E(G)| \geq \frac{nk}{2}$ , since there are  $n$  vertices all with degree  $k$ . Obviously this bound only makes sense when  $nk$  is even, for otherwise this is not an integer.

Furthermore, this minimum bound on the number of edges is achievable, as we can construct a graph which is  $k$ -connected and has this many edges.

We can consider the vertices as being in a circular layout, with the vertices labelled  $(v_0, v_1, \dots, v_{n-1})$  around the circle. To add in the appropriate edges we must consider the cases where  $k$  is even and odd separately.

*Case 1.* If  $k$  is even then the edge set of our  $k$ -connected graph  $G$  can be defined as:

$$E(G) = \left\{ v_a v_b : |a - b| \equiv i \pmod{n}, \quad \forall i \in \left\{ 1, \dots, \frac{k}{2} \right\} \right\}$$

We can see that every vertex is adjacent to  $\frac{k}{2}$  vertices on the left, and  $\frac{k}{2}$  vertices on the right, so this graph is  $k$ -regular, and therefore has  $\frac{kn}{2}$  edges as required.

To prove that this is  $k$ -connected, we can consider some pair of non-adjacent vertices  $v_j$  and  $v_i$ , and show that removing any set of  $k - 1$  other vertices can't disconnect these two vertices.

We have two paths between  $v_i$  and  $v_j$  around the outside of the circle, so to disconnect these two vertices requires that there is no path between them in both directions around the circle. Since  $v_i$  and  $v_j$  are not adjacent, we know that  $(|i - j| \bmod n) > \frac{k}{2}$  by the construction of the graph. If we remove some vertex  $v_t$  and neither of the adjacent vertices, then we know that the edge  $v_{t-1}v_{t+1}$  is in the graph, so we still have a path between  $u$  and  $v$ . Similarly if we remove two adjacent vertices  $v_tv_{t+1}$  we still have the edge  $v_{t-1}v_{t+2}$ .

By the construction of the graph we can see that we need to remove all vertices  $v_tv_{t+1} \dots v_{t+\frac{k}{2}-1}$  to prevent there from being a path from  $v_i$  to  $v_j$  around the circle in the direction of  $v_t$ , which means that we need  $\frac{k}{2}$  vertices to be removed from each of the halves of the circle between  $v_i$  and  $v_j$ . But this requires  $k$  vertices to be removed, meaning that removing only  $k-1$  vertices from this graph won't disconnect the graph. So this graph is  $k$ -connected, and has  $\frac{kn}{2}$  edges.

*Case 2.* If  $k$  is odd then  $n$  must be even for  $kn$  to be even, so we can use the edges as for the even case of  $k-1$  and add edges between vertices which are opposite along the circle. Formally we would define this as:

$$E(G) = \left\{ v_a v_b : |a - b| \equiv i \pmod n, \quad \forall i \in \left\{ 1, \dots, \frac{k-1}{2} \right\} \right\} \cup \left\{ v_a v_b : |a - b| \equiv \frac{n}{2} \right\}$$

As before, we need to show that for any non-adjacent vertices  $v_i$  and  $v_j$  in this graph, there is no set of  $k-1$  vertices whose removal from the graph would disconnect  $v_i$  and  $v_j$ .

Using the reasoning as for the even case, we can see that we need  $\frac{k-1}{2}$  adjacent vertices to be removed along each of the halves of the circle between  $v_i$  and  $v_j$ . However, we also need to consider the edges  $v_i v_{i+\frac{n}{2}}$  and  $v_j v_{j+\frac{n}{2}}$ . Since  $v_i$  and  $v_j$  are not adjacent, both of the vertices  $v_{i+\frac{n}{2}}$  and  $v_{j+\frac{n}{2}}$  are in the same half of the circle.

To separate the vertices  $v_i$  and  $v_j$  we need to remove  $\frac{k-1}{2}$  consecutive vertices from the half of the circle which does not contain  $v_{i+\frac{n}{2}}$  and  $v_{j+\frac{n}{2}}$ , by the same reasoning as for the even case. Furthermore, we need to remove  $\frac{k-1}{2}$  consecutive vertices from the half which does contain  $v_{i+\frac{n}{2}}$  and  $v_{j+\frac{n}{2}}$ , as well as both the vertices  $v_{i+\frac{n}{2}}$  and  $v_{j+\frac{n}{2}}$ .

Since  $v_i$  and  $v_j$  are not connected, it follows that  $v_{i+\frac{n}{2}}$  and  $v_{j+\frac{n}{2}}$  are not connected, so no  $\frac{k-1}{2}$  consecutive vertices can contain both of these two vertices. So we need to remove some  $\frac{k-1}{2}$  vertices which includes either  $v_{i+\frac{n}{2}}$  or  $v_{j+\frac{n}{2}}$  as well as the other of these two vertices. This means that we need to remove  $\frac{k+1}{2}$  vertices from this half, which gives a total of  $k$  vertices that need to be removed from the graph to disconnect  $v_i$  and  $v_j$ . So we can't disconnect the graph with only  $k-1$  vertices, so the graph is  $k$ -connected and contains  $\frac{kn}{2}$  edges.

So the minimum number of edges in a  $k$ -connected graph is  $\frac{kn}{2}$  and if  $kn$  is even

we can construct a  $k$ -connected graph with this many vertices.  $\square$

Another graph property which we discussed earlier is that of planarity, in which case the graph can be embedded in the plane with no edges crossing. We can give a bound on the maximum number of edges in a planar graph.

**Theorem 50.** *For  $n \geq 3$  the maximum number of edges in a planar graph with  $n$  vertices is  $3n - 6$ .*

*Proof.* Consider a maximal planar graph  $G$ . Let  $m$  denote the number of edges in  $G$ , and  $f$  denote the number of faces.

Since  $G$  is maximal, there is no edge that can be added to  $G$  without violating planarity. Every face in  $G$  must have size three, as if there were any face in  $G$  with size larger than three then we could add one of the edges crossing that face to  $G$  without crossing any other edge, contradicting the assumption of maximality.

Since every face is bounded by three edges, and each edge is adjacent to two faces, then the number of faces  $f = \frac{2m}{3}$ .

From Euler's formula we know that for planar graphs  $n - m + f = 2$ . Combining this with the previous relation gives  $n - m + \frac{2m}{3} = n - \frac{m}{3} = 2$ . Rearranging this gives  $m = 3n - 6$  as required.  $\square$

From Theorem 18 we know that graphs with two page book-embeddings are subhamiltonian planar, so this also gives an upper bound on the number of edges in a two page book-embedding.

# Chapter 6

## Pagenumber and Extremal Results

### 6.1 Introduction

The most ubiquitous concept with regards to book-embeddings is that of the pagenumber of graphs. The nature of the relations between the pagenumber of a graph and other graph invariants is a natural question to arise from research in this field. Types of extremal question pertaining to book-embeddings would be with regards to the maximum or minimum pagenumber of graphs with given properties, or the relationship between the pagenumber and some other graph invariants such as chromatic number, or average vertex degree.

