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Computing optimal flow perturbations

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Outline

- Linearised Navier–Stokes equations and temporal behaviour
- Introduce the adjoint Navier–Stokes equations
- Optimal perturbations and optimisation methodology
- Optimal initial (IC) perturbations
  - Optimisation approach
  - Eigenvalue approach
- Optimal inflow boundary (BC) perturbations
  - Optimisation approach
  - Eigenvalue approach
- Applications

Emphasis on analogies and equivalences

Q. What are optimal perturbations?

A. Perturbations that lead to largest kinetic energy growth at finite times.

Q. Why care about optimal perturbations?

A. Extremely large growth can indicate bypass transition to turbulence.

Q. Why worry about optimal inflow boundary perturbations?

A. Because inflow advection is an obvious way for disturbances to enter domain.

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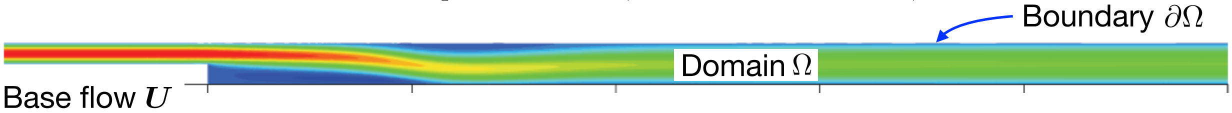
Temporal behaviour of linearized Navier–Stokes operators

(Linearized) Navier–Stokes

$$\partial_t \boldsymbol{u} = -\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{\nabla}p + Re^{-1}\nabla^2\boldsymbol{u}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega,$$



Decompose as $\boldsymbol{u} = \boldsymbol{U} + \boldsymbol{u}'$ where $\boldsymbol{u}'$ is a small/linear perturbation to **base flow** $\boldsymbol{U}$.

Likewise for pressure: $p = P + p'$. $\boldsymbol{U}$ is not necessarily steady in time.

Substitute

$$\partial_t(\boldsymbol{U}+\boldsymbol{u}') = -(\boldsymbol{U}+\boldsymbol{u}')\cdot\boldsymbol{\nabla}(\boldsymbol{U}+\boldsymbol{u}') - \boldsymbol{\nabla}(P+p') + Re^{-1}\nabla^2(\boldsymbol{U}+\boldsymbol{u}'),$$

Expand

$$\partial_t(\boldsymbol{U}+\boldsymbol{u}') = -\boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{U} - \boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{u}' - \boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{U} - \boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}' - \boldsymbol{\nabla}(P+p') + Re^{-1}\nabla^2(\boldsymbol{U}+\boldsymbol{u}'),$$

Split into an equation for base flow and an equation for perturbation:

$$\partial_t\boldsymbol{U} = -\boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{U} - \boldsymbol{\nabla}P + Re^{-1}\nabla^2\boldsymbol{U},$$

$$\partial_t\boldsymbol{u}' = -\boldsymbol{U}\cdot\boldsymbol{\nabla}\boldsymbol{u}' - \boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{U} - u'\cdot\nabla u' - \boldsymbol{\nabla}p' + Re^{-1}\nabla^2\boldsymbol{u}'$$

LNSE

$$\partial_t\boldsymbol{u}' = -\left[\boldsymbol{U}\cdot\boldsymbol{\nabla} + (\boldsymbol{\nabla}\boldsymbol{U})^{\mathrm{T}}\cdot\right]\boldsymbol{u}' - \boldsymbol{\nabla}p' + Re^{-1}\nabla^2\boldsymbol{u}'$$

Shorthand notations for LNSE

Recall $\partial_t \boldsymbol{u}' = -\left[\boldsymbol{U}\cdot\boldsymbol{\nabla} + (\boldsymbol{\nabla}\boldsymbol{U})^{\mathrm{T}}\cdot\right]\boldsymbol{u}' - \boldsymbol{\nabla}p' + Re^{-1}\nabla^2\boldsymbol{u}'$.

Pressure is a constraint field tied to velocity through $\boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0$.

$$p' \equiv \nabla^{-2}\boldsymbol{\nabla}\cdot\left[\boldsymbol{U}\cdot\boldsymbol{\nabla} + (\boldsymbol{\nabla}\boldsymbol{U})^{\mathrm{T}}\cdot\right]\boldsymbol{u}'$$

so, symbolically we may just deal with evolution of the velocity:

$$\partial_t \boldsymbol{u}' = -\left[\boldsymbol{I} - \boldsymbol{\nabla}\nabla^{-2}\boldsymbol{\nabla}\cdot\right]\left[\boldsymbol{U}\cdot\boldsymbol{\nabla} + (\boldsymbol{\nabla}\boldsymbol{U})^{\mathrm{T}}\cdot\right]\boldsymbol{u}' + Re^{-1}\nabla^2\boldsymbol{u}',$$

and arrive at the linear evolution equation $\partial_t \boldsymbol{u}' = L\boldsymbol{u}'$ or $\partial_t \boldsymbol{u}' - L\boldsymbol{u}' = 0$. **LNSE**

For evolution over time interval τ we use the <u>state transition operator</u> $\mathcal{M}(\tau) = \mathrm{e}^{L\tau}$.

$$\boldsymbol{u}'(\tau) = \mathcal{M}(\tau)\boldsymbol{u}'(0) \quad \text{equivalently} \quad \boldsymbol{u}'_\tau = \mathcal{M}(\tau)\boldsymbol{u}'_0$$

This may be applied by integrating (time stepping) the LNSE

$$\boldsymbol{u}'_0 \longrightarrow \boxed{\begin{array}{c}\partial_t \boldsymbol{u}' - L\boldsymbol{u}' = 0\\ \text{integrate forwards}\end{array}} \longrightarrow \boldsymbol{u}'_\tau$$

= apply LNSE = $\boldsymbol{u}'_\tau = \mathcal{M}(\tau)\boldsymbol{u}'_0$

Large-time (asymptotic) linear stability

The eigensystem expansion assumes $\boldsymbol{u}'(\boldsymbol{x},t) = \exp(\lambda_j t)\tilde{\boldsymbol{u}}_j(\boldsymbol{x}) + \text{c.c.}$

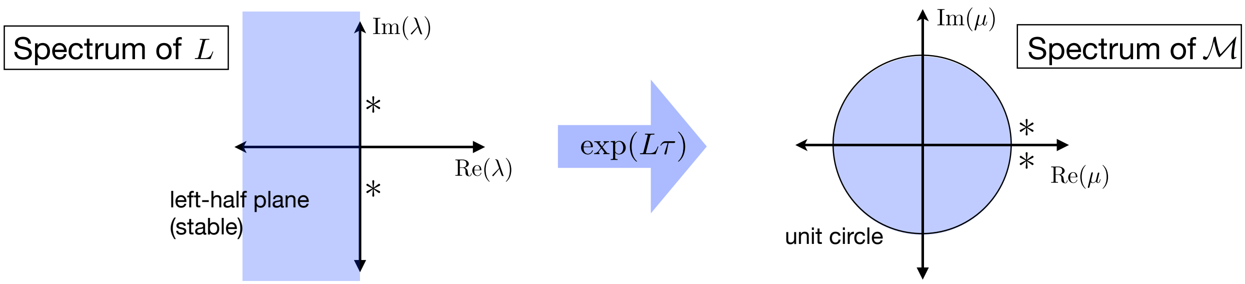
or equivalently $\boldsymbol{u}'(\boldsymbol{x},t+\tau) = \mu_j\tilde{\boldsymbol{u}}_j(\boldsymbol{x}) + \text{c.c.}$

L and $\mathcal{M}(\tau)$ have directly related eigensystems since $\exp L\tau = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{(L\tau)^k}{k!}$

Supposing $\tilde{\boldsymbol{u}}_j$ is an eigenvector of L with corresponding eigenvalue λ_j

$$\mathcal{M}\tilde{\boldsymbol{u}}_j = \exp L\tau\,\tilde{\boldsymbol{u}}_j = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{(L\tau)^k}{k!}\tilde{\boldsymbol{u}}_j = \lim_{N\to\infty}\sum_{k=0}^{N}\frac{(\lambda_j\tau)^k}{k!}\tilde{\boldsymbol{u}}_j = \exp(\lambda_j\tau)\tilde{\boldsymbol{u}}_j$$

i.e. $\tilde{\boldsymbol{u}}_j$ is also an eigenvector of $\mathcal{M}$ and the corresponding eigenvalue is $\mu_j = \exp\lambda_j\tau$.



