A New Exponentially Convergent Spectral Element–Fourier Formulation for Solution of Navier–Stokes Problems in Cylindrical Coordinates

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A number of different formulations for spectral element-based solutions of Navier–Stokes problems in cylindrical coordinates have been proposed.

The radius from the axis appears in the equations, so there has been a tendency to use expansion bases that incorporate radial weighting.

We show that this is not necessary, and with care we can have standard expansion bases, and exponential convergence too.

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Incompressible NSE, primitive variables

$$\partial_t \boldsymbol{u} + \boldsymbol{N}(\boldsymbol{u}) = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0.$$

Coordinates and vector components

$$\boldsymbol{u}(z, r, \theta, t) = (u, v, w)(t)$$

Nonlinear terms (convective form)

$$\begin{aligned}\boldsymbol{N}(\boldsymbol{u}) = (&u\partial_z u + v\partial_r u + \frac{1}{r}[w\partial_\theta u],\\ &u\partial_z v + v\partial_r v + \frac{1}{r}[w\partial_\theta v - ww],\\ &u\partial_z w + v\partial_r w + \frac{1}{r}[w\partial_\theta w + vw])\end{aligned}$$

Standard Step 1

Getting to Fourier space

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Fourier projection/reconstruction in azimuth

$$\hat{\boldsymbol{u}}_k(z,r,t) = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{u}(z,r,\theta,t)\exp(-\mathrm{i}k\theta)\,\mathrm{d}\theta$$

$$\boldsymbol{u}(z,r,\theta,t) = \sum_{k=-\infty}^{\infty} \hat{\boldsymbol{u}}_k(z,r,t)\exp(\mathrm{i}k\theta)$$

Gradient and Laplacian of a complex scalar mode

$$\nabla_k = \left(\partial_z(), \partial_r(), \frac{\mathrm{i}k}{r}()\right), \quad \nabla_k^2 = \partial_z^2() + \frac{1}{r}\partial_r r \partial_r() - \frac{k^2}{r^2}()$$

Divergence of a complex vector mode

$$\nabla\cdot()_k = \partial_z() + \frac{1}{r}\partial_r r() + \frac{\mathrm{i}k}{r}()$$

Fourier-transformed NSE

$$\partial_t \hat{u}_k + [\boldsymbol{N}(\boldsymbol{u})_z]_k^{\wedge} = -\frac{1}{\rho}\partial_z \hat{p}_k + \nu\left(\partial_z^2 + \frac{1}{r}\partial_r r\partial_r - \frac{k^2}{r^2}\right)\hat{u}_k,$$

$$\partial_t \hat{v}_k + [\boldsymbol{N}(\boldsymbol{u})_r]_k^{\wedge} = -\frac{1}{\rho}\partial_r \hat{p}_k + \nu\left(\partial_z^2 + \frac{1}{r}\partial_r r\partial_r - \frac{k^2+1}{r^2}\right)\hat{v}_k - \nu\frac{2\mathrm{i}k}{r^2}\hat{w}_k,$$

$$\partial_t \hat{w}_k + [\boldsymbol{N}(\boldsymbol{u})_\theta]_k^{\wedge} = -\frac{\mathrm{i}k}{\rho r}\hat{p}_k + \nu\left(\partial_z^2 + \frac{1}{r}\partial_r r\partial_r - \frac{k^2+1}{r^2}\right)\hat{w}_k + \nu\frac{2\mathrm{i}k}{r^2}\hat{v}_k.$$

These terms couple the equations

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Standard Step 2

(a) diagonalize
(b) symmetrize

the elliptic operators

Diagonalization: change variables

$$\tilde{v}_k = \hat{v}_k + \mathrm{i}\hat{w}_k \qquad \tilde{w}_k = \hat{v}_k - \mathrm{i}\hat{w}_k$$

which uncouples the linear parts of the NSE

$$\partial_t \hat{u}_k + [\boldsymbol{N}(\boldsymbol{u})_z]_k^{\wedge} = -\frac{1}{\rho}\partial_z \hat{p}_k + \nu\left(\partial_z^2 + \frac{1}{r}\partial_r r \partial_r - \frac{k^2}{r^2}\right)\hat{u}_k,$$