For classes of graphs in which there are known results regarding some other graph invariants, relations between these graph invariants and pagenumber can provide constraints on the number of pages that might be required for book-embeddings of such graphs. Conversely, we can get constraints for other graph invariants for graphs where the pagenumber is known.

In this chapter we give consideration to the types of extremal results which relate other graph invariants to the pagenumber. We give some results which relate pagenumber of a graph to various types of graph thickness parameters. For graphs with bounded pagenumber we give some results relating to the size and chromatic number.

### 6.2 Pagenumber and Size

One of the more common types of extremal questions in graph theory is with regards to the maximum and minimum number of edges in graphs with certain properties. So an obvious question to ask would be the maximum number of edges in an  $n$ -vertex graph such that it can be embedded in  $k$  pages.

**Theorem 51.** [10] *The maximum number of edges in a  $k$ -page book-embedding on*

$n$  vertices is  $(k + 1)n - 3k$ .

*Proof.* Let the vertices of the graph be labelled  $(v_0, v_1, \dots, v_{n-1})$ , in the order of their layout along the outer circle. The edges  $(v_i, v_{i+1}) \pmod{n}$  are the external edges of the book-embedding, with all other edges being internal. The maximum number of external edges in a book-embedding is  $n$ , and we can always add these edges to a page since there are no edges which can cross them.

As any single page of a book-embedding is outerplanar, each page has at most  $2n - 3$  edges, which includes the  $n$  external edges. So the maximum number of internal edges on a page is  $n - 3$ .

Since we can have at most  $n - 3$  internal edges on each of the  $k$  pages, and there are  $n$  external edges, which gives  $k(n - 3) + n = (k + 1)n - 3k$  as the maximum number of edges in a  $k$ -page book-embedding on  $n$  vertices.

This bound is tight, as we can construct  $k$ -page  $n$ -vertex book-embeddings with this many edges.

For each page  $i$ , where  $1 \leq i \leq k$ , we can define the set of edges on the page  $i$  by:

$$E_i = \left\{ v_a v_b : \left\lceil \frac{1}{2}(a + b) \right\rceil \equiv i - 1 \pmod{\frac{n}{2}} \right\}$$

Each of the  $k$  disjoint sets  $E_1, E_2, \dots, E_k$  contains  $n - 3$  internal edges which can be embedded on a single page without crossing. Since the  $n$  external edges can be embedded on any of these pages without conflict, this gives a  $k$ -page  $n$ -vertex book-embedding with a total of  $(k + 1)n - 3k$  edges.  $\square$

As a consequence of this result, we know that any graph on  $n$  vertices with more than  $(k + 1)n - 3k$  edges does not have a book-embedding on  $k$  pages.

### 6.3 Pagenumber and Graph Thickness

The pagenumber of a graph is alternatively referred to as book-thickness, and it can be related to some other measures of the thickness of a graph. There are several types of graph thickness parameters, such as geometric thickness, arboricity and outerplanar thickness. In this section we will consider some relations between the pagenumber with these other graph invariants for measuring graph thickness.

**Definition 52.** The *thickness* of a graph  $G$  is the minimum number of planar subgraphs of  $G$  required such that the union of the subgraphs is equal to  $G$ , and is denoted  $\theta(G)$ .

**Definition 53.** The *outerthickness* of a graph  $G$  is the minimum number of outerplanar subgraphs of  $G$  required such that the union of the subgraphs is equal to  $G$ , and is denoted  $\theta_{op}(G)$ .

The outerthickness of a graph can also be referred to as the graphs *outerplanar thickness* [4].

**Definition 54.** The *geometric thickness* of a graph  $G$  is defined as the minimum number of colours such that there exists a straight line drawing of  $G$  in which all edges that cross are assigned a different colour, and is denoted  $\bar{\theta}(G)$ .

Geometric thickness is also referred to as *real linear thickness*, or *rectilinear thickness* [13].

It is worth noting that as a consequence of Fáry's Theorem [15] that all planar graphs admit a planar straight line drawing, the graphs with thickness of one are exactly the graphs with geometric thickness of one, this being the set of planar graphs.

With regards to these graph properties, the following relations with the pagenum-ber of graphs are known.

**Lemma 55.** [4] *The outerthickness of a graph is at most its pagenumber.*

Intuitively we would expect this to be true, as each page of a graph is outerplanar, but book-embeddings impose greater restrictions on the layout, which would imply that we should require more pages than outerplanar subgraphs.

**Lemma 56.** [4] *The thickness of a graph  $G$  is at most  $\lceil \frac{1}{2}pn(G) \rceil$ .*

This aligns with the previous lemma, as planar graphs which are not outerplanar have pagenumber at least two, implying that the thickness of a graph would be about half its outerthickness.

**Lemma 57.** [13] *The geometric thickness of a graph is at most equal to its pagenumber.*

*Proof.* If we use a circular layout for the book-embedding of a graph  $G$ , we can interpret it as a straight line drawing of  $G$  where each page is assigned a colour. This fits with the definition of geometric thickness of a graph, except with an extra condition imposed on the vertex layout. So we can see that the pagenumber of a graph gives an upper bound on the geometric thickness.  $\square$

From this we can see that the pagenumber of a graph can be interpreted as a restriction of the geometric thickness in which the vertices must be placed in convex position.

More generally, the above thickness parameters have the following order:

$$\theta(G) \leq \bar{\theta}(G) \leq pn(G)$$

$$\theta(G) \leq \theta_{op}(G) \leq pn(G)$$

The pagenumber of a graph imposes greater restrictions on the layout than the other thickness parameters, and is therefore the largest of these values. The thickness is smaller than the other parameters as it is the least restrictive.

Instead of considering partitions of the graph into planar or outerplanar subgraphs, we can consider partitioning the edges into forests. Since all trees are outerplanar, this is a further restriction on the partitions than that of outerplanar thickness.

**Definition 58.** The *arboricity* of a graph  $G$  is the minimum number of forests into which the edges of  $G$  can be partitioned.

**Theorem 59.** [9] For any graph  $G$ , the arboricity of  $G$  is at most  $pn(G) + 1$ .

The arboricity of a graph must be at least the maximum arboricity of its subgraphs. In 1964 Nash-Williams gave a characterisation of arboricity of a graph in terms of the ratio of edges and vertices in any subgraph.

**Theorem 60** (Nash-Williams [36]). For any graph  $G$ , the arboricity of  $G$  is:

$$\max_{S \subseteq G} \left( \left\lfloor \frac{|E(S)|}{|V(S)| - 1} \right\rfloor \right)$$

Since we know that the arboricity of a graph is strictly less than the pagenumber, this gives us a lower bound on the number of pages we might need for a book-embedding.