It is more convenient numerically to search for dominant eigenvalues of $\mathcal{M}(\tau)$ than the most unstable eigenvalues of L. **Either set gives the large-time behaviour.**

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Timestepper approach to eigensystems

Outer loop: based on repeated application of operator $\mathcal{M}$ on an initial vector.

1. Generate a Krylov subspace *T* of dimension *N* × *k* (where *N* >> *k*) by repeated application of $\mathcal{M}$ via inner loop:

$$T = \{\boldsymbol{u}'_0, \mathcal{M}\boldsymbol{u}'_0, \mathcal{M}^2\boldsymbol{u}'_0, \dots, \mathcal{M}^{k-1}\boldsymbol{u}'_0\}$$

$\boldsymbol{u}'_0 \longrightarrow$ [$\partial_t \boldsymbol{u}' - \boldsymbol{L}\boldsymbol{u}' = 0$ integrate forwards] $\longrightarrow \boldsymbol{u}'_\tau = \mathcal{M}\boldsymbol{u}'_0$

2. QR factorize matrix *T*

$$T = QR$$

3. Calculate (*k*×*k*) Hessenberg matrix *H* from *R*

$$h_{i,j} = \frac{1}{r_{j,j}}\left(r_{i,j+1} - \sum_{l=0}^{j-1} h_{i,l} r_{l,j}\right)$$

4. Calculate eigensystem of *H* in *k* × *k* subspace (e.g. LAPACK).
5. If converged, stop and project back to full space, else discard oldest vector in *T*, carry out one more integration of $\mathcal{M}$, go to step 2.

Implicitly-restarted Arnoldi method (ARPACK) gives similar performance.

We can find dominant eigenvalues of an operator without constructing it.

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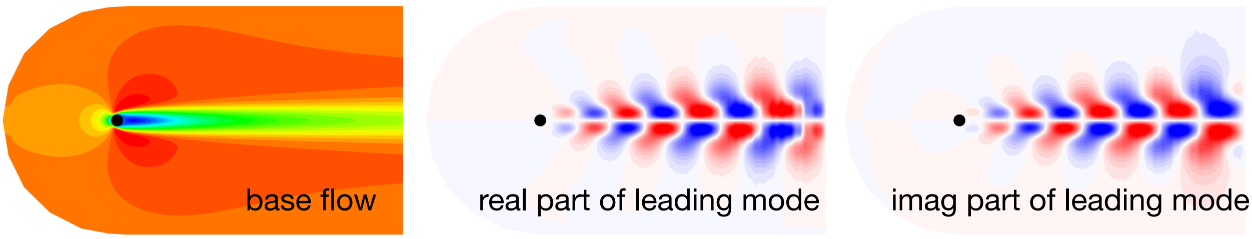
Examples of large-time (eigenmodal) instability

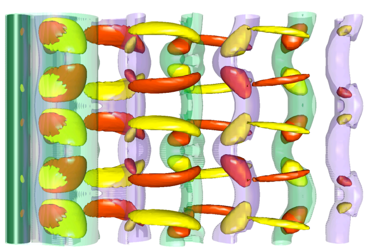
2D instability of steady cylinder wake



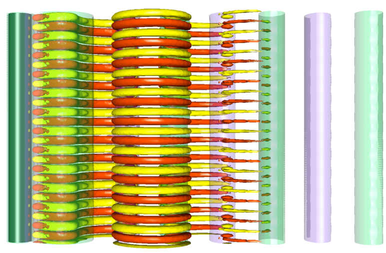
Jackson *JFM* **182** (1987)

3D instability of 2D time-periodic cylinder wake

Mode A

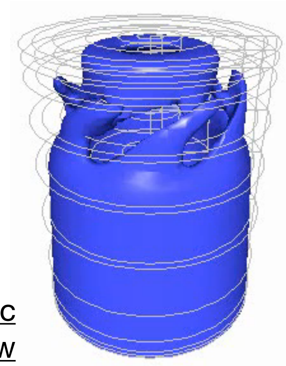


Mode B



Barkley & Henderson *JFM* **322** (1996)

m = 6 mode

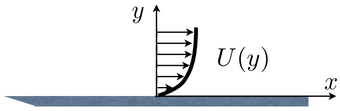


3D instability of 2D time-periodic cylindrical cavity flow

Blackburn *PF* **14** (2002)

Transient growth from initial conditions

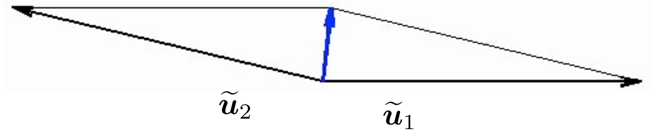
The linearized Navier–Stokes operator is in general non-symmetrical – this is easy to see in the case of parallel shear flows where the base flow ***U***=(*U*(*y*),0,0).



$$\left[\begin{array}{ccc|c} \partial_t + U\partial_x - \nabla^2 & \partial_y U & 0 & \partial_x \\ 0 & \partial_t + U\partial_x - \nabla^2 & 0 & \partial_x \\ 0 & 0 & \partial_t + U\partial_x - \nabla^2 & \partial_x \\ \hline \partial_x & \partial_y & \partial_z & 0 \end{array}\right] \left\{\begin{array}{c} u' \\ v' \\ w' \\ \hline p' \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \end{array}\right\}$$

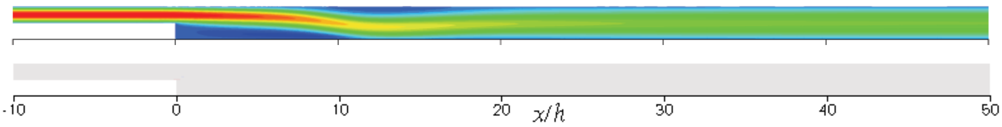
It follows that the eigenmodes of the problem are non-orthogonal and it turns out that <u>even if all modes are stable</u> a perturbation can produce (perhaps very large) algebraic energy *growth* at short times, as opposed to exponential *decay*.

