MTH 1035 – Solutions to Handout 1 – Lipschitz continuity

**Exercise.** For the following claims about real numbers $a$, $b, c, d$, which always hold? Which hold under extra assumptions?

Note: for this question (and only this question), it is enough just to say which are true without full justification or proof. In fact, some of these are *axioms* of the real numbers, meaning they cannot be proved and instead must just be assumed.

(1) If $a < b$, then $a + c < b + c$.
(2) If $a \le b$, then $a + c \le b + c$.
(3) If $a \le b$, then $a \cdot c \le b \cdot c$.
(4) If $a \le b$, then $a^2 \le b^2$.
(5) If $a^2 \le b^2$, then $a \le b$.
(6) $\sqrt{a^2} = |a|$.
(7) $(\sqrt{a})^2 = |a|$.
(8) $|a + b| = |a| + |b|$.
(9) $|a + b| \le |a| + |b|$.
(10) $|a - b + c - d| \le |a - b| + |c - d|$.
(11) $|ab| = |a|\,|b|$
(12) If $a \le b$, then $\frac{1}{b} \le \frac{1}{a}$.

*Answer.* Items 1,2,6,9,10,11 always hold. Item 3 holds if $c \ge 0$. Items 4,5 hold if $a, b \ge 0$. Item 7 requires $a \ge 0$ to be well-defined. Item 8 holds if $a, b \ge 0$ or $a, b \le 0$. Item 12 holds if $a, b > 0$ or $a, b < 0$. Also, items 3,4,5,12 hold if $a = b$.

Which number lies halfway between 1 and 9? In an arithmetic sense, $5 = \frac{1}{2}(1+9)$. In a geometric sense, $3 = 3^1$ is halfway between $1 = 3^0$ and $9 = 3^2$. For positive numbers $x$ and $y$, define $\frac{1}{2}(x + y)$ as the *arithmetic mean* and $\sqrt{xy}$ as the *geometric mean*.

**Exercise.** Show the AM-GM inequality: $\sqrt{xy} \le \frac{1}{2}(x + y)$.
*Answer.* Observe for $x, y > 0$ that

$$
\begin{aligned}
\sqrt{xy} \le \tfrac{1}{2}(x+y) \quad &\Leftrightarrow \\
2\sqrt{xy} \le x + y \quad &\Leftrightarrow \\
(2\sqrt{xy})^2 \le (x+y)^2 \quad &\Leftrightarrow \\
4xy \le x^2 + 2xy + y^2 \quad &\Leftrightarrow \\
4xy - 4xy \le x^2 + 2xy + y^2 - 4xy \quad &\Leftrightarrow \\
0 \le x^2 - 2xy + y^2 \quad &\Leftrightarrow \\
0 \le (x-y)^2 &
\end{aligned}
$$

and the last line is always true.

**Exercise.** Show that if $1 \le x, y \le 2$, then $|x^2 - y^2| \le 4|x - y|$.

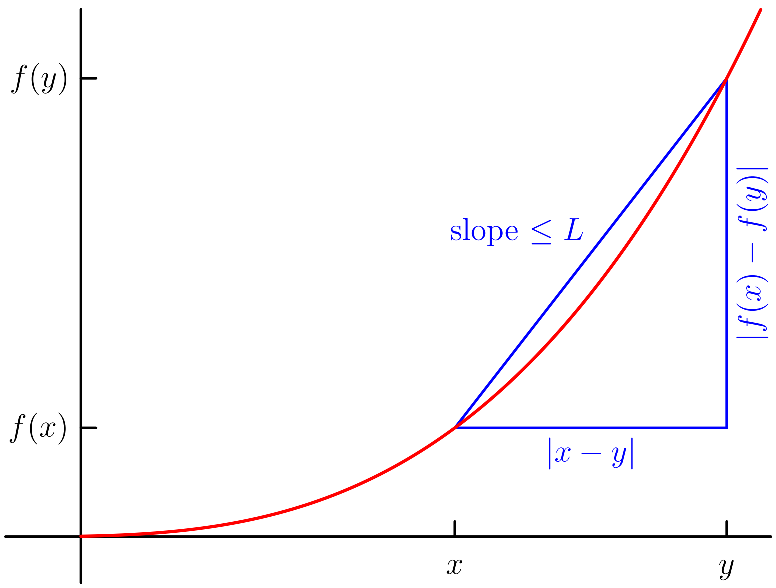
Note: the notation $1 \le x, y \le 2$ means that *both* $x$ and $y$ are between 1 and 2.
*Answer.* Since $x^2 - y^2 = (x + y)(x - y)$. It follows that