## 6.4 Pagenumber and Other Graph Invariants

Given the pagenumber of a graph, one might wonder what further information can be inferred regarding other invariants for that graph. Here we will present some results relating the pagenumber of a graph to other invariants such as the average vertex degree and chromatic number. We will also consider relations between the pagenumber and several other measures of the thickness of a graph, such as thickness, outerplanar thickness and arboricity.

**Definition 61.** The *average vertex degree* of a graph  $G$  is defined as the average number of edges incident to any vertex in  $G$ , and can be calculated as  $\frac{2m}{n}$ , where  $m$  is the number of edges, and  $n$  the number of vertices in the graph.

For graphs with bounded pagenumber we can also give a bound on the average vertex degree.

**Theorem 62.** [4] For a graph  $G$  and some integer  $k > 0$ , if  $pn(G) \leq k$  then the average degree of  $G$  is less than  $2k + 2$ .



*Proof.* Consider a  $k$ -page book embedding of a graph  $G$ , and let  $m = |E(G)|$  and  $n = |V(G)|$ . There are at most  $n$  external edges in the embedding, and at most  $n - 3$  other edges in each of the pages. Therefore  $m \leq n + k(n - 3)$  which gives the relation  $pn(G) \leq \frac{m-n}{n-3}$ . From this we know that the average degree is  $\frac{2m}{n} \leq 2\frac{(k+1)n-3k}{n} < 2k + 2$ .

The only assumption was that  $G$  has a  $k$ -page book-embedding, which means that  $pn(G) \leq k$  implies that the average degree is less than  $2k + 2$ .  $\square$

Since book-embeddings can be considered in terms of colouring the chords of a circular embedding of a graph, it seems natural to relate this to the idea of graph colourings. The following result relating the pagenumber and chromatic number of a graph follows directly from Theorem 62 on the average vertex degree.

**Theorem 63.** [4] *For a graph  $G$  and some integer  $k$ , if  $pn(G) \leq k$  then  $\chi(G) \leq 2k + 2$ .*

*Proof.* From Theorem 62 we know that the average degree of a graph with pagenumber at most  $k$  is less than  $2k + 2$ . We show using induction that this means the chromatic is at most  $2k + 2$ .

For small graphs, for example with  $n = 3$ , if the average degree is less than some value  $d$  then we can see that the chromatic number is at most  $d$ . Assume that graphs on  $n - 1$  vertices and average degree  $d - 1$  have chromatic number at most  $d$ . Consider some graph  $G$  of size  $n$  with average degree  $d - 1$ , which means that there exists some vertex  $v$  with degree at most  $d - 1$ . Since the graph  $G \setminus \{v\}$  has chromatic number at most  $d$ , and the vertex  $v$  has degree at most  $d - 1$ , we can add the vertex  $v$  to  $G \setminus \{v\}$  without increasing the number of colours used. So  $G$  has chromatic number  $d$ .

So if we have some graph with pagenumber at most  $k$ , then the average degree is less than  $2k + 2$ . Since the chromatic number is at most one more than the average degree, this means that graphs with pagenumber at most  $k$  have chromatic number at most  $2k + 2$ .  $\square$

There are examples of graph with pagenumber  $k$  and chromatic number  $2k$ . For example, from Theorem 21 we know that  $K_{2k}$  has pagenumber  $k$ , and the chromatic number of this graph is  $2k$ . So this relation is not far from optimal.

# Chapter 7

## Maximal Book-Embeddings

### 7.1 Introduction

When considering extremal questions with regards to book-embeddings, it seems natural to look at graphs which have book-embeddings with  $k$ -pages and which have the maximum number of edges. In Theorem 51 we showed that the maximum number of edges in a book-embedding on  $k$ -pages with  $n$  vertices is  $(k + 1)n - 3k$ . Since there are maximal  $k$ -page book-embeddings with fewer than the maximum number of edges, we can define the sub-class of book-embeddings which are edge-maximal for a given number of pages.

**Definition 64.** An *edge-maximal  $k$ -page book-embedding* is a book-embedding on  $k$ -pages such that no edge can be added on any of the  $k$  pages.

For convenience, when discussing this type of book-embedding we will usually be using the circular layout for book-embeddings, and interpreting each of the pages as a colouring of the edges.

In this chapter, some results on graphs of this class will be discussed, particularly questions with regards to the size of such graphs. The original results for this thesis give bounds on the minimum number of edges in such graphs. This is the first work to be done on the minimum number of edges in edge-maximal book-embeddings.

### 7.2 Size of Maximal Book-Embeddings

Determining the minimum number of edges which can be in a  $k$ -page maximal book-embedding is a complicated question, as an edge can only be absent from such a graph if it would cross an edge on every page. For the cases of one and two page book-embeddings, the number of edges in an edge-maximal book-embedding is exactly given by the maximum number of edges.

**Lemma 65.** For  $k < 3$  and  $n \geq 3$ , the number of edges in a  $k$ -page edge-maximal book-embedding with  $n$  vertices is  $(k + 1)n - 3k$ .

*Proof.* It follows from Theorem 51 that the maximum number of edges in a  $k$ -page book-embedding with  $n$  vertices is  $(k + 1)n - 3k$ . So we need to show that for  $k < 3$  the total number of edges in an edge-maximal book-embedding can be no less than this.

The maximum number of edges occurs when all the pages are triangulated. If the total number of edges is less than the upper bound, then there is some face on a page of the book-embedding which is not triangulated. For the edges inside this face to be absent from the page, they must be embedded on one of the other pages of the book-embedding. Since any non-triangular face has two crossing edges inside it, we require at least two other pages to embed the edges inside this face. So if  $k < 3$  then there aren't enough pages to embed these edges, so there can't be any faces of size greater than three.

So for  $k < 3$  all pages in an edge-maximal book-embedding must be triangulated, so the total number of edges is  $(k + 1)n - 3k$ .  $\square$

When considering the number of edges in a book-embedding, it can be of interest to consider the number of edges that can be of the same page, or colour. We have already noted that the maximum number of internal edges on the same page is  $n - 3$ , which occurs when that page is triangulated. A slightly less obvious result is the minimum number of edges which can be embedded on a single page of an edge-maximal book-embedding.

**Definition 66.** A *monochromatic face* is a face in a book-embedding of a graph for which all edges are on the same page, that is, all the edges of the face have the same colour.

It should be noted that as external edges are considered as being on all pages of the book-embedding they are included in a face on each of the  $k$  pages.

An edge  $e = (u, v)$  is *internal* to a face if  $u$  and  $v$  are vertices bounding that face but  $e$  is not an edge which bounds the face. Two faces are considered to *cross* when there are edges bounding one of the faces which are internal to the other face.

From this concept we get a simple result which is quite useful when considering the structure of edge-maximal book-embeddings.

**Theorem 67.** For  $k \geq 3$ , the largest possible size for a monochromatic face in a  $k$ -page book-embedding is  $2k - 2$ .

