

History is doomed to

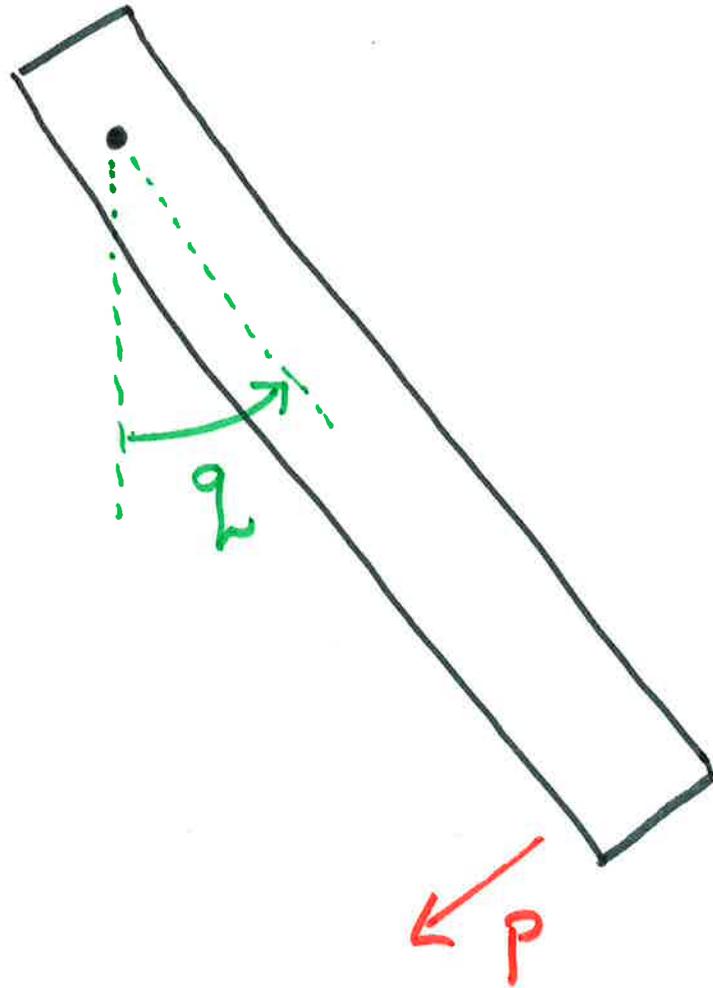
almost

repeat itself

over and over again.

Andy Hammerlindl.

History Repeats Itself!
At least for a pendulum.

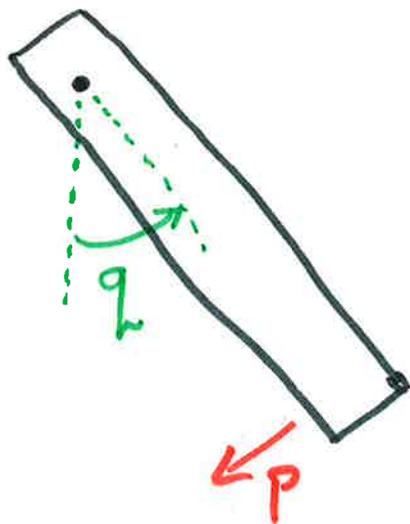


If we know the angle q
and angular momentum P
of a pendulum,

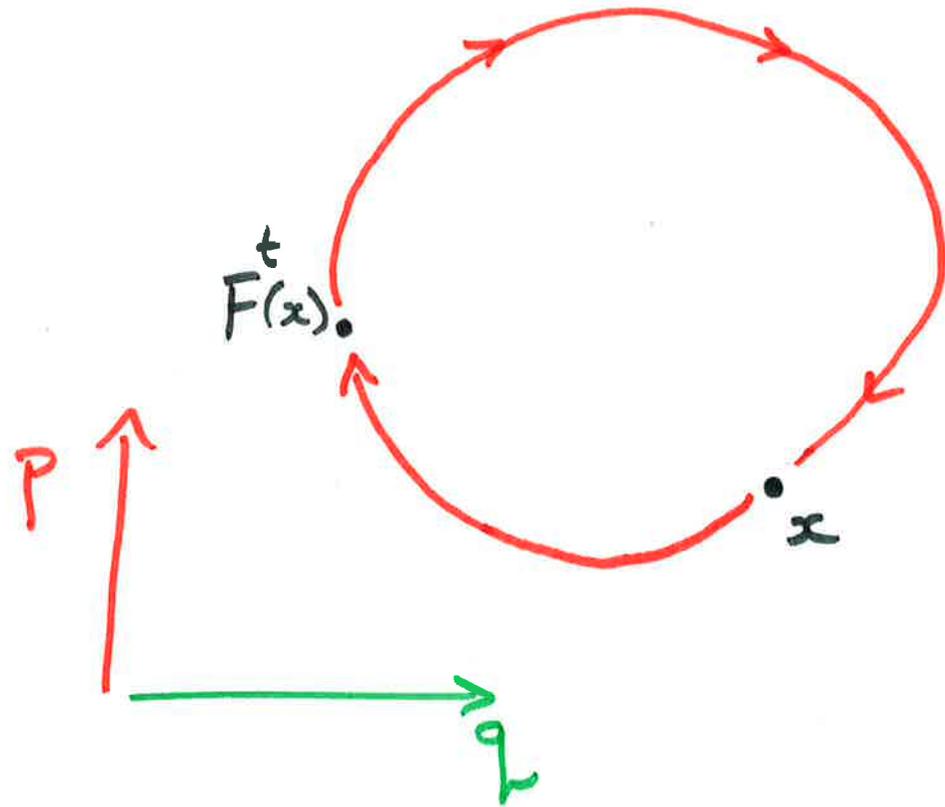
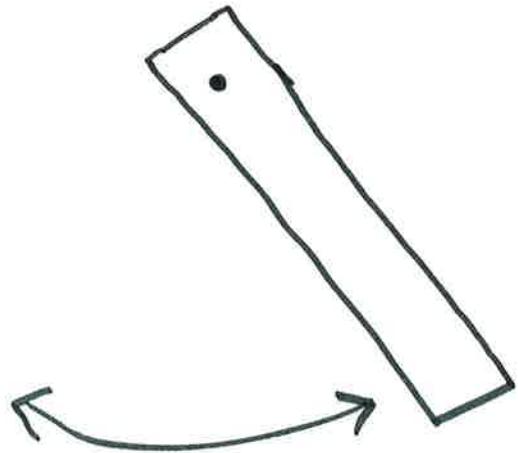
physics tells us the future behaviour.