$$\partial_t \tilde{v}_k + [\boldsymbol{N}(\boldsymbol{u})_r]_k^{\sim} = -\frac{1}{\rho}\left(\partial_r - \frac{k}{r}\right)\hat{p}_k + \nu\left(\partial_z^2 + \frac{1}{r}\partial_r r \partial_r - \frac{[k+1]^2}{r^2}\right)\tilde{v}_k,$$

$$\partial_t \tilde{w}_k + [\boldsymbol{N}(\boldsymbol{u})_\theta]_k^{\sim} = -\frac{1}{\rho}\left(\partial_r + \frac{k}{r}\right)\hat{p}_k + \nu\left(\partial_z^2 + \frac{1}{r}\partial_r r \partial_r - \frac{[k-1]^2}{r^2}\right)\tilde{w}_k,$$

$$\partial_z \hat{u}_k + \frac{1}{r}\partial_r r \hat{v}_k + \frac{\mathrm{i}k}{r}\hat{w}_k = 0$$

Symmetrize elliptic operators: multiply NSE by r

$$\partial_t r\hat{u}_k + r[\boldsymbol{N}(\boldsymbol{u})_z]^{\wedge}_k = -\frac{1}{\rho} r\partial_z \hat{p}_k + \nu\left(\partial_z r\partial_z + \partial_r r\partial_r - \frac{k^2}{r}\right)\hat{u}_k,$$

$$\partial_t r\hat{v}_k + r[\boldsymbol{N}(\boldsymbol{u})_r]^{\sim}_k = -\frac{1}{\rho}(r\partial_r - k)\,\hat{p}_k + \nu\left(\partial_z r\partial_z + \partial_r r\partial_r - \frac{[k+1]^2}{r}\right)\tilde{v}_k,$$

$$\partial_t r\hat{w}_k + r[\boldsymbol{N}(\boldsymbol{u})_\theta]^{\sim}_k = -\frac{1}{\rho}(r\partial_r + k)\,\hat{p}_k + \nu\left(\partial_z r\partial_z + \partial_r r\partial_r - \frac{[k-1]^2}{r}\right)\tilde{w}_k,$$

$$\partial_z r\hat{u}_k + \partial_r r\hat{v}_k + \mathrm{i}k\hat{w}_k = 0,$$

where we use $\partial_z r = 0$.

At this point, geometrically singular terms are at worst of type 1/r.

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Conditions at the Axis

(a) Boundary conditions
(b) Nonlinear terms

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Fourier mode (k) dependence of boundary conditions at the axis:

$$k=0: \partial_r \hat{u}_0 = \tilde{v}_0 = \tilde{w}_0 = \partial_r \hat{p}_0 = 0;$$

$$k=1: \hat{u}_1 = \tilde{v}_1 = \partial_r \tilde{w}_1 = \hat{p}_1 = 0;$$

$$k>1: \hat{u}_k = \tilde{v}_k = \tilde{w}_k = \hat{p}_k = 0.$$

All zero

k > 1

Some come from solvability requirements, some from parity.

Values for k=0 are standard for axisymmetric flows.

In particular, $\tilde{w}_1 \neq 0$ allows flow to cross the axis.

Fourier transformed nonlinear terms

$$[\boldsymbol{N}(\boldsymbol{u})]_k^{\wedge} = \{(\hat{u} \circledast \partial_z \hat{u})_k + (\hat{v} \circledast \partial_r \hat{u})_k + \frac{1}{r}[(\hat{w} \circledast \widehat{\partial_\theta u})_k],$$
$$(\hat{u} \circledast \partial_z \hat{v})_k + (\hat{v} \circledast \partial_r \hat{v})_k + \frac{1}{r}[(\hat{w} \circledast \widehat{\partial_\theta v})_k - (\hat{w} \circledast \hat{w})_k],$$
$$(\hat{u} \circledast \partial_z \hat{w})_k + (\hat{v} \circledast \partial_r \hat{w})_k + \frac{1}{r}[(\hat{w} \circledast \widehat{\partial_\theta w})_k + (\hat{v} \circledast \hat{w})_k]\}$$