*Proof.* For a monochromatic face of size  $f$  to be in a maximal  $k$ -page book-embedding, all of the edges internal to the face must be able to be embedded in the remaining  $k - 1$  pages without crossings. This corresponds with a complete subgraph of size

$f$  which must have pagenumber less than  $k - 2$ . So this problem is equivalent to finding the maximum value of  $f$  such that  $K_f$  can be embedded in  $k - 1$  pages.

From Theorem 21 we know that a complete graph of size  $f$  has pagenumber  $\lceil \frac{f}{2} \rceil$ . So for our edge-maximal book-embedding to have a monochromatic face of size  $f$ , the number of other pages in the book must satisfy  $k - 1 \geq \lceil \frac{f}{2} \rceil$ .

This gives  $f \leq 2k - 2$ , so the maximum size of a monochromatic face in a  $k$ -page book-embedding is  $2k - 2$ .  $\square$

We can see that increasing the size of the cycles in a given colour corresponds with decreasing the number of edges which are embedded on the corresponding page, as the edges internal to that face are not on that page. If a graph has a monochromatic face of size  $f$  then there are  $f - 3$  edges less than if that face were to be triangulated.

So we can use the maximum size of the monochromatic faces to characterise the minimum width of a page in an edge-maximal book-embedding.

**Theorem 68.** *For  $k \geq 3$ , the minimum number of internal edges embedded on any single page of an edge-maximal  $k$ -page book-embedding on  $n$  vertices is  $\lceil \frac{n-2}{2k-4} \rceil - 1$ .*

*Proof.* From Theorem 67, the maximum size of a face on a single page of a maximal book-embedding with  $k$  pages is  $2k - 2$ . Maximising the size of the faces on a page is equivalent to minimising the number of edges on that page, since all the edges internal to that face are absent from the page.

If we consider a page with only the external edges, then there is exactly one internal face on the page. For each internal edge we add, the number of internal faces increases by one. If we denote the number of internal edges by  $e$  then this gives the total number of internal faces as  $e + 1$ . If we denote the size of face  $i$  by  $f_i$ , then from Theorem 67 we know that the size of each internal face on the page is at most  $2k - 2$ , so for all  $i$  we have  $f_i \leq 2k - 2$ .

If we sum up the size of each face, then since internal edges bound two faces, and external edges bound one face, this gives:

$$n + 2e = \sum_{i=1}^{e+1} f_i \leq (e + 1)(2k - 2)$$

Rearranging this gives  $e \geq \frac{n-2}{2k-4} - 1$  as a lower bound on the number of internal edges.

We note that when  $n - 2$  is not a multiple of  $2k - 4$  then we have one face smaller than the maximal size, and  $\lceil \frac{n-2}{2k-4} \rceil - 1$  edges.

So when the size of the faces on a page is maximal, then if the graph has order  $n$  that gives a total of  $\lceil \frac{n-2}{2k-4} \rceil - 1$  edges and this is the minimal number of edges which can be on a page of an edge-maximal book-embedding.  $\square$

### 7.3 Minimum Edge Number

Another question with regards to edge-maximal  $k$ -page book-embeddings is under what conditions can an edge be absent from such a graph. This would be when that edge can not be placed on any page of the graph without crossing some edge already on that page. So the positions of all edges not in that graph must be crossed by at least  $k$  edges that are in the graph, with at least one of these edges on each of the  $k$  pages. This leads in to the question of the minimum number of edges that must be in an edge-maximal  $k$ -page book-embedding.

First we will give a constructive proof for a minimal edge-maximal 3-page book-embedding, this being the first value for which there is a difference between the upper and lower bounds on the number of edges.

From Theorem 67 we know that the maximum size of a face for a 3-page edge-maximal book-embedding is  $2k - 2 = 4$ . In Figure 7.1 we can see that if there were a 5-face in one colour, then there would be an internal edge that could still be added in that colour, which violates the assumption of maximality.

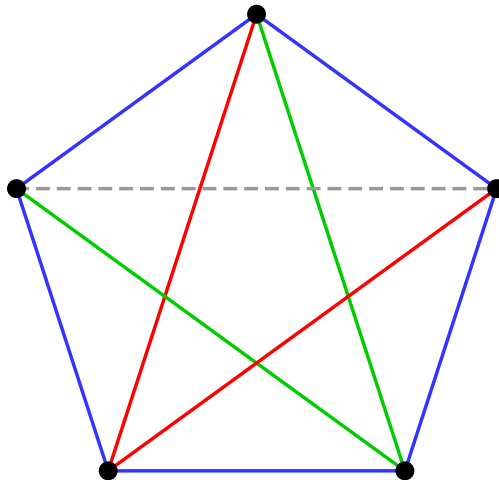


Figure 7.1: A 3-page book-embedding with a monochromatic 5-face is not edge-maximal

We shall refer to faces of size four as 4-faces, and use the following terminology.

**Definition 69.** A page is a *quadrangulation* when every face is a 4-face, except for one 3-face if there are an odd number of vertices.

We will show that the minimum number of edges in a 3-page edge-maximal book-embedding occurs when one page is a quadrangulation, and that this required the remaining two pages to be triangulated.

**Theorem 70.** *Every 3-page edge-maximal book-embedding on  $n$  vertices has at least  $\lceil \frac{7n}{2} \rceil - 8$  edges, with equality occurring if and only if one of the pages is a quadrangulation, with the other two pages triangulations.*

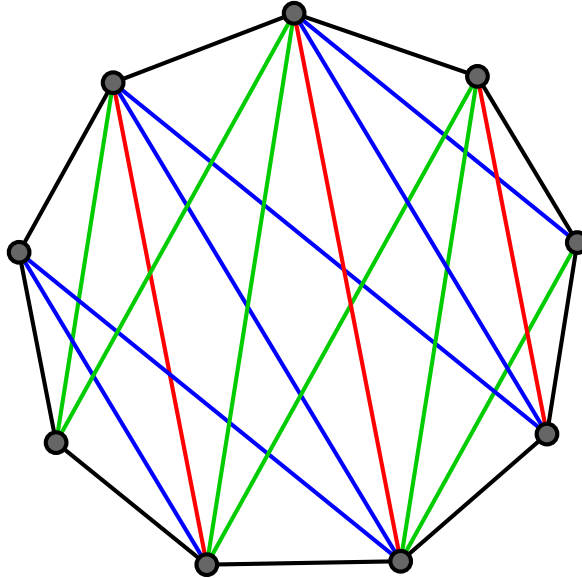


Figure 7.2: A maximal three page book-embedding. The quadrangulated page is represented by the red edges, with external edges represented as black as they can be considered to be embedded on any of the pages

*Proof.* From Theorem 68 we know that the minimum number of edges on a single page of a 3-page edge-maximal book-embedding is  $\lceil \frac{n-2}{2k-4} \rceil - 1$ , which gives the number of internal faces as  $\lceil \frac{n-2}{2k-4} \rceil$ . For  $k = 3$  this gives  $\lceil \frac{n-2}{2} \rceil = \lfloor \frac{n}{2} \rfloor - 1$  internal faces. When we have three pages and there are an odd number of vertices, then it is not possible for all faces of a page to be 4-faces, so there would always be at least one 3-face. We can see that this gives the number of 4-faces in a quadrangulation as  $\lfloor \frac{n}{2} \rfloor - 1$  when  $n$  is odd.