This point $x = (q, P) \in \mathbb{R}^2$

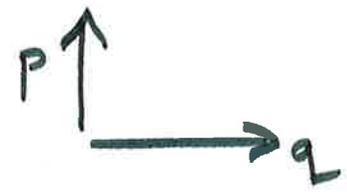
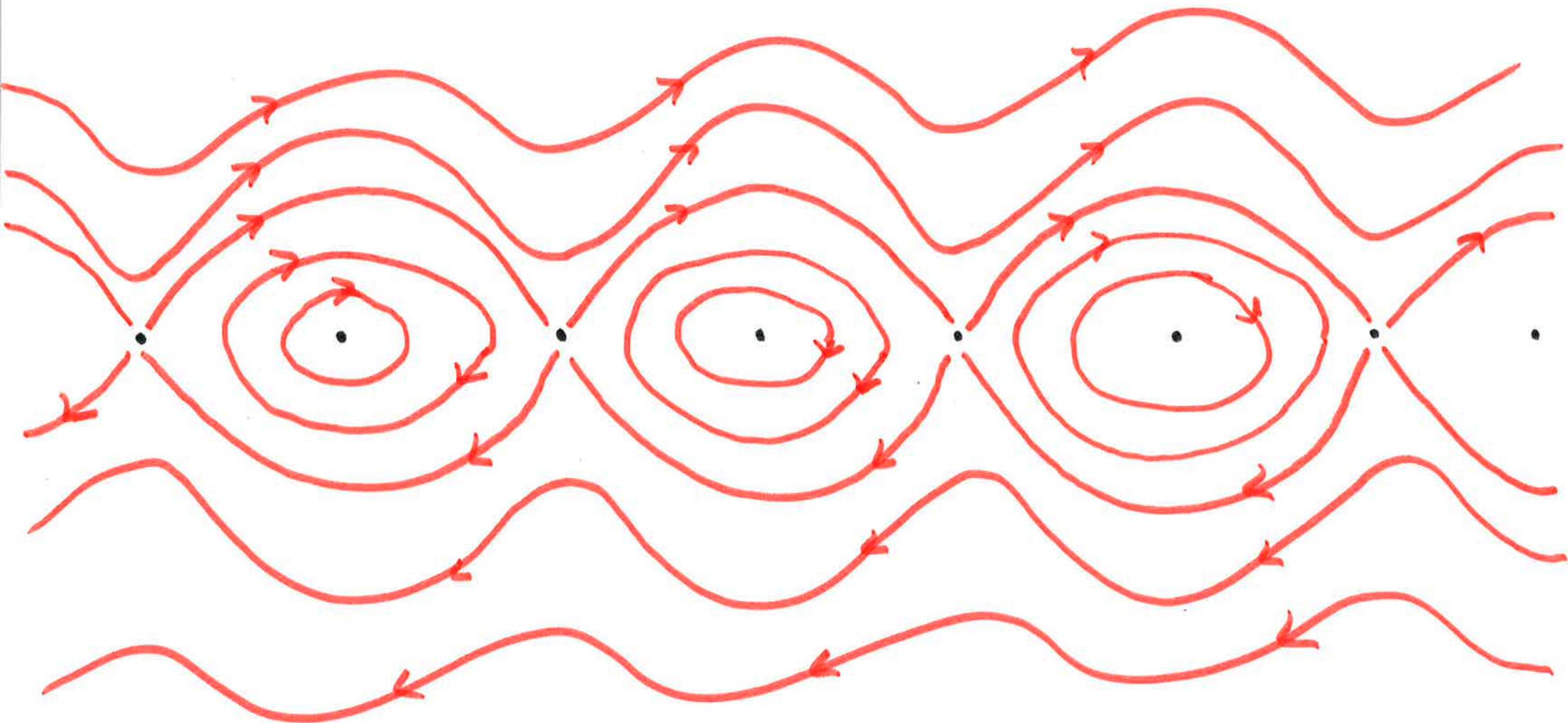
is in the phase space of the system.



Given a point x in phase space,
let $F^t(x)$ be the state of the
system after t units of time.