Using BCs and the convolution theorem,

$$\hat{c}_k = \widehat{ab}_k = (\hat{a} \circledast \hat{b})_k = \sum_{p+q=k} \hat{a}_p \hat{b}_q, \qquad k,p,q \in \mathbb{I}$$

these are all zero at the axis for |k| > 2

For the 1/r type–terms, we also want to know *how* they go to zero with r

Radial variation of nonlinear terms at axis (1)

First, the 1/r–premultiplied terms:

	$k=0$	$k=1$	$k=2$	$k>2$
$\widehat{[N(u)_z]}_k\big\vert_{r=0}:$	quadratic	linear	linear	quadratic,
$\widehat{[N(u)_r]}_k\big\vert_{r=0}:$	quadratic	quadratic	quartic	quadratic,
$\widehat{[N(u)_\theta]}_k\big\vert_{r=0}:$	quadratic	linear	quartic	quadratic,

so after multiplication by 1/r:

$\widehat{[N(u)_z]}_k\big\vert_{r=0}:$	linear	finite	finite	linear,
$\widehat{[N(u)_r]}_k\big\vert_{r=0}:$	linear	linear	cubic	linear,
$\widehat{[N(u)_\theta]}_k\big\vert_{r=0}:$	linear	finite	cubic	linear.

Radial variation of nonlinear terms at axis (2)

Now, the remaining terms

	$k=0$	$k=1$	$k=2$	$k>2$
$\widehat{[N(u)_z]}_k\big\vert_{r=0}:$	0	$\tilde{w}_1\partial_r\hat{u}_0$	0	$0,$
$\widehat{[N(u)_r]}_k\big\vert_{r=0}:$	$\mathrm{Re}(\tilde{w}_1\partial_r\tilde{w}_1)/2$	$\hat{u}_0\partial_z\tilde{w}_1/2$	$\tilde{w}_1\partial_r\tilde{w}_1/4$	$0,$
$\widehat{[N(u)_\theta]}_k\big\vert_{r=0}:$	0	$\hat{u}_0\partial_z\tilde{w}_1/2$	$\tilde{w}_1\partial_r\tilde{w}_1/4$	$0.$

Note this finite term at the axis for k = 0

Discretisation, I

Galerkin treatment
of elliptic operators
(simplified variant)

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After symmetrisation, all elliptic scalar operators in NSE are of form

$$\partial_z r \partial_z \hat{c}_k + \partial_r r \partial_r \hat{c}_k - \frac{\sigma^2}{r}\hat{c}_k = r\hat{f}_k$$

where σ^2 is a real Fourier-mode constant

We convert this to weak form, with weight function ϕ

$$\int_\Omega r\partial_z\phi\partial_z\hat{c}_k + r\partial_r\phi\partial_r\hat{c}_k + \frac{\sigma^2}{r}\phi\hat{c}_k \,\mathrm{d}\Omega = -\int_\Omega r\phi\hat{f}_k \,\mathrm{d}\Omega + \int_{\Gamma_\mathrm{N}} r\phi h \,\mathrm{d}\Gamma$$

where h
represents Neumann BCs on boundary segment Γ_N

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$$\int_\Omega r\partial_z\phi\partial_z\hat{c}_k + r\partial_r\phi\partial_r\hat{c}_k + \frac{\sigma^2}{r}\phi\hat{c}_k \, d\Omega = -\int_\Omega r\phi\hat{f}_k \, d\Omega + \int_{\Gamma_N} r\phi h \, d\Gamma$$

This is the only set of terms that can create singularity problems

The axial BCs are all homogeneous/0, and either of Dirichlet or Neumann type

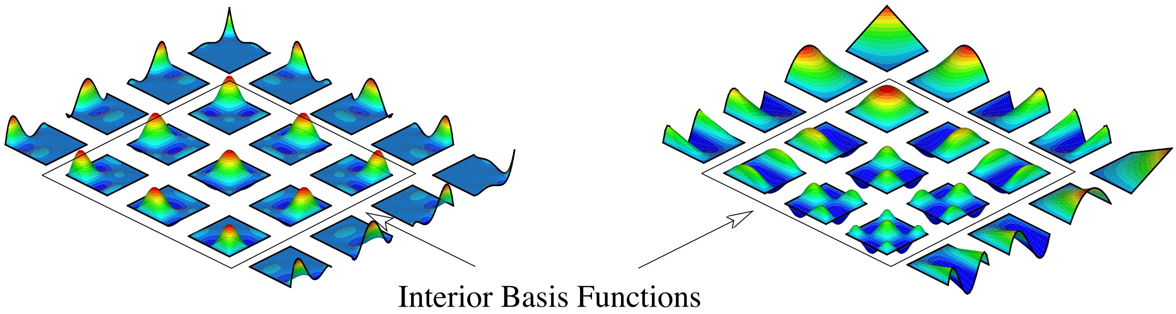
For the Dirchlet axial BCs, we use strong enforcement, meaning the shape functions are zero at r=0