For mode shapes, non-orthogonality means $\int_\Omega \tilde{\boldsymbol{u}}_i \cdot \tilde{\boldsymbol{u}}_j \, \mathrm{d}\Omega \neq 0$



(Base flow)

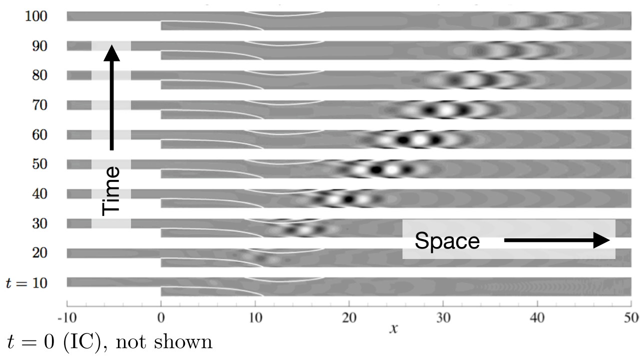
Perturbation vorticity



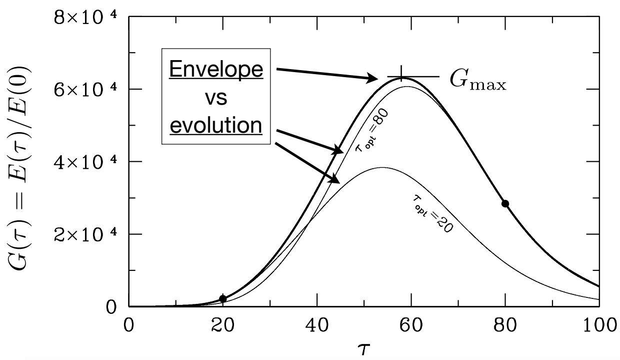
Focus changes from long-time growth to transient growth though ultimately we still expect to see exponential (eigensystem) behaviour as $t \rightarrow \infty$.

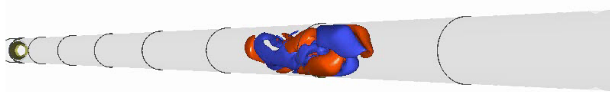
Transient growth (from ICs) 101

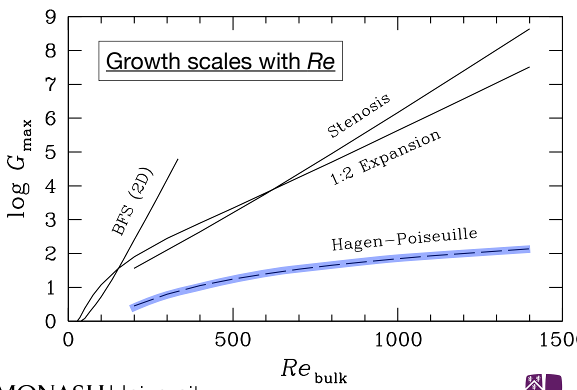
<u>Perturbation vorticity evolution, optimal IC</u>

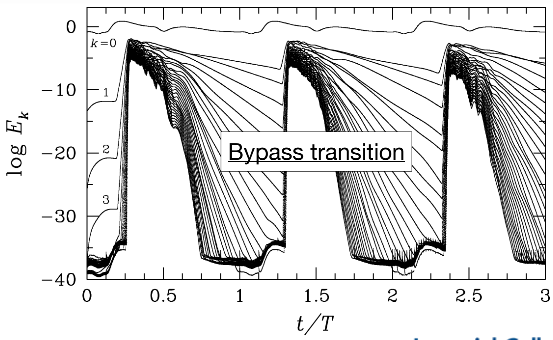


$t = 0$ (IC), not shown









Optimisation toolkit

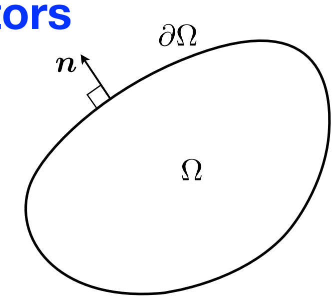
Adjoint variables and operators

For steady applications, the adjoint variable $\boldsymbol{v}^*$ and operator A^* are defined such that

$$(\boldsymbol{v}^*, A\boldsymbol{v}) = (A^*\boldsymbol{v}^*, \boldsymbol{v})$$

where $(\boldsymbol{a}, \boldsymbol{b}) = \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b} \, \mathrm{d}V$ on domain Ω

and where $\boldsymbol{v}$ and $\boldsymbol{v}^*$ have 'compact support' in Ω



For unsteady problems (NSE) we have also to consider the temporal domain, say [0, τ] so the overall domain is Ω × [0, τ] and now

$$\langle \boldsymbol{u}^*, \partial_t \boldsymbol{u}' - L\boldsymbol{u}' \rangle = \langle \boldsymbol{u}', \partial_t \boldsymbol{u}^* + L^*\boldsymbol{u}^* \rangle$$

where $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \int_0^\tau \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b} \, \mathrm{d}V \mathrm{d}t$

Starting from Linearised Navier–Stokes equations (LNSE)

$$\partial_t \boldsymbol{u}' = -\boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{u}' - (\boldsymbol{\nabla}\boldsymbol{U})^{\mathrm{T}} \cdot \boldsymbol{u}' - \nabla p' + Re^{-1}\nabla^2 \boldsymbol{u}' \text{ with } \boldsymbol{\nabla} \cdot \boldsymbol{u}' = 0$$

$$\boldsymbol{u}'_\tau = \mathcal{M}(\tau)\boldsymbol{u}'_0$$

Short form
$\partial_t \boldsymbol{u}' - L\boldsymbol{u}' = 0$

Integration by parts → Adjoint NSE (ANSE)

$$-\partial_t \boldsymbol{u}^* = +\boldsymbol{U} \cdot \boldsymbol{\nabla} \boldsymbol{u}^* - \boldsymbol{\nabla}\boldsymbol{U} \cdot \boldsymbol{u}^* - \nabla p^* + Re^{-1}\nabla^2 \boldsymbol{u}^* \text{ with } \boldsymbol{\nabla} \cdot \boldsymbol{u}^* = 0$$

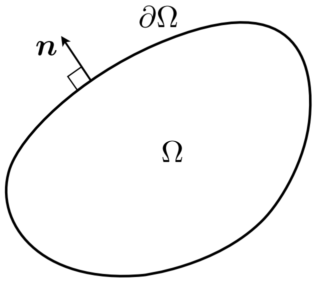
Short form
$\partial_t \boldsymbol{u}^* + L^*\boldsymbol{u}^* = 0$

NB: $\boldsymbol{u}^*_0 \longleftarrow$ $\partial_t \boldsymbol{u}^* + L^*\boldsymbol{u}^* = 0$ integrate backwards $\longleftarrow \boldsymbol{u}^*_\tau$

$$\boldsymbol{u}^*_0 = \mathcal{M}^*(\tau)\boldsymbol{u}^*_\tau$$

Boundary conditions

Recall $(\boldsymbol{a},\boldsymbol{b}) = \int_\Omega \boldsymbol{a}\cdot\boldsymbol{b}\,\mathrm{d}V \qquad \langle \boldsymbol{a},\boldsymbol{b}\rangle = \int_0^T\int_\Omega \boldsymbol{a}\cdot\boldsymbol{b}\,\mathrm{d}V\mathrm{d}t$



Compact support allowed IBP without regard to space-time boundary conditions.