So a quadrangulated page in an edge-maximal 3-page book-embedding has  $\lfloor \frac{n}{2} \rfloor - 1$  faces of size four.

In any 3-page book-embedding, all 4-faces must be disjoint. This is because in any monochromatic 4-face, edges of both other colours must cross that face. So if any two 4-faces were to cross, one of them could not be crossed by both other colours, meaning that one of those 4-faces could be triangulated. But that would mean the book-embedding is not edge maximal, so all 4-faces must be disjoint.

It is not possible for 4-faces to cross, as if an edge which is internal to some 4-face were to be part of another 4-face then there would be some internal edge in one of these faces which could only be included in the same colour as the face it is crossing.

We can see in Figure 7.3 that if one of the edges crossing the red 4-face were to bound another 4-face then there is always an edge internal to that 4-face that can only be included in the same colour as the face it crosses. So for the book-embedding to be maximal, this face must be a triangle. This includes the case in which one of the bounding edges of the faces is external, which is represented as a black edge

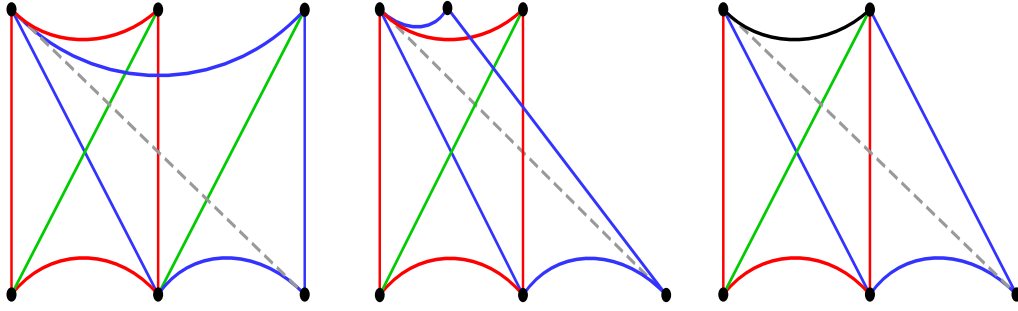


Figure 7.3: Crossing 4-faces - the grey edge must be added to the blue face for the book-embedding to be maximal

bounding both of the 4-faces. So any edges internal to a 4-face must be 3-faces to maintain the maximality of the book-embedding.

Since no two 4-faces can cross there can be at most  $\lfloor \frac{n}{2} \rfloor - 1$  such faces, with the maximal case when all 4-faces are the same colour, which corresponds with one face being a quadrangulation.

Denote the number of faces of size four on page  $i$  by  $f_i$ . So our upper bound on the total number of 4-faces gives:

$$\sum_{i \in \{1,2,3\}} f_i \leq \lfloor \frac{n}{2} \rfloor - 1$$

Denote the number of internal edges on page  $i$  by  $p_i$ . Since the maximum number of internal edges on a page is  $n - 3$ , and this is decreased by one for every 4-face, we have  $p_i = n - 3 - f_i$ , for  $i \in \{1, 2, 3\}$ .

So the total number of edges in the maximal book-embedding is the sum of all the internal edges on each page, along with the  $n$  external edges.

Summing over all the pages gives:

$$\begin{aligned} |E(G)| &= n + \sum_{i \in \{1,2,3\}} p_i = n + \sum_{i \in \{1,2,3\}} (n - 3 - f_i) \\ &= 4n - 9 - \sum_{i \in \{1,2,3\}} f_i \\ &\geq 4n - 9 - \lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{7n}{2} \rceil - 8 \end{aligned}$$

This bound is tight, since we can construct graphs with one page quadrangulated and the other two pages triangulated.

So  $E \geq \lceil \frac{7n}{2} \rceil - 8$ , with equality occurring when one page is a quadrangulation.  $\square$

We can extend the construction used for the three page case to edge-maximal book-embeddings with  $k$  pages, where  $k - 2$  of the pages are mostly quadrangulated,

and the remaining two pages are triangulated. This gives an upper bound on the minimum number of edges in such graphs.

**Theorem 71.** *The minimum possible number of edges in an edge-maximal  $k$ -page  $n$ -vertex book-embedding is fewer than  $\frac{(k+4)n}{2} - \frac{3k}{2}$ .*

*Proof.* We prove this by construction of an edge-maximal  $k$ -page book-embedding on  $n$  vertices with the required number of edges. We triangulate two of the pages, with the remaining  $k - 2$  pages quadrangulations with as many as two triangular faces.

We consider the vertices to be arranged in a circle and labelled in circular order as  $v_1, v_2, \dots, v_n$ . External edges can be embedded on any page without crossing, so we consider them to be present on all pages.

For  $1 \leq i \leq k - 2$  we define the set of edges on page  $i$  as:

$$E_i = \left\{ v_a v_b : (a + b) \equiv i \pmod{\frac{n}{2}} \right\}$$

For the remaining two colours  $k$  and  $k - 1$  the edges on associated pages are:

$$E_{k-1} = \left\{ v_a v_b : \left\lceil \frac{1}{2}(a + b) \right\rceil \equiv \left\lceil \frac{k-1}{2} \right\rceil \pmod{\frac{n}{2}} \right\}$$

$$E_k = \left\{ v_a v_b : \left\lceil \frac{1}{2}(a + b) \right\rceil \equiv 0 \pmod{\frac{n}{2}} \right\}$$

These last two pages are triangulations. Assume that there is some non-triangular face on one of these pages, for example on page  $k$ . If we consider some edge  $v_a v_b$  on page  $k$ , then we know that  $\left\lceil \frac{1}{2}(a + b) \right\rceil \equiv 0 \pmod{\frac{n}{2}}$ . Depending on the parity of  $a + b$  either  $\left\lceil \frac{1}{2}(a + b - 1) \right\rceil \equiv 0 \pmod{\frac{n}{2}}$  or  $\left\lceil \frac{1}{2}(a + 1 + b) \right\rceil \equiv 0 \pmod{\frac{n}{2}}$ , so one of these edges is on page  $k$ . This gives us a triangular face, either  $v_a v_{a+1} v_b$  or  $v_a v_{b-1} v_b$ .

The remaining  $k - 2$  pages are either quadrangulated, or mostly quadrangulated with two triangular faces. For  $1 \leq i \leq k - 2$ , if we consider edge  $v_a v_b$  on page  $i$  then we know that  $(a + b) \equiv i \pmod{\frac{n}{2}}$ . Since  $a + 1 + b - 1 = a - 1 + b + 1 = i$ , the edges  $v_{a+1} v_{b-1}$  and  $v_{a-1} v_{b+1}$  are also on page  $i$ , which gives us the 4-faces  $v_a v_b v_{b-1} v_{a+1}$  and  $v_a v_b v_{b+1} v_{a-1}$  on this page.