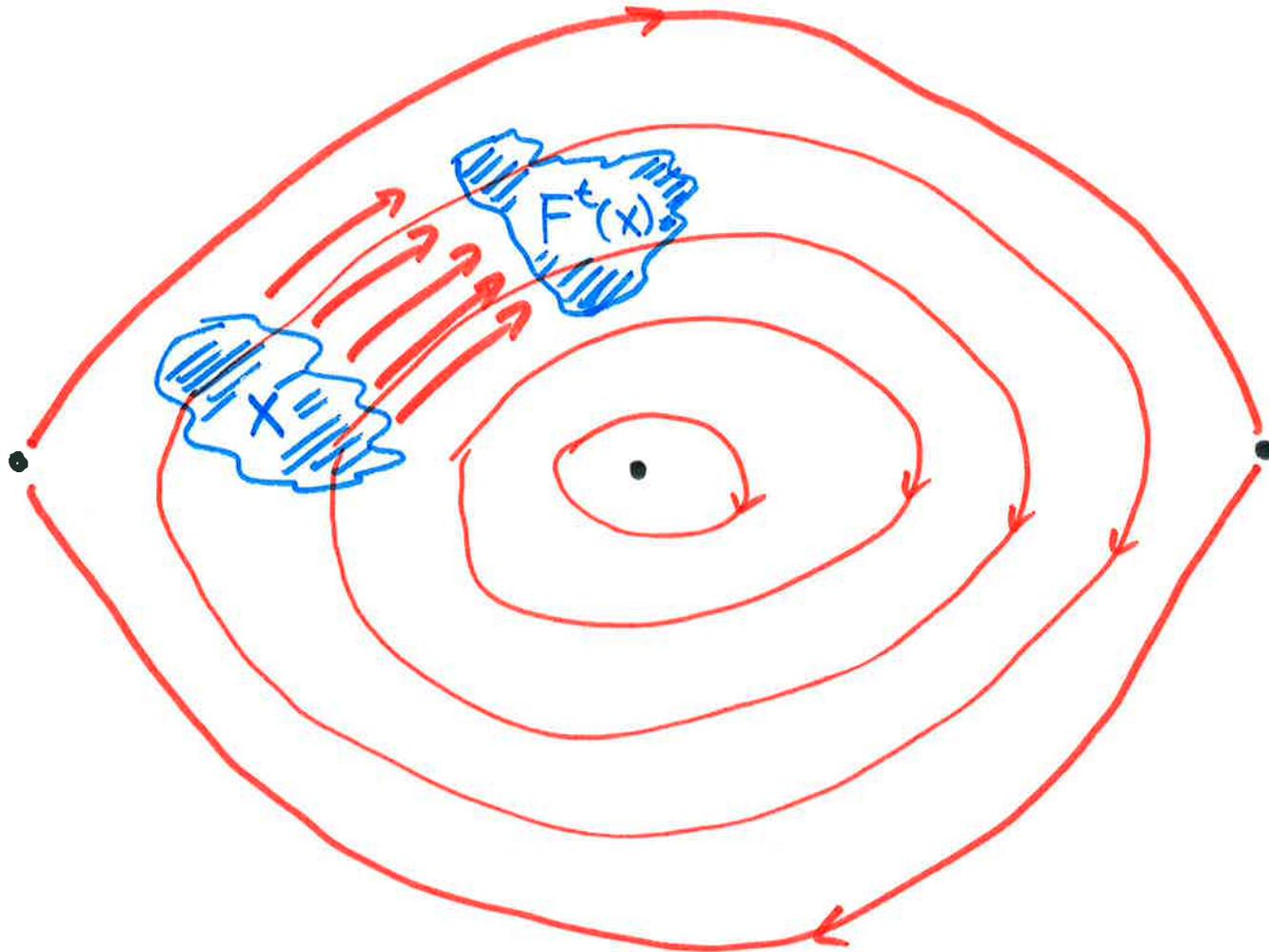
This defines a flow on the phase space
(in this case \mathbb{R}^2).



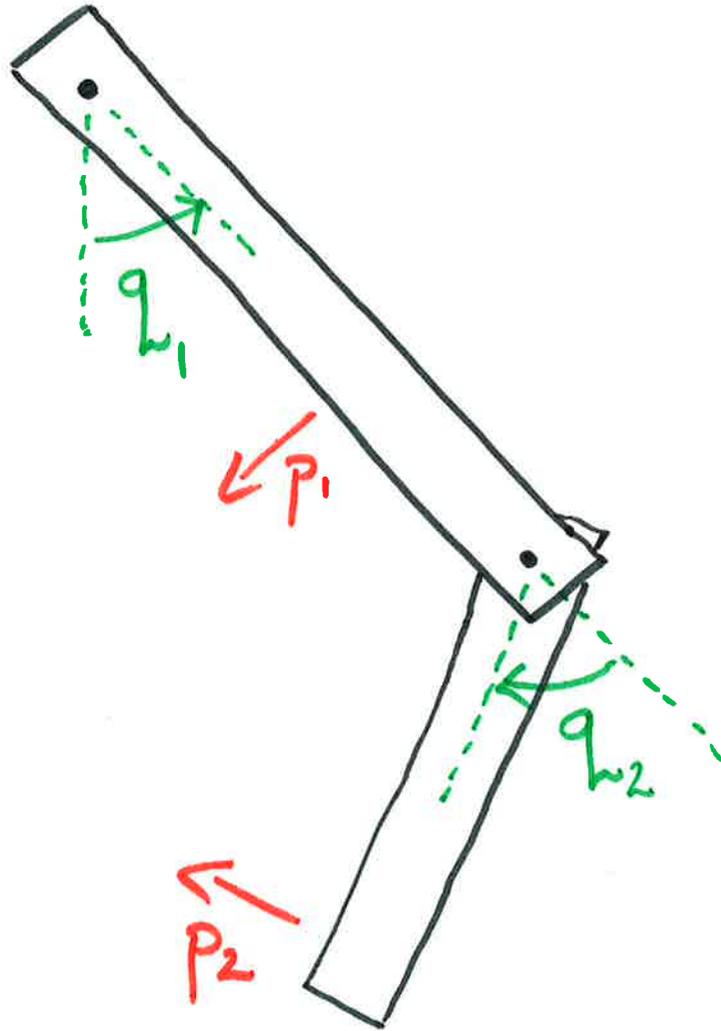
For a pendulum

(ignoring friction, air resistance, etc.)

this flow preserves area.



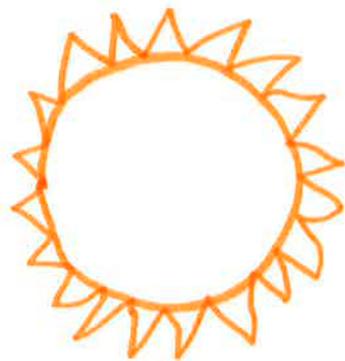
Consider a double pendulum.



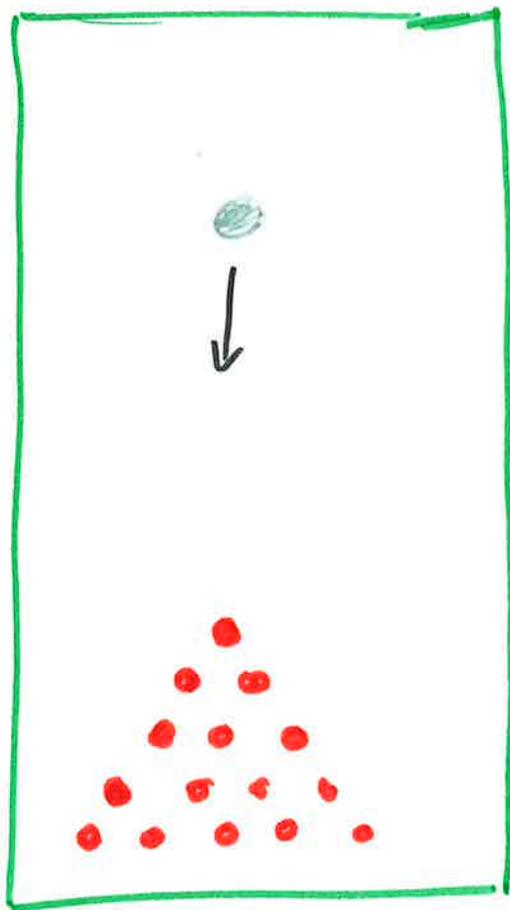
Its phase space
is 4-dimensional,
and the flow
 F^t preserves
4-dimensional
volume.