Re-introducing terminal, boundary conditions:

Apply IBP to $-\langle \boldsymbol{u}^*, \partial_t\boldsymbol{u}' - L\boldsymbol{u}'\rangle$

$$-\langle \boldsymbol{u}^*, \partial_t\boldsymbol{u}' - L\boldsymbol{u}'\rangle = \langle \boldsymbol{u}', \partial_t\boldsymbol{u}^* + L^*\boldsymbol{u}^*\rangle + \int_0^\tau\int_\Omega -\partial_t(\boldsymbol{u}'\cdot\boldsymbol{u}^*)\,\mathrm{d}V\mathrm{d}t$$

$$+\int_0^\tau\int_\Omega \boldsymbol{\nabla}\cdot\{-\boldsymbol{U}(\boldsymbol{u}'\cdot\boldsymbol{u}^*)+\boldsymbol{u}'p^*-\boldsymbol{u}^*p'+Re^{-1}[(\boldsymbol{\nabla}\boldsymbol{u}')\cdot\boldsymbol{u}^*-(\boldsymbol{\nabla}\boldsymbol{u}^*)\cdot\boldsymbol{u}']\}\,\mathrm{d}V\mathrm{d}t$$

Exchange order of integration and apply divergence theorem

$$-\langle \boldsymbol{u}^*, \partial_t\boldsymbol{u}' - L\boldsymbol{u}'\rangle = \langle \boldsymbol{u}', \partial_t\boldsymbol{u}^* + L^*\boldsymbol{u}^*\rangle - (\boldsymbol{u}^*_\tau, \boldsymbol{u}'_\tau) + (\boldsymbol{u}^*_0, \boldsymbol{u}'_0)$$

Volume integrals involving terminal conditions

$$+\int_0^\tau\int_{\partial\Omega} \boldsymbol{n}\cdot\{-\boldsymbol{U}(\boldsymbol{u}'\cdot\boldsymbol{u}^*)+\boldsymbol{u}'p^*-\boldsymbol{u}^*p'+Re^{-1}[(\boldsymbol{\nabla}\boldsymbol{u}')\cdot\boldsymbol{u}^*-(\boldsymbol{\nabla}\boldsymbol{u}^*)\cdot\boldsymbol{u}']\}\,\mathrm{d}S\mathrm{d}t$$

Surface integral involving boundary conditions

As far as possible we will choose terminal and boundary conditions to suit us.

Two optimal energy functionals

The two kinds of optimisations we consider:

1. Initial flow perturbation $\boldsymbol{u}'_0$ that produces maximum kinetic energy growth at time τ.

Growth $G = \max_{\boldsymbol{u}'_0} \frac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{(\boldsymbol{u}'_0, \boldsymbol{u}'_0)} = \max_{\boldsymbol{u}'_0} \frac{E(\tau)}{E(0)}$

Final energy

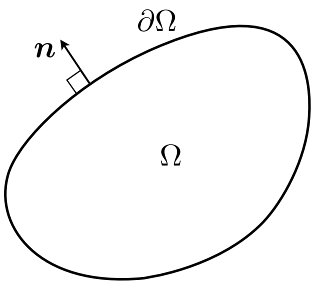
Initial energy

2. Boundary flow perturbation $\boldsymbol{u}'_c$ that produces maximum kinetic energy gain at time τ.

We need definitions for boundary integrals:

$$[\boldsymbol{a},\boldsymbol{b}] = D\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{b}\,\mathrm{d}S \qquad \{\boldsymbol{a},\boldsymbol{b}\} = D\int_0^\tau\int_{\partial\Omega}\boldsymbol{a}\cdot\boldsymbol{b}\,\mathrm{d}S\mathrm{d}t$$

where *D* is a representative length.



Gain $K = \max_{\boldsymbol{u}'_c} \frac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{\{\boldsymbol{u}'_c, \boldsymbol{u}'_c\}}$

Final energy

Boundary energy

Optimisation in both cases is constrained: solutions have to obey LNSE.

Constrained optimisation

Generalising the kinetic energy functionals to be optimised as $\mathcal{J}(\boldsymbol{u}'_{\text{opt}})$

$$\mathcal{J}(\boldsymbol{u}'_{\text{opt}}) \equiv G = \max_{\boldsymbol{u}'_0} \frac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{(\boldsymbol{u}'_0, \boldsymbol{u}'_0)} \quad \text{or} \quad K = \max_{\boldsymbol{u}_c} \frac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{\{\boldsymbol{u}'_c, \boldsymbol{u}'_c\}}$$

we will construct an augmented/Lagrangian functional

$$\mathcal{L} = \mathcal{J}(\boldsymbol{u}'_{\text{opt}}) - \langle \boldsymbol{u}^*, \partial_t \boldsymbol{u}' - L\boldsymbol{u}' \rangle$$

for which we will find extrema.

Constrains solutions to satisfy LNSE

$\boldsymbol{u}^*$ plays the role of a Lagrange multiplier.

We have converted a constrained optimisation problem into an unconstrained optimisation problem – but with more variables.

We allow arbitrary variations of the Lagrangian with respect to all the variables and ensure that all gradients are simultaneously zero.

The standard tool for this job is the Gateaux differential

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{q}} = \lim_{\epsilon \to 0} \frac{\mathcal{L}(\boldsymbol{q} + \epsilon \delta \boldsymbol{q}) - \mathcal{L}(\boldsymbol{q})}{\epsilon} \equiv \langle \nabla_{\boldsymbol{q}} \mathcal{L}, \delta \boldsymbol{q} \rangle,$$

which identifies the directional derivative of $\mathcal{L}$ with respect to arbitrary variation in variable $\boldsymbol{q}$.

Gunzburger, SIAM, 2003.

Noack *et al*, Springer, 2011.

Optimal INITIAL perturbations

Calculus of variations for optimal ICs

Set boundary perturbations to zero, and seek the initial perturbation $\boldsymbol{u}_0'$ that provides maximum energy growth for a given time horizon τ.

$$\mathcal{L}_0 = \frac{(\boldsymbol{u'}_\tau, \boldsymbol{u'}_\tau)}{(\boldsymbol{u'}_0, \boldsymbol{u'}_0)} - \langle \boldsymbol{u}^*, \partial_t \boldsymbol{u}' + L(\boldsymbol{u}') \rangle$$

$$\equiv \frac{(\boldsymbol{u'}_\tau, \boldsymbol{u'}_\tau)}{(\boldsymbol{u'}_0, \boldsymbol{u'}_0)} + \langle \boldsymbol{u}', \partial_t \boldsymbol{u}^* + L^*(\boldsymbol{u}^*) \rangle - (\boldsymbol{u}_\tau^*, \boldsymbol{u'}_\tau) + (\boldsymbol{u}_0^*, \boldsymbol{u'}_0)$$

removed using zero BCs

$$+ \int_0^T \int_{\partial\Omega} \boldsymbol{n} \cdot \{ -U(\boldsymbol{u}' \cdot \boldsymbol{u}^*) + \boldsymbol{u}' p^* - \boldsymbol{u}^* p' + Re^{-1} [(\nabla \boldsymbol{u}') \cdot \boldsymbol{u}^* - (\nabla \boldsymbol{u}^*) \cdot \boldsymbol{u}'] \} \, \mathrm{d}S \mathrm{d}t$$

$$\frac{\delta \mathcal{L}_0}{\delta \boldsymbol{u}^*} = 0 \implies \partial_t \boldsymbol{u}' - L\boldsymbol{u}' = 0$$

$$\frac{\delta \mathcal{L}_0}{\delta \boldsymbol{u}'} = 0 \implies \partial_t \boldsymbol{u}^* + L^* \boldsymbol{u}^* = 0$$

$$\frac{\delta \mathcal{L}_0}{\delta \boldsymbol{u}_\tau'} = 0 \implies \boldsymbol{u}_\tau^* = 2 \frac{\boldsymbol{u}_\tau'}{(\boldsymbol{u}_0', \boldsymbol{u}_0')}$$

$$\frac{\delta \mathcal{L}_0}{\delta \boldsymbol{u}_0'} = 0 \implies \boldsymbol{u}_0^* = 2 \frac{(\boldsymbol{u}_\tau', \boldsymbol{u}_\tau')}{(\boldsymbol{u}_0', \boldsymbol{u}_0')^2} \boldsymbol{u}_0'$$

i.e. $\nabla_{\boldsymbol{u}_0'} \mathcal{L}_0 = \boldsymbol{u}_0^* - 2 \frac{(\boldsymbol{u}_\tau', \boldsymbol{u}_\tau')}{(\boldsymbol{u}_0', \boldsymbol{u}_0')^2} \boldsymbol{u}_0'$