We can consider a face  $v_a v_{a+1} v_{b-1} v_b$  on page  $i$  for  $1 \leq i \leq k - 2$ , where  $a + b = a + 1 + b - 1 \equiv i \pmod{\frac{n}{2}}$ . This face is crossed by the edges  $v_a v_{b-1}$  and  $v_{a+1} v_b$ . Since  $(a + b - 1) \equiv i - 1 \pmod{\frac{n}{2}}$  and  $(a + 1 + b) \equiv i + 1 \pmod{\frac{n}{2}}$  we know from the definitions of the edge sets that these edges are already present on the pages corresponding to  $i + 1$  and  $i - 1 \pmod{k}$ , as the 4-faces on page 1 are crossed by the edges on page  $k$  from the definition of  $E_k$ . So no edges can be added to any of the pages, and therefore this defines an edge-maximal book-embedding.



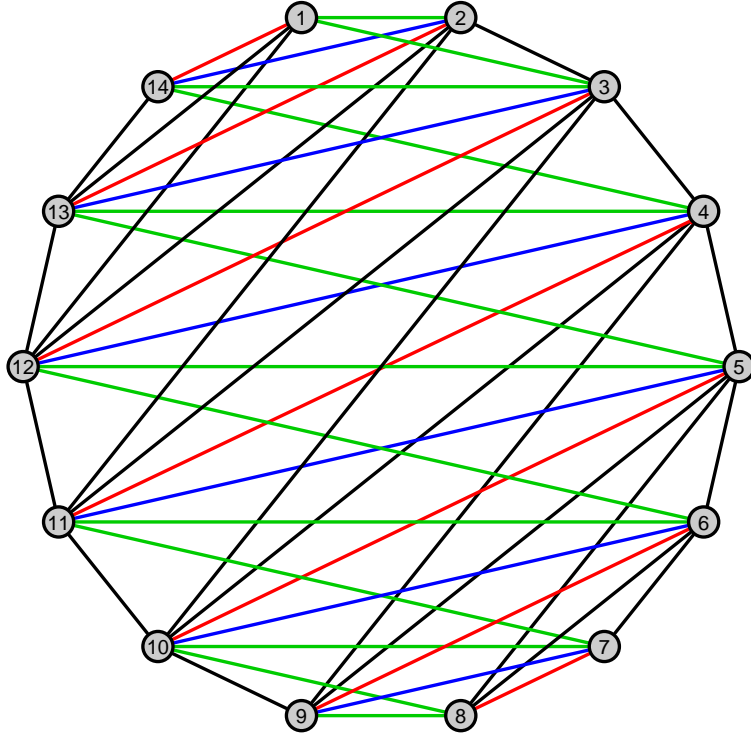


Figure 7.4: A maximal four page book-embedding. The quadrangulated pages are represented by the red and blue edges, with the black and green pages triangulated.

We can see this construction illustrated for the case of  $k = 4$  and  $n = 14$  in Figure 7.4, where the red and blue faces are both maximal since the edges internal to the 4-faces are present on other pages. The red and blue faces are quadrangulated, with the exception of the two triangular faces on the blue page since there are an even number of vertices. The page for colour  $k = 4$  is represented in black, and can be seen to be triangulated with some edges crossing the red 4-faces. Similarly, the page for  $k - 1 = 3$  is represented as the triangulated green page.

The number of internal edges in each of these sets depends on the parity of  $n$ .

*Case 1.* If  $n$  is odd, then the first  $k - 2$  edge sets are always quadrangulations, with  $\lfloor \frac{n}{2} \rfloor - 1$  4-faces on each page, and therefore  $\lfloor \frac{n}{2} \rfloor - 2 = \frac{n-3}{2}$  edges on each of these  $k - 2$  pages. The remaining two pages are triangulated, with  $n - 3$  internal edges each, and there are  $n$  external edges. So the total number of edges is:

$$\frac{n-3}{2}(k-2) + n + 2(n-3) = \frac{k+2}{2}(n-3) + n = \frac{(k+4)n}{2} - \frac{3k}{2} - 3$$

*Case 2.* If  $n$  is even, then we get two cases for the number of edges on a page. For even values of  $i$  we have a quadrangulation on the corresponding page, which gives the number of edges on that page as  $\frac{n}{2} - 2$ . For odd values of  $i$  the page is quadrangulated except for two triangular faces, so we have  $\frac{n}{2} - 1$  edges on these pages.

So there are  $\lfloor \frac{k-2}{2} \rfloor$  pages with  $\frac{n}{2} - 2$  internal edges, and  $\lceil \frac{k-2}{2} \rceil$  pages with  $\frac{n}{2} - 1$  internal edges, which adds up to  $\frac{(k-2)n}{2} - \lfloor \frac{3k}{2} \rfloor + 3$  internal edges on these pages. On the remaining two pages we have  $n - 3$  internal edges, and there are  $n$  external edges. So the total number of edges is:

$$\frac{(k-2)n}{2} - \left\lfloor \frac{3k}{2} \right\rfloor + 3 + 2(n-3) + n = \frac{(k+4)n}{2} - \left\lfloor \frac{3k}{2} \right\rfloor - 3$$

For both of these cases we can see that the total number of edges in the constructed graph is less than  $\frac{(k+4)n}{2} - \frac{3k}{2}$ .  $\square$

For our lower bound on the minimum number of edges in edge-maximal  $k$ -page book-embeddings, we give a restriction on the minimum vertex degree for such graphs, and use this to bound the number of edges relative to the number of vertices.

**Theorem 72.** *The minimum degree of a vertex in a  $k$ -page edge-maximal book-embedding is at least  $\sqrt{2k}$  for all  $k \leq n$ .*

*Proof.* We consider external edges to be embedded on all pages. Let  $v_0 \in V(G)$  be a vertex such that  $\deg(v_0)$  is minimal amongst the vertices of the graph. If there is no vertex to which  $v_0$  is not adjacent, then since  $\deg(v_0)$  is minimal this means that we have a complete graph, in which case the average degree is  $n - 1 \geq \sqrt{2k}$ . So now we consider the case in which there is at least one edge not present in the  $k$ -page edge-maximal book-embedding, and therefore that there is some vertex which is not adjacent to  $v_0$ .

Consider the vertices to have a circular layout  $v_0, v_1, v_2, \dots, v_{n-1}$  in a clockwise order. Define the set of vertices which are adjacent to  $v_0$  as  $N(v_0)$ , the neighbours of  $v_0$ , and note that  $|N(v_0)| = \deg(v_0)$ . This gives the number of edges between vertices in  $N(v_0)$  as  $\binom{\deg(v_0)}{2}$ .

For each of the  $k$  colours of the graph there is at least one monochromatic face of that colour which includes the vertex  $v_0$ , and from this we will prove that there is at least one internal edge of that colour between the neighbours of  $v_0$ .