The solar system has 1 sun and ~~9~~ 8 planets, each with a position in \mathbb{R}^3 and a momentum in \mathbb{R}^3 .

The phase space is $(8+1) \cdot (3+3) = 54$ dimensional and the flow due to gravity preserves 54-dimensional volume.

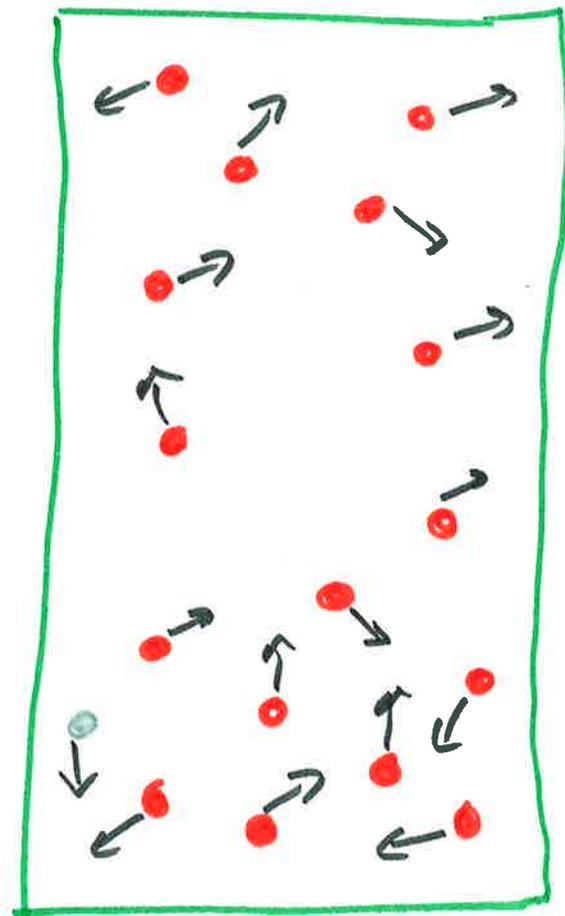


Billiards

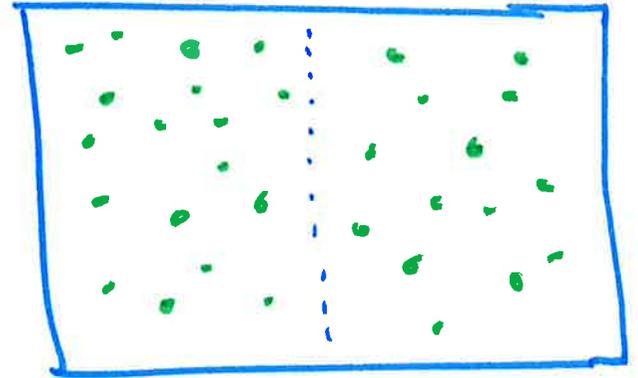
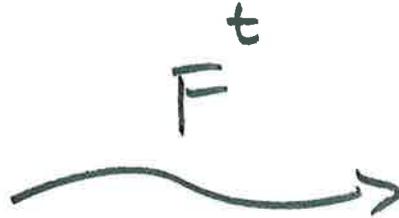
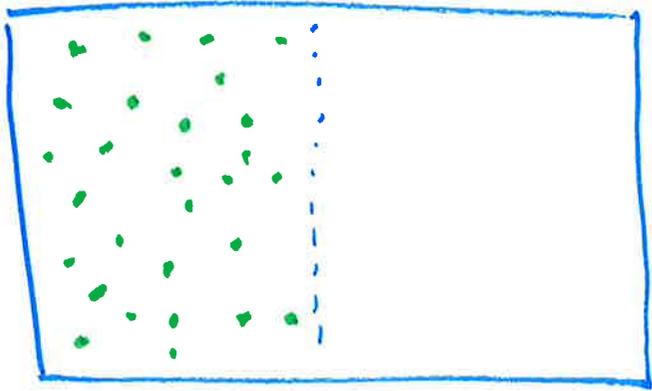


$F t$

A wavy arrow pointing from the left diagram to the right diagram, with the text $F t$ written above it.



Ideal Gases



Theorem (Liouville 1838)

For a system in classical mechanics,

the resulting flow F^t on phase

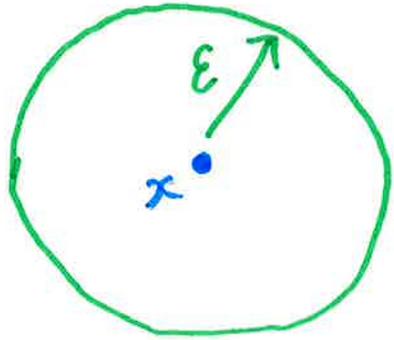
space preserves volume.

Does history

almost

repeat itself?