In each case we use

$$\frac{\delta \mathcal{L}}{\delta \boldsymbol{q}} = \lim_{\epsilon \to 0} \frac{\mathcal{L}(\boldsymbol{q} + \epsilon \delta \boldsymbol{q}) - \mathcal{L}(\boldsymbol{q})}{\epsilon} \equiv \langle \nabla_{\boldsymbol{q}} \mathcal{L}, \delta \boldsymbol{q} \rangle$$

Optimisation approach for initial perturbation

Calculus of variations gave four outcomes:

Evolution equations

$$\partial_t \boldsymbol{u}' - L\boldsymbol{u}' = 0$$

$$\partial_t \boldsymbol{u}^* + L^* \boldsymbol{u}^* = 0$$

Terminal condition

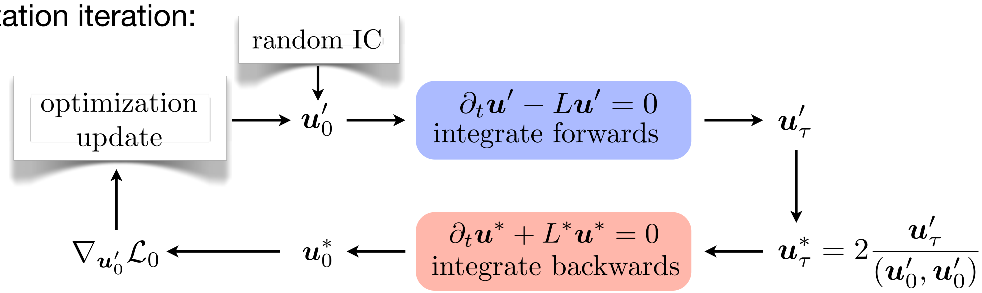
$$\boldsymbol{u}_\tau^* = 2 \frac{\boldsymbol{u}_\tau'}{(\boldsymbol{u}_0', \boldsymbol{u}_0')}$$

Optimality condition

$$\nabla_{\boldsymbol{u}_0'} \mathcal{L}_0 = \boldsymbol{u}_0^* - 2 \frac{(\boldsymbol{u}_\tau', \boldsymbol{u}_\tau')}{(\boldsymbol{u}_0', \boldsymbol{u}_0')^2} \boldsymbol{u}_0' = 0$$

Solve by optimization iteration:

(e.g. Schmid 2007)



The 'optimisation update' could be steepest descent or other appropriate method.

At convergence we have IC $\boldsymbol{u}_0'$ that maximizes $(\boldsymbol{u}_\tau', \boldsymbol{u}_\tau') \,/\, (\boldsymbol{u}_0', \boldsymbol{u}_0')$ subject to constraints.

Eigenvalue approach for initial perturbation

Recall $\boldsymbol{u}'_\tau = \mathcal{M}_0 \boldsymbol{u}'_0$ obtained by forward integration of LNSE

$\boldsymbol{u}^*_0 = \mathcal{M}^*_0 \boldsymbol{u}^*_\tau$ obtained by backward integration of ANSE

Now compare

$$\mathcal{L}_0 = \frac{(\boldsymbol{u'}_\tau, \boldsymbol{u'}_\tau)}{(\boldsymbol{u'}_0, \boldsymbol{u'}_0)} - \langle \boldsymbol{u}^*, \partial_t \boldsymbol{u}' + L(\boldsymbol{u}') \rangle$$

$$\equiv \frac{(\boldsymbol{u'}_\tau, \boldsymbol{u'}_\tau)}{(\boldsymbol{u'}_0, \boldsymbol{u'}_0)} + \langle \boldsymbol{u}', \partial_t \boldsymbol{u}^* + L^*(\boldsymbol{u}^*) \rangle - (\boldsymbol{u}^*_\tau, \boldsymbol{u'}_\tau) + (\boldsymbol{u}^*_0, \boldsymbol{u'}_0)$$

If LNSE and ANSE are always satisfied, $(\boldsymbol{u}^*_\tau, \boldsymbol{u}'_\tau) = (\boldsymbol{u}^*_0, \boldsymbol{u}'_0)$

equivalently $(\boldsymbol{u}^*_\tau, \mathcal{M}_0 \boldsymbol{u}'_0) = (\mathcal{M}^*_0 \boldsymbol{u}^*_\tau, \boldsymbol{u}'_0)$, meaning $\mathcal{M}^*_0$ is the operator adjoint to $\mathcal{M}_0$ with respect to the inner product (,).

So now

$$\mathcal{L}_0 = \frac{(\boldsymbol{u'}_\tau, \boldsymbol{u'}_\tau)}{(\boldsymbol{u'}_0, \boldsymbol{u'}_0)} = \frac{(\mathcal{M}_0 \boldsymbol{u'}_0, \mathcal{M}_0 \boldsymbol{u'}_0)}{(\boldsymbol{u'}_0, \boldsymbol{u'}_0)} = \frac{(\boldsymbol{u'}_0, \mathcal{M}^*_0 \mathcal{M}_0 \boldsymbol{u'}_0)}{(\boldsymbol{u'}_0, \boldsymbol{u'}_0)}$$

which is maximised when $\boldsymbol{u}'_0$ is the eigenvector of joint symmetric operator $\mathcal{M}^*_0 \mathcal{M}_0$ corresponding to the largest eigenvalue. This eigenvalue is *G.*

Compute eigensystem by timestepping

$$\mathcal{M}^*_0 \mathcal{M}_0 \boldsymbol{u}'_0 = G(\tau) \boldsymbol{u}'_0$$

To be able to solve the eigensystem of $\mathcal{M}^*_0 \mathcal{M}_0$ we need only to be able to apply the operator to a vector.

Solve by Krylov method with inner loop:

$\boldsymbol{u}'_0 \longrightarrow$ [$\partial_t \boldsymbol{u}' - L\boldsymbol{u}' = 0$ integrate forwards] $\longrightarrow \boldsymbol{u}'_\tau$

$\mathcal{M}^*_0 \mathcal{M}_0 \boldsymbol{u}'_0 = \boldsymbol{u}^*_0 \longleftarrow$ [$\partial_t \boldsymbol{u}^* + L^* \boldsymbol{u}^* = 0$ integrate backwards] $\longleftarrow \boldsymbol{u}^*_\tau$

Same outer loop as for instability solution, different inner (operator) loop.

Generates Krylov sequence $T = \{\boldsymbol{u}'_0, (\mathcal{M}^*\mathcal{M})\boldsymbol{u}'_0, (\mathcal{M}^*\mathcal{M})^2 \boldsymbol{u}'_0, \dots, (\mathcal{M}^*\mathcal{M})^{k-1} \boldsymbol{u}'_0\}$

This means we can compute optimal initial conditions via either optimisation-based or eigensystem solvers – great for bootstrapping new codes!