$$
\begin{aligned}
|x^2 - y^2| &\le |x + y| \cdot |x - y| \quad \Rightarrow \\
|x^2 - y^2| &\le (|x| + |y|) \cdot |x - y| \quad \Rightarrow \\
|x^2 - y^2| &\le (2 + 2) \cdot |x - y|.
\end{aligned}
$$

Let $X$ be a subset of $\mathbb{R}$. (For example, $X = [1,2]$ or $X = [0,\infty)$ or $X = \mathbb{R}$.) A function $f : X \to \mathbb{R}$ is called *Lipschitz* if there is a constant $L > 0$ such that

$$|f(x) - f(y)| \le L|x - y|$$

for any two numbers $x, y \in X$.
For instance, the previous exercise shows that the function $f : [1,2] \to \mathbb{R}$ defined by $f(x) = x^2$ is Lipschitz with constant $L = 4$.




**Exercises.**

(1) Show that the function $g : [1,2] \to \mathbb{R}$ defined by $g(x) = x^3$ is Lipschitz.
*Answer.* Since $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$. It follows that

$$|g(x) - g(y)| \leq |x^2 + xy + y^2| \cdot |x - y| \quad \Rightarrow$$
$$|g(x) - g(y)| \leq (x^2 + |xy| + y^2) \cdot |x - y| \quad \Rightarrow$$
$$|g(x) - g(y)| \leq (4 + 4 + 4) \cdot |x - y|,$$

so $g$ is Lipschitz with constant 12.

(2) Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is **NOT** Lipschitz. That is, for any $L > 0$, find $x, y \in \mathbb{R}$ such that

$$|f(x) - f(y)| > L|x - y|.$$

*Answer.* Take $x = 2L$ and $y = L$. Then $|f(x) - f(y)| = 3L^2 > L^2 = L|x - y|$.

(3) Prove the following: if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions, then the composition $f \circ g : \mathbb{R} \to \mathbb{R}$ is Lipschitz.
*Answer.* Let $L_f$ and $L_g$ be the Lipschitz constants of $f$ and $g$ respectively. That is, $|f(x) - f(y)| \leq L_f|x - y|$ and $|g(x) - g(y)| \leq L_g|x - y|$ for any $x, y \in \mathbb{R}$. Then,

$$\begin{aligned} |(f \circ g)(x) - (f \circ g)(x)| &= |f(g(x)) - f(g(y))| \\ &\leq L_f|g(x) - g(y)| \\ &\leq L_f L_g|x - y|, \end{aligned}$$

so $f \circ g$ is Lipschitz with constant $L = L_f L_g$.

(4) Prove the following: if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions, then the sum $f + g : \mathbb{R} \to \mathbb{R}$ defined by $(f + g)(x) = f(x) + g(x)$ is Lipschitz.

*Answer.* With $L_f$ and $L_g$ as above,

$$\begin{aligned}|(f+g)(x)-(f+g)(y)| &= |f(x)+g(x)-f(y)-g(y)| \\ &= |f(x)-f(y)+g(x)-g(y)| \\ &\le |f(x)-f(y)|+|g(x)-g(y)| \\ &\le L_f|x-y|+L_g|x-y| \\ &= (L_f+L_g)|x-y|,\end{aligned}$$

so $f+g$ is Lipschitz with constant $L = L_f + L_g$.

(5) A function $f : X \to \mathbb{R}$ is *bounded* if there is a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in M$.
For instance, sin is bounded, since $|\sin(x)| \le 1 = M$ for all $x \in \mathbb{R}$.
Give an example of a function which is Lipschitz, but not bounded.
*Answer.* Try $f(x) = x$.