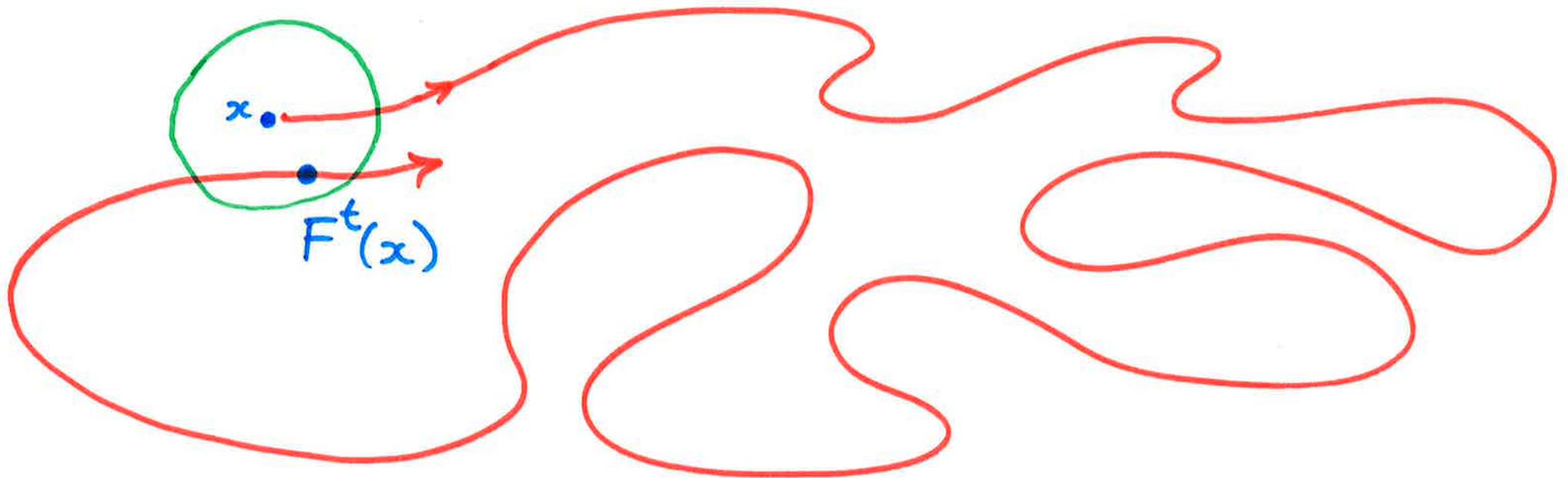
Consider a point x in phase space
and a ball of radius ϵ around x .



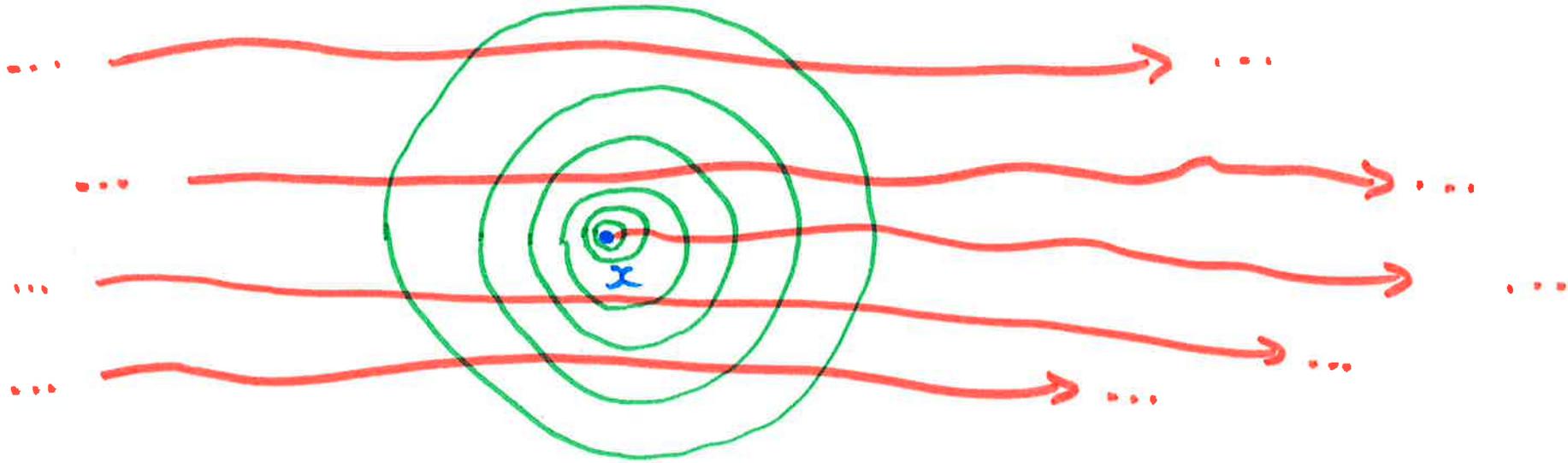
The point x is

ϵ -recurrent

if the orbit of x visits
the ball at some future time $t \geq 1$.



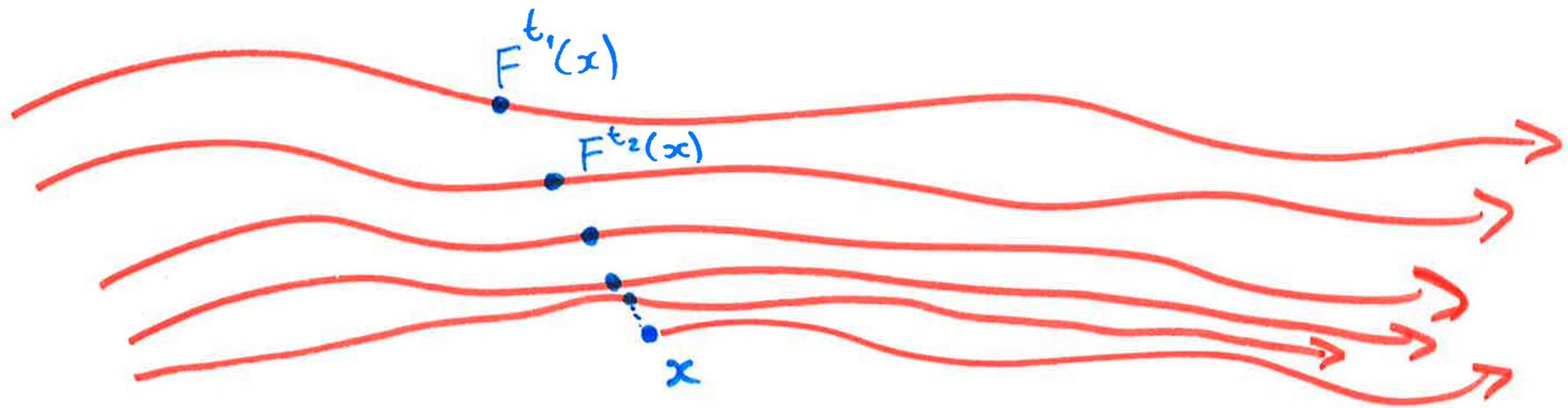
A point is recurrent if it is ϵ -recurrent for every positive ϵ .