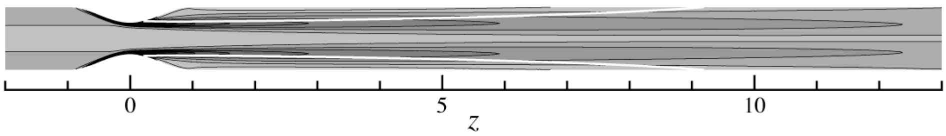
The only issue then is relative performance - with most of the work being in time-integration as the inner loop is basically identical in each case.

But first we'll look at some outcomes.

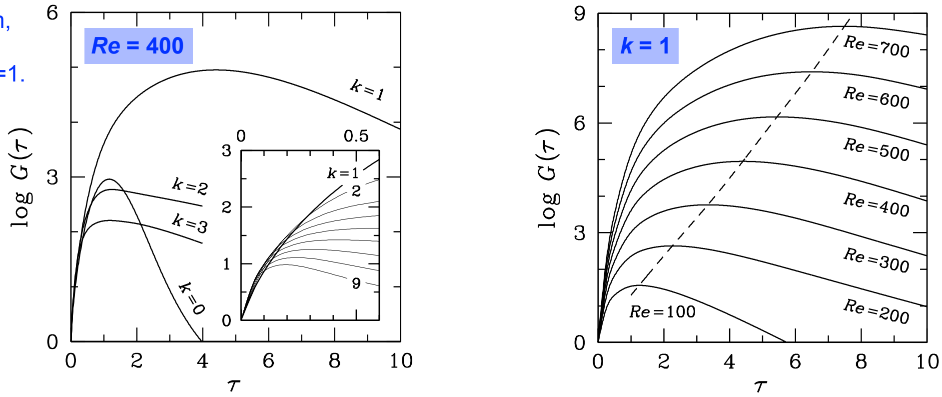
Steady, stable, stenotic flow

Re = 400

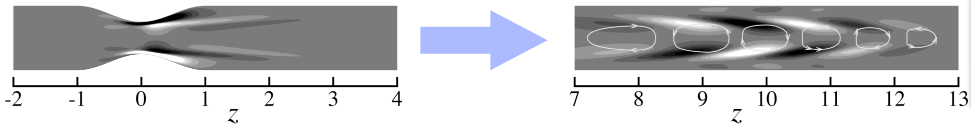
Base flow



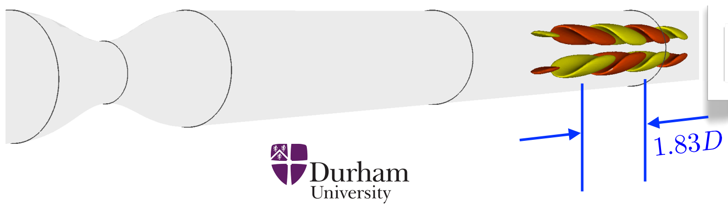
Global optimum, *Re*=400: G_{max}=8.9×10^4, *k*=1.



Optimal initial



Optimal max



Optimal max – axial velocity isosurfaces

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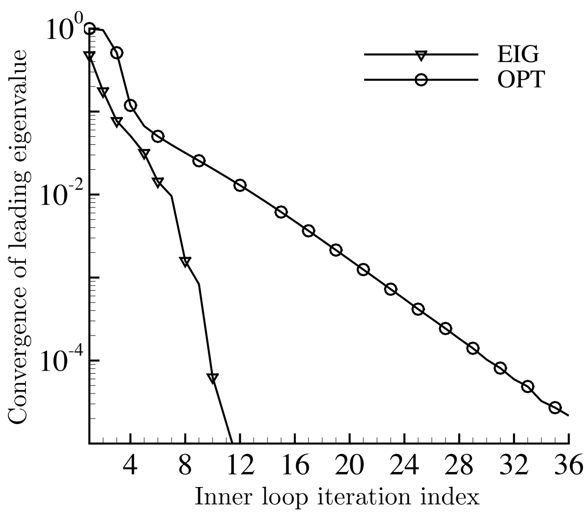
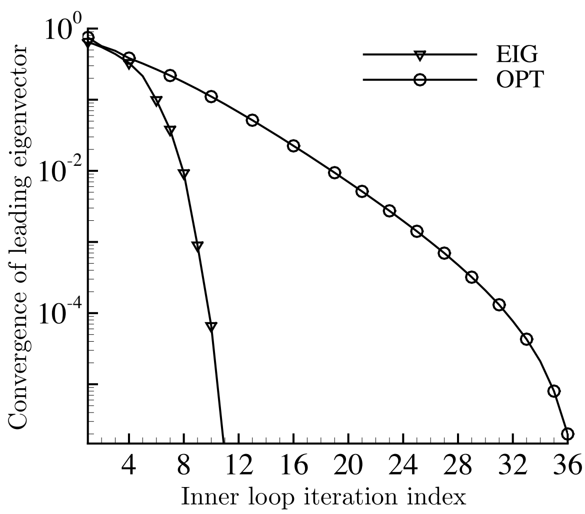
Relative performance





This suggests that the eigensystem approach is generally preferable when computing optimal initial conditions.

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Optimal BOUNDARY perturbations

Calculus of variations for optimal BCs

Set the initial condition $\boldsymbol{u}'_0 = 0$ and compute a boundary perturbation $\boldsymbol{u}_c$ on part of domain boundary, $\boldsymbol{x}_c$, that maximizes the kinetic energy gain at fixed τ.

General $$K = \max_{\boldsymbol{u}'_c} \frac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{\{\boldsymbol{u}'_c, \boldsymbol{u}'_c\}} \quad \text{where} \quad \{\boldsymbol{a}, \boldsymbol{b}\} = D \int_0^\tau \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{b} \, \mathrm{d}S \mathrm{d}t$$

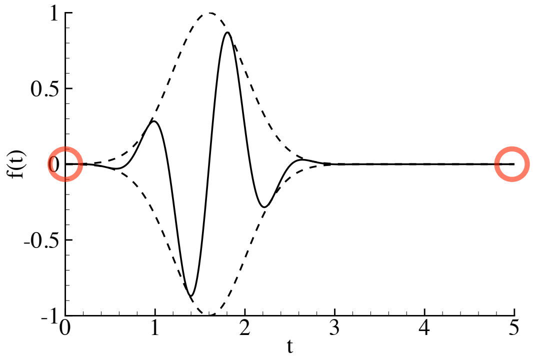
Separation-of-variables approach: $\boldsymbol{u}'_c(\boldsymbol{x}_c, t) = \widehat{\boldsymbol{u}}_c(\boldsymbol{x}_c) f(t, \omega)$

where $f(t, \omega)$ is a prescribed function of time and ω is a parameter (frequency).

Simplification $$K = \max_{\widehat{\boldsymbol{u}}_c} \frac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{[\widehat{\boldsymbol{u}}_c, \widehat{\boldsymbol{u}}_c]} \quad \text{where} \quad [\boldsymbol{a}, \boldsymbol{b}] = D \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{b} \, \mathrm{d}S$$

Looking for an optimal spatial distribution of BC disturbance.

Optimisation is specific to the particular choice made for $f(t, \omega)$.