(6) Show that any Lipschitz function $f : [a,b] \to \mathbb{R}$ defined on an interval of the form $[a,b]$ is a bounded function
*Answer.* For any $x \in [a,b]$,

$$|f(x)-f(a)| \le L|x-a| \le L|b-a|,$$

and

$$\begin{aligned}|f(x)| &= |f(x)-f(a)+f(a)| \\ &\le |f(x)-f(a)|+|f(a)| \\ &\le L|b-a|+|f(a)|.\end{aligned}$$

so take $M = L|b-a|+|f(a)|$.

(7) Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are both Lipschitz functions and bounded functions. Show that the product $f \cdot g : \mathbb{R} \to \mathbb{R}$ defined by $(f \cdot g)(x) = f(x)g(x)$ is Lipschitz.
*Answer.* Let $L_f$ and $L_g$ be the Lipschitz constants and $M_f$ and $M_g$ the bounds. Then

$$\begin{aligned}(f\cdot g)(x)-(f\cdot g)(y) &= f(x)g(x)-f(y)g(y) \\ &= f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y),\end{aligned}$$

so

$$\begin{aligned}|(f\cdot g)(x)-(f\cdot g)(y)| &= |f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y)| \\ &\le |f(x)g(x)-f(x)g(y)|+|f(x)g(y)-f(y)g(y)| \\ &\le |f(x)|\cdot|g(x)-g(y)|+|f(x)-f(y)|\cdot|g(y)| \\ &\le M_fL_g|x-y|+L_f|x-y|M_g \\ &= (M_fL_g+L_fM_g)|x-y|.\end{aligned}$$

This shows $f \cdot g$ is Lipschitz with constant $L = M_fL_g + L_fM_g$.

(8) Show that $h : [0,1] \to \mathbb{R}$ given by $h(x) = \sqrt{x}$, is bounded, but not Lipschitz.
*Answer.* If $0 \le x \le 1$, then $0 \le \sqrt{x} \le 1$ and therefore $|h(x)| \le 1$. For a constant $0 < L < 1$, take $x = 0$ and $y = 1$ and observe that

$$|h(x)-h(y)| = 1 > L = L|x-y|.$$

For a constant $L > 1$, take $x = 0$ and $y = \frac{1}{4L^2}$ and observe that

$$|h(x) - h(y)| = \frac{1}{2L} > \frac{1}{4L} = L|x - y|.$$

(9) Show that the absolute value function $f : \mathbb{R} \to [0, \infty)$ given by $f(x) = |x|$ is Lipschitz, but is not differentiable at zero.

*Answer.*

We will argue case by case that $\big||y| - |x|\big| \leq |y - x|$. From this, it follows that the absolute value function is Lipschitz with constant $L = 1$. Note that since

$$\big||x| - |y|\big| = \big||y| - |x|\big| \quad \text{and} \quad |x - y| = |y - x|,$$

we are free to exchange $x$ and $y$. Therefore, we may freely assume that $x \leq y$. This reduces the problem to three cases. In case one, if $0 \leq x \leq y$, then

$$\big||y| - |x|\big| = y - x = |y - x|.$$

In case two, if $x < 0 \leq y$, then

$$\big||y| - |x|\big| = y + x < y - x = |y - x|.$$

Finally in case three, if $x \leq y < 0$, then

$$\big||y| - |x|\big| = y - x = |y - x|.$$

This establishes

$$\big||y| - |x|\big| \leq |y - x|$$

in all cases and so the function is Lipschitz.

One can see that the absolute value function $f(x) = |x|$ is not differentiable at zero. In particular,

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 1 \quad \text{and} \quad \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = -1,$$

so the two-sided limit

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist.

On an interesting historical note, Isaac Newton first used the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h}$$

as the definition of the derivative. Under this definition, the absolute value function would be differentiable everywhere. However, using this definition would cause serious problems later on in calculus and so it was soon replaced with the definition of the derivative we have today.

**Bonus Exercises.**

(1) Let $X$ be a subset of $\mathbb{R}$. A function $f : X \to \mathbb{R}$ is called *Hölder* if there are constants $H > 0$ and $0 < \theta < 1$ such that

$$|f(x) - f(y)| \leq H|x - y|^\theta$$

for any two numbers $x, y \in X$ with $|x - y| < 1$. Try the exercises in the handout again with Hölder in place of Lipschitz.

(2) Suppose $f$ is everywhere differentiable. Show that $f$ is Lipschitz if and only if its derivative is bounded.