Equivalently, x is recurrent if there is a sequence of future times

$$t_1 < t_2 < t_3 < t_4 < \dots$$

such that $F^{t_k}(x)$ converges to x .



Theorem (Poincaré 1890)

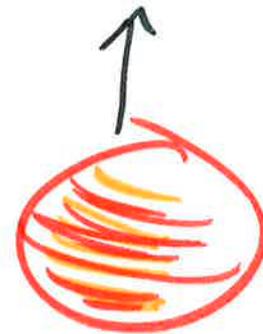
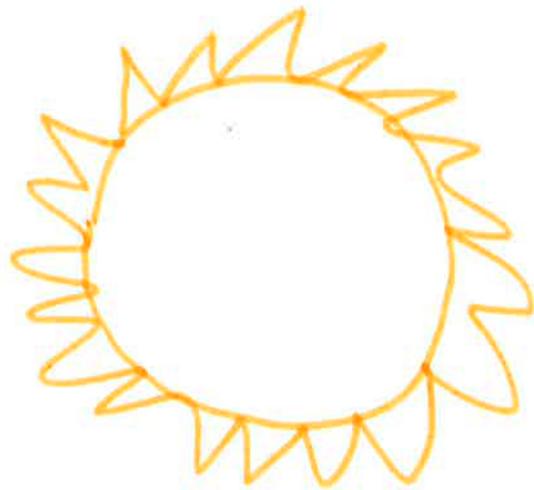
Let M be a phase space with finite volume
and F^t a volume-preserving flow on M .

Then almost every point is recurrent.

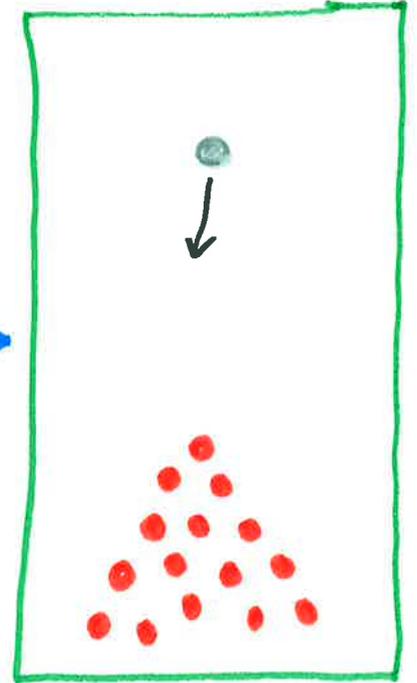
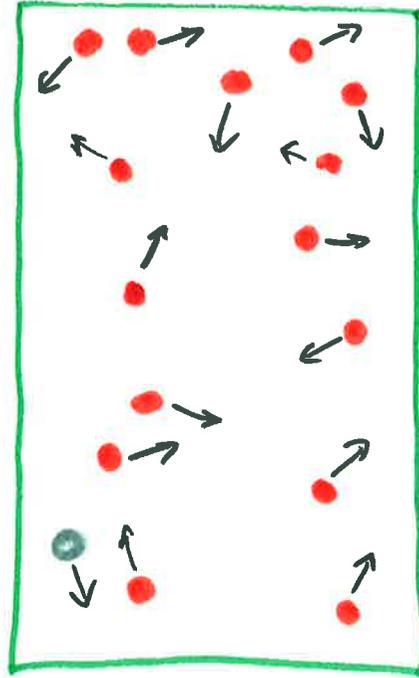
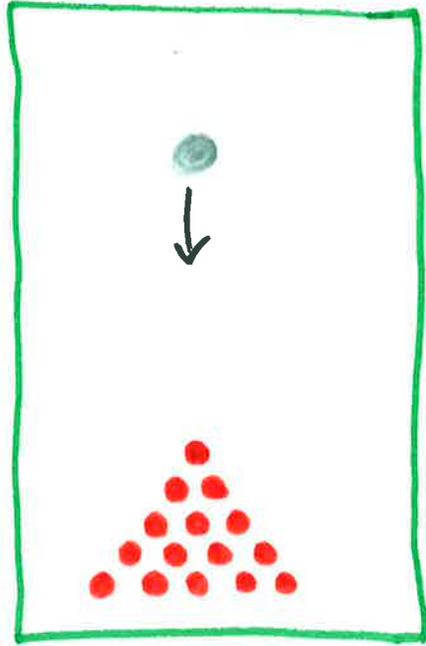
(That is, the set of non-recurrent points
has zero volume.)