Optimal BOUNDARY perturbations

$$\mathcal{L}_c = \frac{(\boldsymbol{u'}_\tau, \boldsymbol{u'}_\tau)}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]} - \langle \boldsymbol{u}^*, \partial_t \boldsymbol{u}' - L(\boldsymbol{u}') \rangle$$

choose $\boldsymbol{u}_0' = 0$

$$\mathcal{L}_c = \frac{(\boldsymbol{u'}_\tau, \boldsymbol{u'}_\tau)}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]} + \langle \boldsymbol{u}', \partial_t \boldsymbol{u}^* + L^*(\boldsymbol{u}^*) \rangle + (\boldsymbol{u}_0^*, \boldsymbol{u}_0') - (\boldsymbol{u}_\tau^*, \boldsymbol{u}_\tau')$$

choose $\boldsymbol{u}^*(\partial\Omega) = 0$

$$+ \int_0^\tau \int_{\partial\Omega} \boldsymbol{n} \cdot \{ -U(\boldsymbol{u}' \cdot \boldsymbol{u}^*) + \boldsymbol{u}' p^* - \boldsymbol{u}^* p' + Re^{-1} [(\nabla \boldsymbol{u}') \cdot \boldsymbol{u}^* - (\boldsymbol{\nabla} \boldsymbol{u}^*) \cdot \boldsymbol{u}'] \} \, \mathrm{d}S \mathrm{d}t$$

Applying the tools:

$$\frac{\delta \mathcal{L}_c}{\delta \boldsymbol{u}^*} = 0 \implies \partial_t \boldsymbol{u}' - L(\boldsymbol{u}') = 0,$$

$$\frac{\delta \mathcal{L}_c}{\delta \boldsymbol{u}'} = 0 \implies \partial_t \boldsymbol{u}^* + L^*(\boldsymbol{u}^*) = 0,$$

$$\frac{\delta \mathcal{L}_c}{\delta \boldsymbol{u'}_\tau} = 0 \implies \boldsymbol{u}_\tau^* = \frac{2\boldsymbol{u'}_\tau}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]},$$

$$\frac{\delta \mathcal{L}_c}{\delta \hat{\boldsymbol{u}}_c} = 0 \implies \nabla_{\hat{\boldsymbol{u}}_c} \mathcal{L}_c = \frac{-2(\boldsymbol{u}_\tau', \boldsymbol{u}_\tau') \hat{\boldsymbol{u}}_c}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]^2} + \int_0^\tau f(t,\omega)(p^* \boldsymbol{n} - Re^{-1} \boldsymbol{\nabla}_{\boldsymbol{n}} \boldsymbol{u}^*) \, \mathrm{d}t.$$

$$\nabla_{\hat{\boldsymbol{u}}_c} \mathcal{L}_c = \frac{-2(\boldsymbol{u}_\tau', \boldsymbol{u}_\tau') \hat{\boldsymbol{u}}_c}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]^2} + \boldsymbol{g}(\boldsymbol{u}^*, p^*, \omega)$$

Optimisation approach for boundary perturbation

Analogous to approach for optimal initial condition:

Evolution equations

$$\partial_t \boldsymbol{u}' - L\boldsymbol{u}' = 0$$

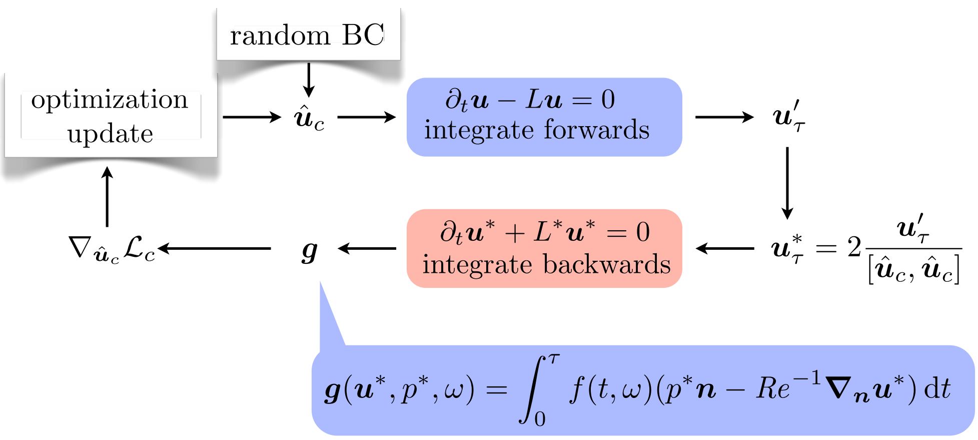
$$\partial_t \boldsymbol{u}^* + L^* \boldsymbol{u}^* = 0$$

Terminal condition

$$\boldsymbol{u}_\tau^* = 2 \frac{\boldsymbol{u}_\tau'}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]}$$

Optimality condition

$$\nabla_{\hat{\boldsymbol{u}}_c} \mathcal{L}_c = \frac{-2(\boldsymbol{u}_\tau', \boldsymbol{u}_\tau') \hat{\boldsymbol{u}}_c}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]^2} + \boldsymbol{g}(\boldsymbol{u}^*, p^*, \omega)$$



Eigenvalue approach for boundary perturbation

$$\boldsymbol{u}'_\tau = \mathcal{M}_c \hat{\boldsymbol{u}}_c \quad \text{obtained by forward integration of LNSE}$$

$$\boldsymbol{g}(\boldsymbol{u}^*, p^*, \omega) = \mathcal{M}_c^* \boldsymbol{u}_\tau^* \quad \text{obtained during backward integration of ANSE}$$

Now compare

$$\mathcal{L}_c = \frac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]} - \langle \boldsymbol{u}^*, \partial_t \boldsymbol{u}' + L\boldsymbol{u}' \rangle$$

$$\equiv \frac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{[\hat{\boldsymbol{u}}_c \hat{\boldsymbol{u}}_c,]} + \langle \boldsymbol{u}', \partial_t \boldsymbol{u}^* + L^* \boldsymbol{u}^* \rangle - (\boldsymbol{u}_\tau^*, \boldsymbol{u}'_\tau) + \Big[\underbrace{\int_0^\tau f(t,\omega)(p^* \boldsymbol{n} - Re^{-1} \boldsymbol{\nabla}_{\boldsymbol{n}} \boldsymbol{u}^*) \, \mathrm{d}t}_{\boldsymbol{g}(\boldsymbol{u}^*, p^*, \omega)}, \hat{\boldsymbol{u}}_c \Big]$$

If LNSE and ANSE are always satisfied, $(\boldsymbol{u}_\tau^*, \boldsymbol{u}'_\tau) = [\boldsymbol{g}, \hat{\boldsymbol{u}}_c]$

equivalently $(\boldsymbol{u}_\tau^*, \mathcal{M}_c \hat{\boldsymbol{u}}_c) = [\mathcal{M}_c^* \boldsymbol{u}_\tau^*, \hat{\boldsymbol{u}}_c]$ i.e. $(\boldsymbol{a}, \mathcal{M}_c \boldsymbol{b}) = [\mathcal{M}_c^* \boldsymbol{a}, \boldsymbol{b}]$

So now $\mathcal{L}_c = \dfrac{(\boldsymbol{u}'_\tau, \boldsymbol{u}'_\tau)}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]} = \dfrac{(\mathcal{M}_c \hat{\boldsymbol{u}}_c, \mathcal{M}_c \hat{\boldsymbol{u}}_c)}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]} = \dfrac{[\mathcal{M}_c^* \mathcal{M}_c \hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]}{[\hat{\boldsymbol{u}}_c, \hat{\boldsymbol{u}}_c]}$

which is maximised when $\hat{\boldsymbol{u}}_c$ is the eigenvector of joint operator $\mathcal{M}_c^* \mathcal{M}_c$ corresponding to the largest eigenvalue. This eigenvalue is *K.*

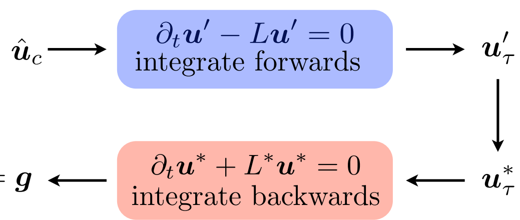
Analogous to approach for optimal initial condition.