For cases with Neumann axial BCs, (i.e. with possibly non-zero values), it happens that $\sigma^2 = 0$

Consequently there are no problems with axial singularity — no need for special shape functions

Candidate shape functions are sets with an interior–exterior decomposition

nodal modal



Nodal and modal spectral elements make natural choices — but standard finite elements would work too.

Discretisation, 2

Time integration —
velocity correction scheme
(aka "stiffly stable" integration)

1. Pressure PPE, pressure gradient update

$$r\boldsymbol{u}^{*} = -\sum_{q=1}^{J} \alpha_q r\boldsymbol{u}^{(n-q)} - \Delta t \sum_{q=0}^{J-1} \beta_q r\boldsymbol{N}(u^{(n-q)}),$$

$$r\nabla^2 p^{(n+1)} = \frac{\rho}{\Delta t} r\boldsymbol{\nabla}\cdot\boldsymbol{u}^{*}, \quad \text{with}$$

$$r\partial_n p^{(n+1)} = -r\rho\,\boldsymbol{n}\cdot\sum_{q=0}^{J-1}\beta_q\big(\boldsymbol{N}(\boldsymbol{u}^{(n-q)}) + \nu\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{u}^{(n-q)} + \partial_t\boldsymbol{u}^{(n-q)}\big),$$

$$r\boldsymbol{u}^{**} = r\boldsymbol{u}^{*} - \frac{\Delta t}{\rho} r\boldsymbol{\nabla} p^{(n+1)},$$

$(\hat{u}_k, \hat{v}_k, \hat{w}_k, \hat{p}_k)$

Primitive variables

2. Viscous correction

$$r\nabla^2\boldsymbol{u}^{(n+1)} - \frac{r\alpha_0}{\nu\Delta t}\boldsymbol{u}^{(n+1)} = -\frac{r\boldsymbol{u}^{**}}{\nu\Delta t}$$

$(\hat{u}_k, \tilde{v}_k, \tilde{w}_k, \hat{p}_k)$

Diagonalising variables

That completes the algorithm

- All geometric singularities resolved
- No need for special expansions

But: care is needed with this equation:

$$r\nabla^2 p^{(n+1)} = \frac{\rho}{\Delta t} r \boldsymbol{\nabla} \cdot \boldsymbol{u}^*$$

because the RHS is divergence of a vector

$$\frac{\rho}{\Delta t}\left(\partial_z r\hat{u}_k^* + \partial_r r\hat{v}_k^* + \mathrm{i}k\hat{w}_k^*\right) = \frac{\rho}{\Delta t}\left(\partial_z r\hat{u}_k^* + r\partial_r\hat{v}_k^* + \hat{v}_k^* + \mathrm{i}k\hat{w}_k^*\right)$$

incorporating the nonlinear terms.

Specifically, $\hat{v}_0^*$ is non-zero at the axis from $\mathrm{Re}(\tilde{w}_1 \partial_r \tilde{w}_1)/2$

This means we cannot incorporate r into our quadrature.

Test case

Need cross-axial flow to exercise all terms

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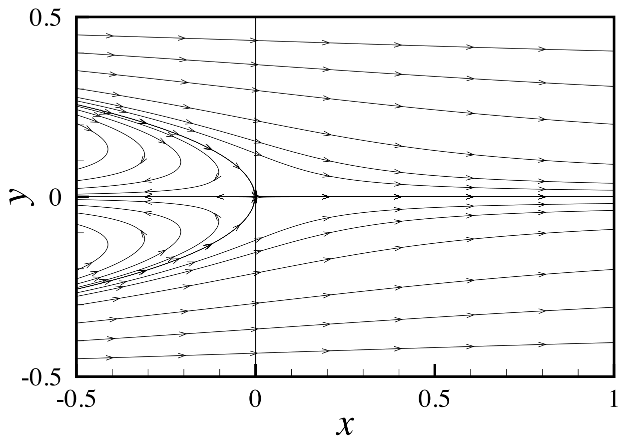
The Kovasznay flow