Examples again

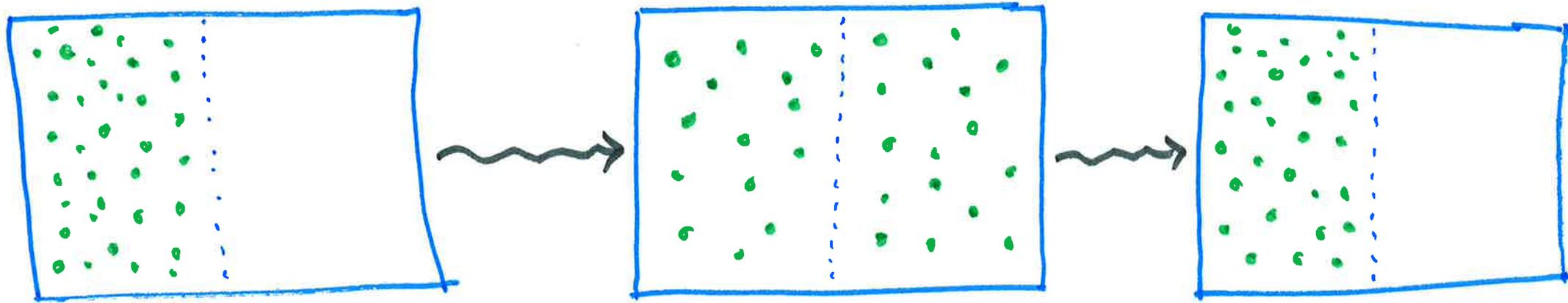
The Solar System



Billiards



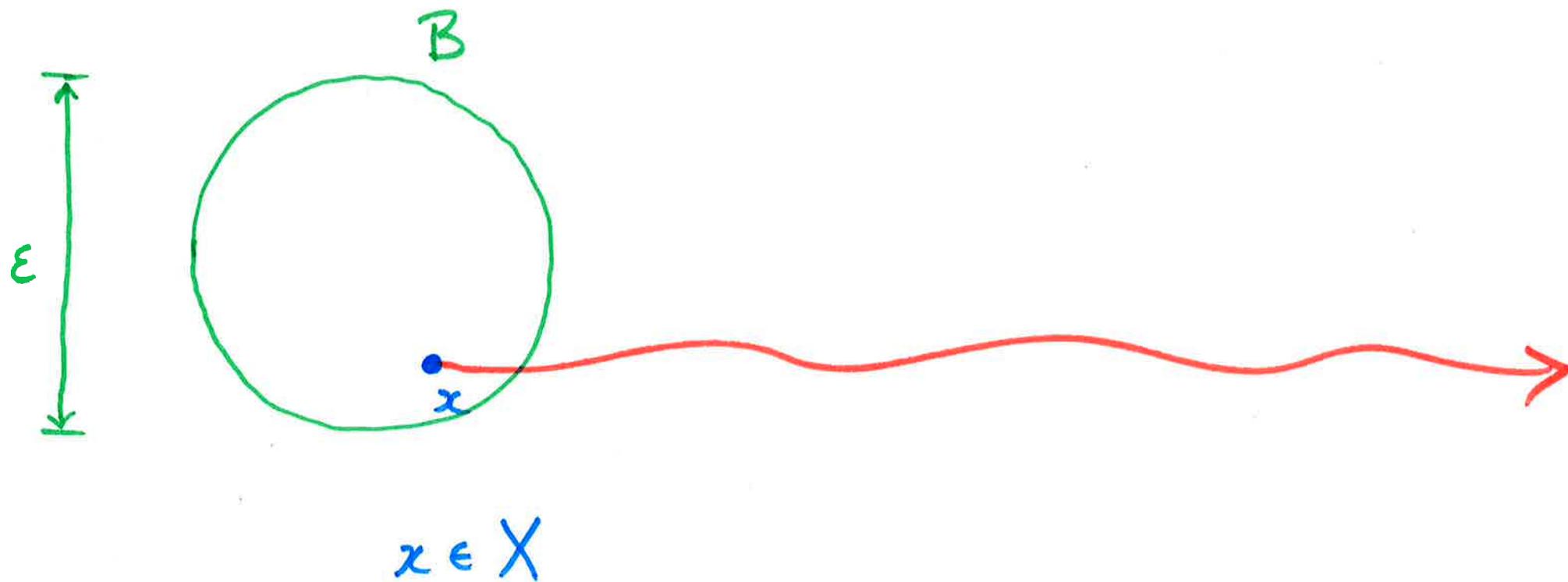
Ideal Gas

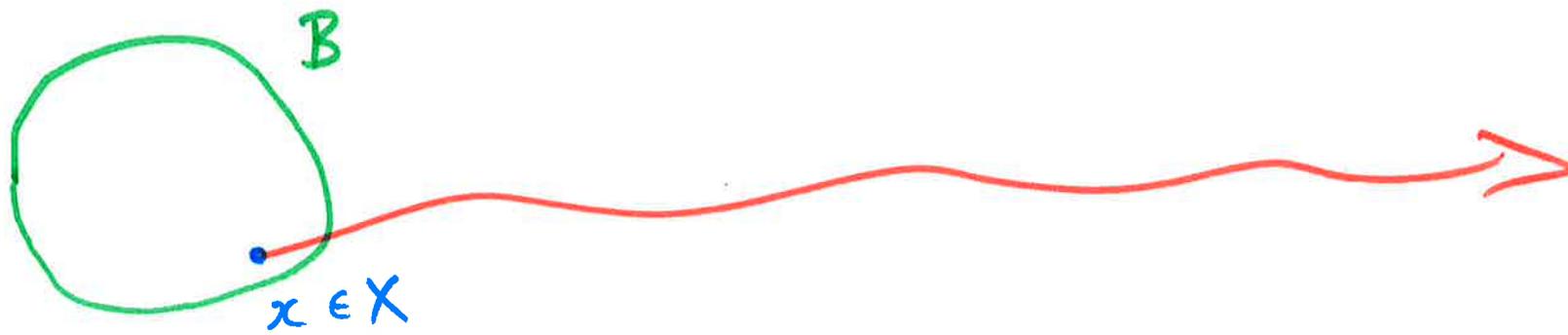


Idea of proof

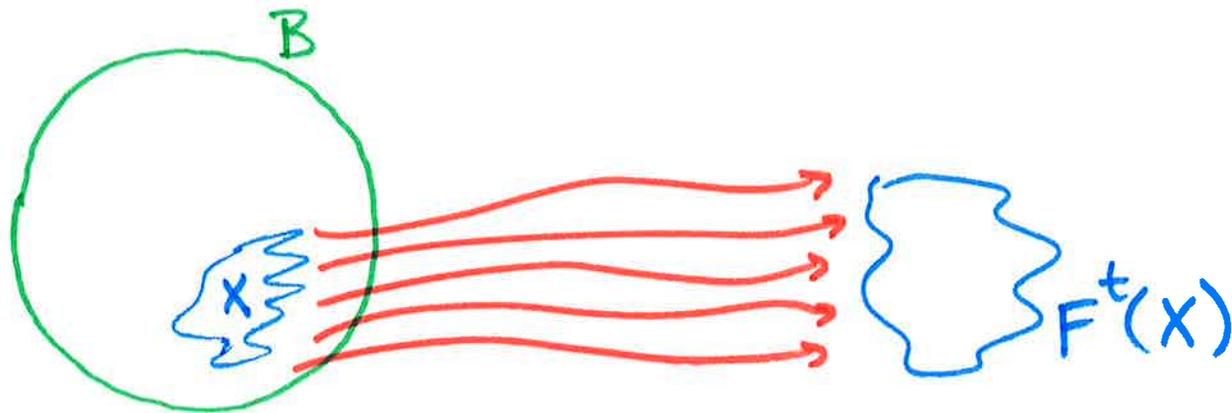
Consider a ball B
of diameter ϵ .