Compute eigensystem by timestepping

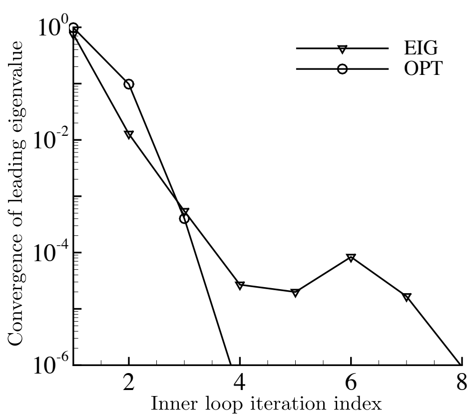
$$\mathcal{M}_c^* \mathcal{M}_c \hat{\boldsymbol{u}}_c = K(\tau) \hat{\boldsymbol{u}}_c$$

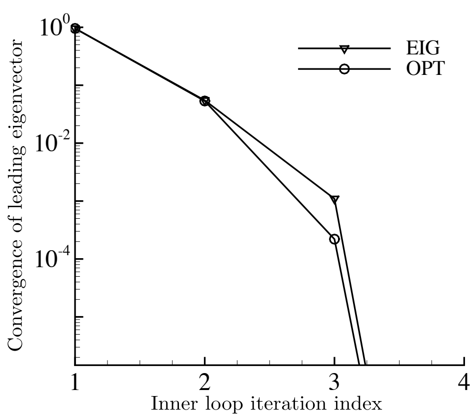
To be able to solve the eigensystem of $\mathcal{M}_c^* \mathcal{M}_c$ we need only to be able to apply the operator to a vector.

Solve by Krylov method with inner loop:



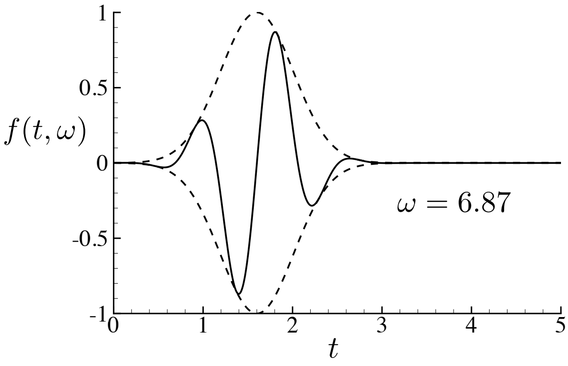
Again, analogous to approach for optimal initial condition

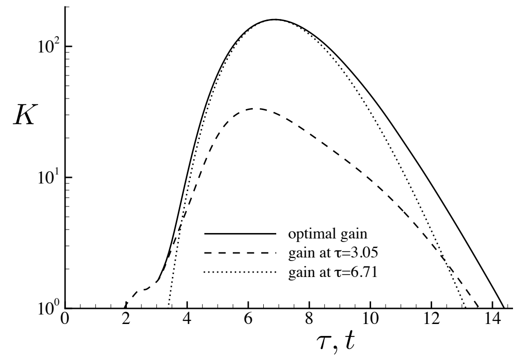


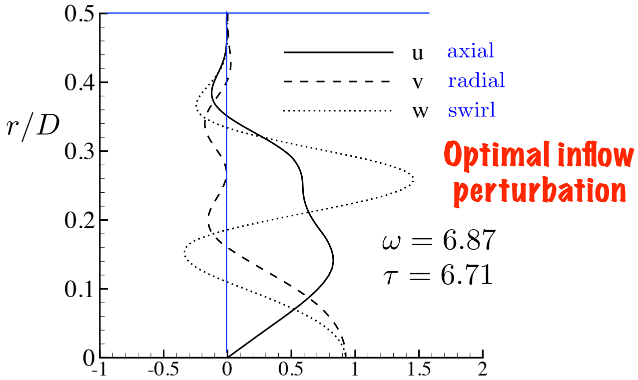


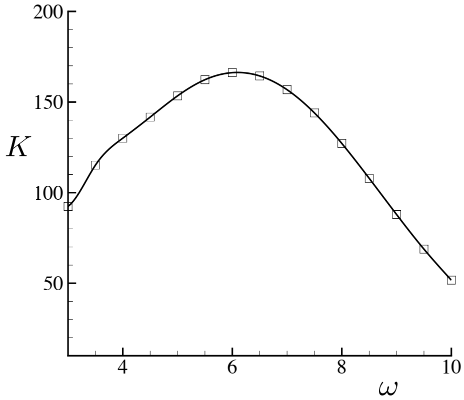
Example: Steady stenotic flow, *Re* = 400

azimuthal Fourier mode $m = 1$









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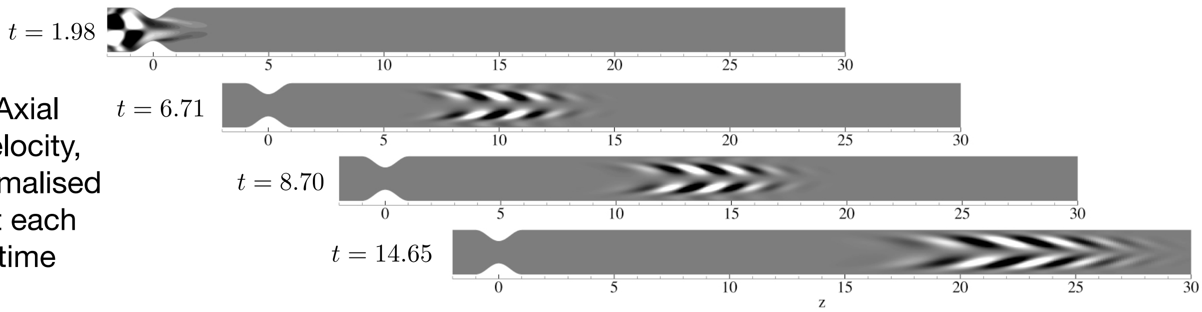
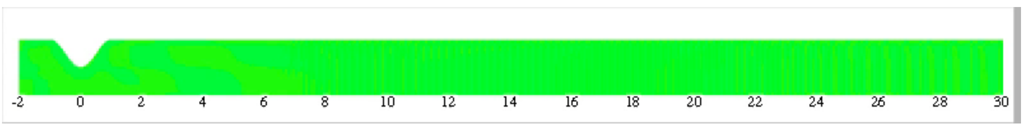
Steady stenotic flow, *Re* = 400, *m* = 1, τ = 6.71

Axial velocity, normalised at each time



Azimuthal vorticity



Axial velocity at *t* = τ = 6.71

Outcome of boundary perturbation

Axial velocity resulting from optimal IC

Outcome of initial perturbation

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Summary

- Timestepper approach to linear systems involving LNSE. Krylov methods for eigensystem solutions – cheaper than direct methods.
- Eigenmodal (large-time) vs transient growth (finite time) growth from initial conditions.
- Optimal growth (ICs) vs optimal gain (BCs).
- Tools for calculus of variations: Gateaux differential, adjoint variable and operators, integration by parts.
- Optimal transient growth via optimisation or eigensystem methods.
- Optimal transient gain via optimisation or eigensystem methods.
- Close relationship between optimal growth and gain outcomes.