$$u = 1 - \exp(\lambda x)\cos(2\pi y),$$
$$v = (2\pi)^{-1}\lambda \exp(\lambda x)\sin(2\pi y),$$
$$p = (1 - \exp \lambda x)/2,$$

$$\lambda = Re/2 - (Re^2/4 + 4\pi^2)^{1/2}, \quad Re \equiv 1/\nu.$$



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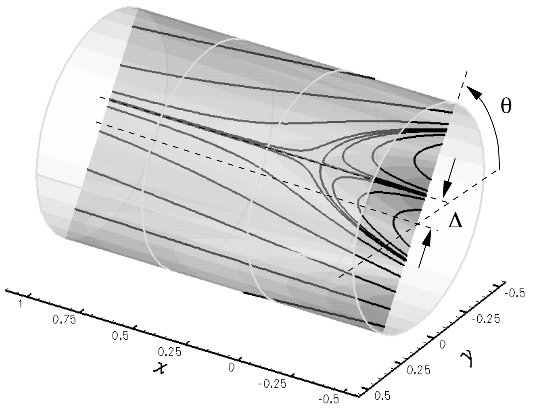
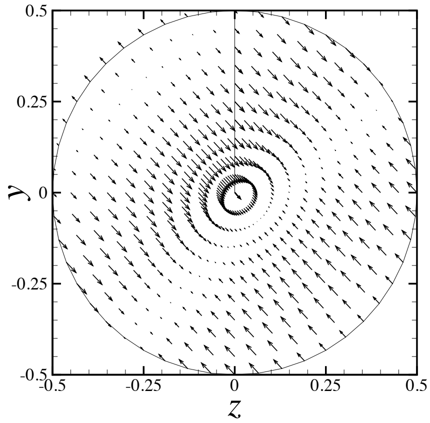
In cylindrical coordinates

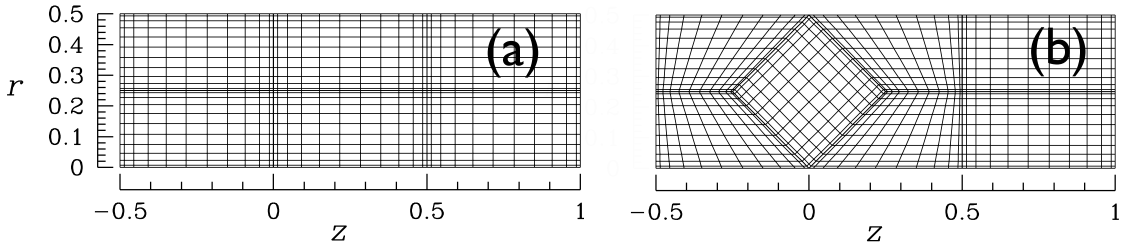
$$u = 1 - \exp(\lambda z)\cos\left(2\pi[r\cos(\theta+\Theta)+\Delta]\right),$$
$$v = (2\pi)^{-1}\lambda\exp(\lambda z)\sin\left(2\pi[r\cos(\theta+\Theta)+\Delta]\right)\cos(\theta+\Theta),$$
$$w = -(2\pi)^{-1}\lambda\exp(\lambda z)\sin\left(2\pi[r\cos(\theta+\Theta)+\Delta]\right)\sin(\theta+\Theta),$$
$$p = (1-\exp\lambda z)/2.$$

Where Δ shifts, and Θ rotates, the solution w.r.t. the axis.