Let $X \subset B$ be the set of all points
which leave B and never return.

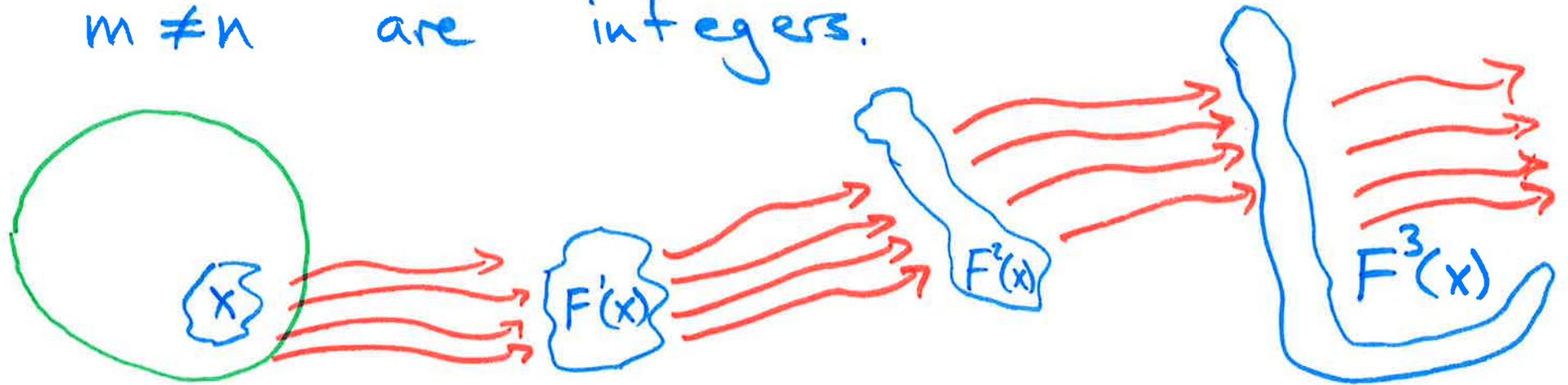




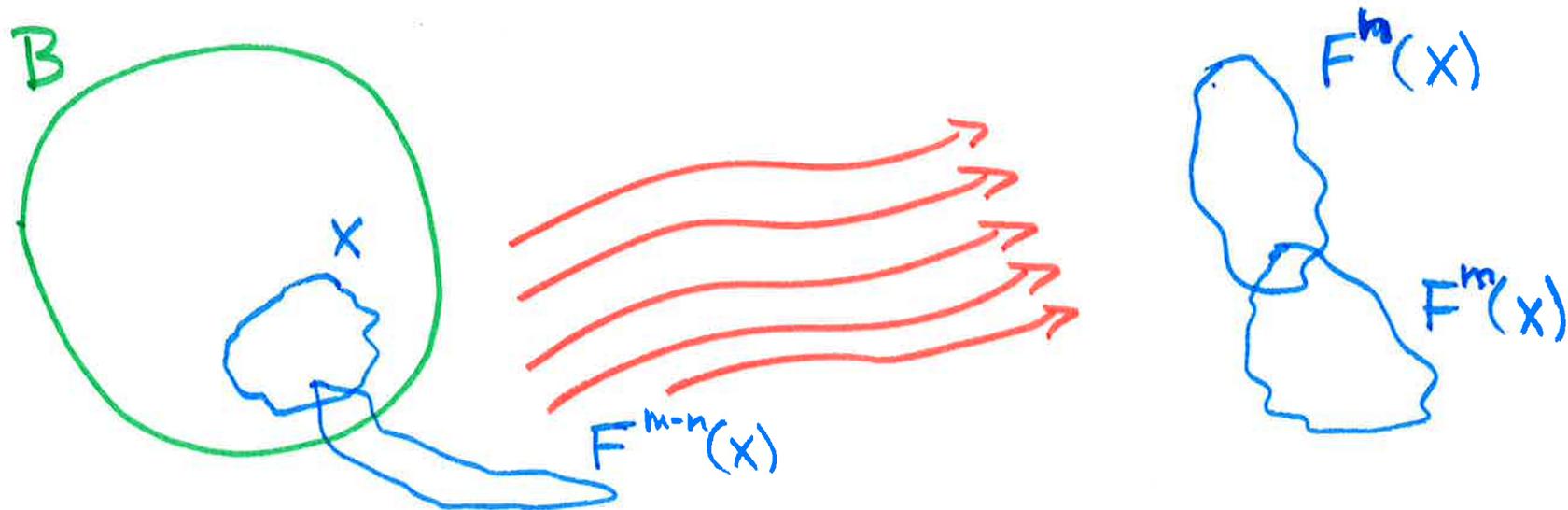
Note that $F^t(X)$ does not intersect B
for any $t \geq 1$.



Consider $F^m(x)$ and $F^n(x)$ where $m \neq n$ are integers.



If $F^m(x)$ intersected $F^n(x)$, we could flow back to show $F^{m-n}(x)$ intersected $X \subset B$.



So the sets

$X, F^1(X), F^2(X), F^3(X), \dots$

are pairwise disjoint.

Note that $\text{volume}(X) = \text{volume}(F^1(X)) = \text{volume}(F^2(X))$
and so on.

If this volume is positive, the set

$X \cup F^1(X) \cup F^2(X) \cup F^3(X) \cup \dots$

is an infinite volume subset of the
finite volume phase space.



Hence, $\text{Volume}(X) = 0$.

\Rightarrow Almost every point in B is ϵ -recurrent.

\Rightarrow Almost every point in the phase space is ϵ -recurrent.

Take a sequence $\epsilon_n \downarrow 0$.

Then, almost every point in phase space is recurrent.

QED