Consider the page of colour  $i$ . If there is no internal edge of colour  $i$  adjacent to  $v_0$ , then we consider the monochromatic face of colour  $i$  which includes the vertex  $v_0$ . All the edges internal to this face must be on other pages of the book-embedding for it to be maximal, and there must be some internal edge of colour  $i$  in this face, as otherwise we could add edges in colour  $i$  from  $v_0$  to all vertices not adjacent to  $v_0$ , contradicting the assumption that there exist vertices not adjacent to  $v_0$ . So there is at least one internal edge of colour  $i$  on the vertices in  $N(v_0)$ .

If there are edges of colour  $i$  adjacent to vertex  $v_0$ , then since  $\deg(v_0)$  is minimal there is some vertex  $v_t$  which is not adjacent to  $v_0$ . Consider the edges  $v_0v_j$  and  $v_0v_k$  of colour  $i$  such that there is no edge  $v_0v_s$  of colour  $i$  where  $j < s < t$  or  $t < s < k$ .

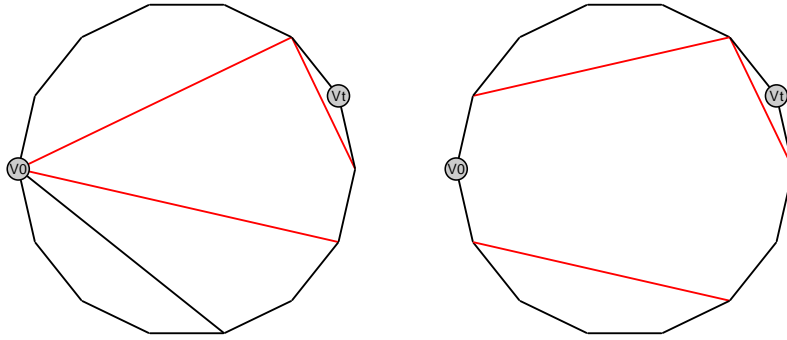


Figure 7.5: There is always an internal edge on a face with vertex  $v_0$ .

Consider the monochromatic face of colour  $i$  which includes the edges  $v_0v_j$  and  $v_0v_k$ . If this face is a triangle, then the edge  $v_jv_k$  means that we have an internal edge on the neighbours of  $v_0$ .

If this face has size greater than three, then there are some vertices between  $v_j$  and  $v_k$  in the monochromatic face. For the book-embedding to be edge-maximal, all of the edges internal to the monochromatic face must be on other pages of the book-embedding, so all of the edges bounding the face between  $v_j$  and  $v_k$  are between vertices adjacent to  $v_0$ . Since vertex  $v_t$  is not adjacent to  $v_0$  and is between  $v_j$  and  $v_k$ , at least one of the edges in this face is an internal edge of the book-embedding.

So for any colour  $i$  there is an internal edge of colour  $i$  between vertices in  $N(v_0)$ .

Since the total number of edges between the neighbours of  $v_0$  is  $\binom{\deg(v_0)}{2}$ , this means that  $k \leq \binom{\deg(v_0)}{2}$ , which can be rearranged to give  $\deg(v_0) > \sqrt{2k}$ .  $\square$

**Theorem 73.** *The minimum number of edges in an edge-maximal  $k$ -page book-embedding is at least  $\sqrt{\frac{k}{2}}n$ .*

*Proof.* From Theorem 72 we know that every vertex in a  $k$ -page edge-maximal book-embedding has degree greater than  $\sqrt{2k}$ . If there are  $n$  vertices in the graph, then counting the edges from each vertex gives:

$$|E(G)| > \frac{\sqrt{2kn}}{2} = \sqrt{\frac{k}{2}}n$$

So a lower bound on the number of edges in a  $k$ -page edge-maximal book-embedding with  $n$  vertices is  $\sqrt{\frac{k}{2}}n$ .  $\square$

So we have a linear upper bound and a square root lower bound on the average degree of vertices in an edge-maximal  $k$ -page book-embedding, in terms of the number of pages. The upper bound gives a construction for a maximal book-embedding with relatively few edges, while there may not exist maximal book-embeddings which achieve the lower bound.

# pages	min # edges	max # edges
1	$2n - 3$	$2n - 3$
2	$3n - 6$	$3n - 6$
3	$\lceil \frac{7n}{2} \rceil - 8$	$4n - 9$
$k$	$\geq \sqrt{\frac{k}{2}}n$ $\leq \frac{(k+4)n}{2} - \frac{3k}{2}$	$(k + 1)n - 3k$

Table 7.1: Upper and lower bounds on the number of edges in a  $k$ -page edge-maximal book-embedding

## 7.4 Conclusion

Book-embeddings of graphs pose many interesting questions in terms of both optimal graph layout and the relation to other classes of graphs. We have considered many results on the pagewidth of major classes of graphs such as outerplanar, planar and complete graphs. We have also considered the relationship between the pagewidth of graphs with several other graph invariants, as part of our consideration of extremal types of questions that might be posed with regards to this topic.

As the original work in this thesis, we established upper and lower bounds on the minimum number of edges in edge-maximal  $k$ -page book-embeddings, giving exact results for up to three pages, with three being the first value for which the minimum number of edges diverges from the maximum. Future research might improve upon these bounds, with the aim of finding an exact solution for the minimum number of edges in these graphs.

In Table 7.1 we can see the current bounds on the number of edges in a  $k$ -page edge-maximal book-embedding. For  $k \leq 2$  there is no variance between the maximum and minimum number of edges, and for three pages we have exact bounds on the extremal values. For general  $k$  the maximum number of edges is known, but the bounds on the minimum number of edges are asymptotically different.

We conclude with a conjecture on the minimum number of edges in  $k$ -page edge-maximal book-embeddings with  $n$  vertices.

**Conjecture 74.** The minimum number of edges in an  $n$ -vertex  $k$ -page edge-maximal book-embedding is  $\Omega(kn)$ .

# Bibliography

- [1] Martin Aigner. Turán’s graph theorem. *Amer. Math. Monthly*, 102(9):808–816, 1995.
- [2] K.I. Appel and W. Haken. *Every planar map is four colorable*, volume 98. American Mathematical Society, 1989.
- [3] G.A. Atneosen. *On the embeddability of compacta in  $n$ -books: intrinsic and extrinsic properties*. PhD thesis, Michigan State University. Dept. of Mathematics, 1968.
- [4] Frank Bernhart and Paul C. Kainen. The book thickness of a graph. *J. Combin. Theory Ser. B*, 27(3):320–331, 1979.
- [5] R.P. Brent and HT Kung. On the area of binary tree layouts. *Information Processing Letters*, 11(1):46–48, 1980.
- [6] J.F. Buss and P.W. Shor. On the pagewidth of planar graphs. In *Proceedings of the Sixteenth Annual ACM Symposium on Theory of Computing*, pages 98–100. ACM, 1984.
- [7] Fan R. K. Chung, Frank Thomson Leighton, and Arnold L. Rosenberg. Embedding graphs in books: a layout problem with applications to VLSI design. *SIAM J. Algebraic Discrete Methods*, 8(1):33–58, 1987.
- [8] P. Clote, S. Dobrev, I. Dotu, E. Kranakis, D. Krizanc, and J. Urrutia. On the page number of RNA secondary structures with pseudoknots. *Journal of Mathematical Biology*, pages 1–21, 2011.
- [9] A.M. Dean and J.P. Hutchinson. Relations among embedding parameters for graphs. *Graph theory, Combinatorics, and Applications*, 1:287–296, 1988.
- [10] Vida Dujmović and David R. Wood. On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):339–357, 2004.