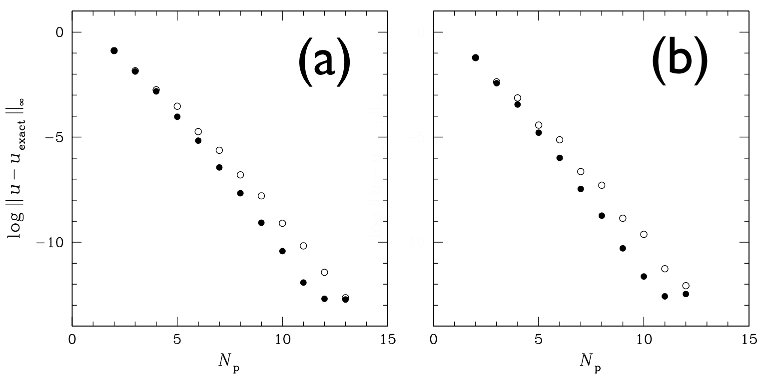




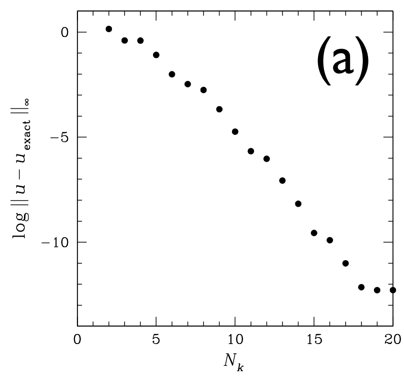
Meshes



In-plane p-convergence

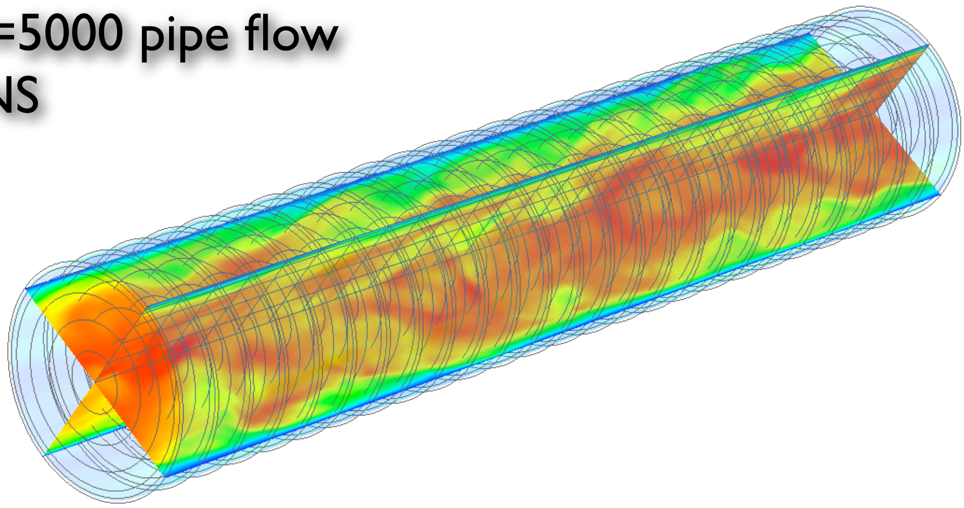


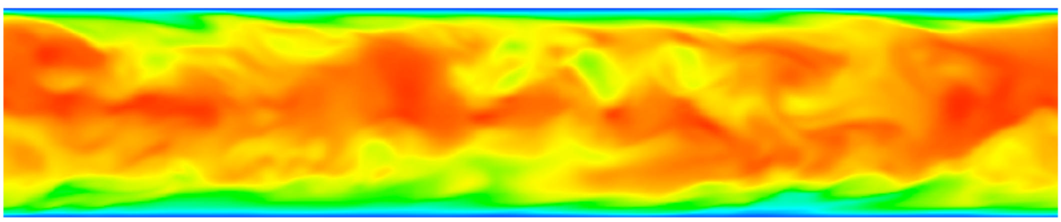
Azimuthal (Fourier)



Applications

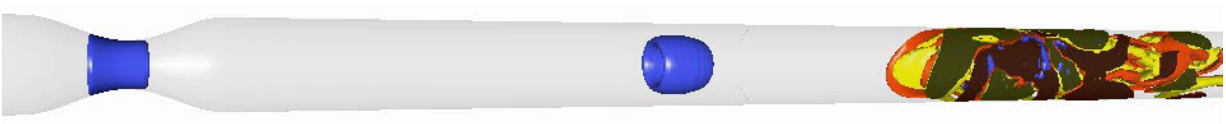
Re=5000 pipe flow
DNS





No sign of axis artifacts

DNS of pulsatile stenotic flow



Thank you

Details of this work appear in JCP **197** (2004).

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