- [11] H. Enomoto and M.S. Miyauchi. Embedding graphs into a three page book with  $O(M \log N)$  crossings of edges over the spine. *SIAM Journal on Discrete Mathematics*, 12(3):337–341, 1999.
- [12] H. Enomoto, T. Nakamigawa, and K. Ota. On the pagewidth of complete bipartite graphs. *Journal of Combinatorial Theory, Series B*, 71(1):111–120, 1997.
- [13] D. Eppstein. Separating thickness from geometric thickness. *Contemporary Mathematics*, 342:75–86, 2004.
- [14] S. Even and A. Itai. Queues, stacks, and graphs. In *Proc. International Symp. on Theory of Machines and Computations*, pages 71–86, 1971.
- [15] István Fáry. On straight line representation of planar graphs. *Acta Univ. Szeged. Sect. Sci. Math.*, 11:229–233, 1948.
- [16] Zvi Galil, Ravi Kannan, and Endre Szemerédi. On nontrivial separators for  $k$ -page graphs and simulations by nondeterministic one-tape Turing machines. *J. Comput. System Sci.*, 38(1):134–149, 1989. 18th Annual ACM Symposium on Theory of Computing (Berkeley, CA, 1986).
- [17] Richard A. Games. Optimal book embeddings of the FFT, benes, and barrel shifter networks. *Algorithmica*, 1(1):233–250, 1986.
- [18] M.R. Garey, D.S. Johnson, G.L. Miller, and C.H. Papadimitriou. The complexity of coloring circular arcs and chords. *SIAM Journal on Algebraic and Discrete Methods*, 1:216, 1980.
- [19] Christian Haslinger and Peter Stadler. RNA structures with pseudo-knots: Graph-theoretical, combinatorial, and statistical properties. *Bulletin of Mathematical Biology*, 61:437–467, 1999. 10.1006/bulm.1998.0085.
- [20] L. Heath. Embedding planar graphs in seven pages. In *25th Annual Symposium on Foundations of Computer Science*, pages 74 – 83, 1984.
- [21] L. Heath. Embedding outerplanar graphs in small books. *SIAM Journal on Algebraic Discrete Methods*, 8(2):198–218, 1987.
- [22] L.S. Heath and S. Istrail. The pagewidth of genus  $g$  graphs is  $O(g)$ . *Journal of the ACM*, 39(3):479–501, 1992.
- [23] L.S. Heath, F.T. Leighton, and A.L. Rosenberg. Comparing queues and stacks as machines for laying out graphs. *SIAM Journal on Discrete Mathematics*, 5:398, 1992.

- [24] L.S. Heath and A.L. Rosenberg. Laying out graphs using queues. *SIAM Journal on Computing*, 21:927, 1992.
- [25] J. Hopcroft and R. Tarjan. Efficient planarity testing. *Journal of the ACM*, 21(4):549–568, 1974.
- [26] S. Istrail. An algorithm for embedding planar graphs in six pages. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Sect. I a Mat.*, 34(4):329–341, 1988.
- [27] G. Jacobson. Space-efficient static trees and graphs. In *30th Annual Symposium on Foundations of Computer Science*, pages 549–554. IEEE, 1989.
- [28] P.C. Kainen and S. Overbay. Book embeddings of graphs and a theorem of Whitney. *Recall*, 13(12):8, 2003.
- [29] R.M. Karp. Reducibility among combinatorial problems. *Complexity of Computer Computations*, 40(4):85–103, 1972.
- [30] Dénes König. Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre. *Math. Ann.*, 77(4):453–465, 1916.
- [31] Seth M. Malitz. Genus  $g$  graphs have pagenumbers  $O(\sqrt{g})$ . *J. Algorithms*, 17(1):85–109, 1994.
- [32] Seth M. Malitz. Graphs with  $E$  edges have pagenumbers  $O(\sqrt{E})$ . *J. Algorithms*, 17(1):71–84, 1994.
- [33] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10:60–61, 1907.
- [34] M.S. Miyauchi. An  $O(mn)$  algorithm for embedding graphs into a 3-page book. *IEICE TRANSACTIONS on Fundamentals of Electronics, Communications and Computer Sciences*, 77(3):521–526, 1994.
- [35] D.J. Muder, M.L. Weaver, and D.B. West. Pagenumbers of complete bipartite graphs. *Journal of Graph Theory*, 12(4):469–489, 1988.
- [36] C.S.J.A. Nash-Williams. Decomposition of finite graphs into forests. *Journal of the London Mathematical Society*, 1(1):12–12, 1964.
- [37] L.T. Ollmann. On the book thicknesses of various graphs. In *Proc. 4th Southeastern Conference on Combinatorics, Graph Theory and Computing*, volume 8, page 459, 1973.
- [38] G. Ringel and JWT Youngs. Solution of the heawood map-coloring problem. *Proceedings of the National Academy of Sciences of the United States of America*, 60(2):438, 1968.

- [39] A.L. Rosenberg. The DIOGENES approach to testable fault-tolerant arrays of processors. *Computers, IEEE Transactions on*, 100(10):902–910, 1983.
- [40] R.P. Swaminathan, D. Giriraj, and DK Bhatia. Thepagenumber of the class of bandwidth- $k$  graphs is  $k - 1$ . *Information Processing Letters*, 55(2):71–74, 1995.
- [41] Robert Tarjan. Sorting using networks of queues and stacks. *J. Assoc. Comput. Mach.*, 19:341–346, 1972.
- [42] Paul Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941.
- [43] Hassler Whitney. A theorem on graphs. *Ann. of Math. (2)*, 32(2):378–390, 1931.
- [44] A. Wigderson. The complexity of the hamiltonian circuit problem for maximal planar graphs. Technical report, Tech. Rep. EECS 198, Princeton University, USA, 1982.
- [45] Mihalis Yannakakis. Linear and book embeddings of graphs. In *VLSI algorithms and architectures (Loutraki, 1986)*, volume 227 of *Lecture Notes in Comput. Sci.*, pages 226–235. Springer, Berlin, 1986.
- [46] Mihalis Yannakakis. Embedding planar graphs in four pages. *J. Comput. System Sci.*, 38(1):36–67, 1989.