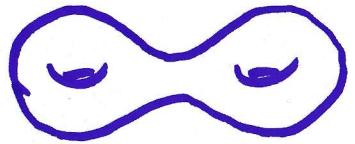
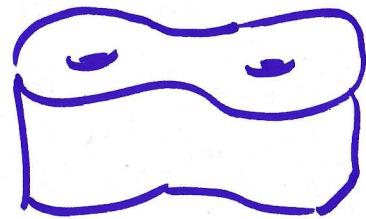


# Ergodicity and Accessibility in dimension 3

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$\Sigma$  - surface of curvature  $K = -1$



$\varphi^t$  - geodesic flow on the unit tangent bundle

Thm [Grayson-Pugh-Shub, 1994]

The time-one map  $\varphi^1$  is stably ergodic  
as a diffeo.

two keys tools

- 1) partial hyperbolicity
- 2) accessibility

# partial hyperbolicity:

diff<sup>o</sup> f: M<sup>S</sup>

$$TM = E^u \oplus E^c \oplus E^s$$

Diagram illustrating the decomposition of the tangent bundle  $TM$  into three subbundles:  $E^u$  (red),  $E^c$  (green), and  $E^s$  (blue). The decomposition is shown as a direct sum ( $\oplus$ ). Above each subbundle, a curved arrow labeled  $Df$  indicates the action of the derivative of the map  $f$ . Below each subbundle, an upward arrow indicates the direction of flow or behavior:

- $E^u$  (expanding): Red arrow pointing upwards.
- $E^c$  (dominated): Green arrow pointing upwards.
- $E^s$  (contracting): Blue arrow pointing downwards.

accessibility : for any  $x, y \in M$

$x$

$y$

# The Pugh-Shub Conjectures:

Among vol. pres. partially hyperbolic  $C^r$  diffeos ( $r \geq 2$ )

- 1) accessibility is open and dense
- 2) accessibility implies ergodicity
- 3) ergodicity is open and dense

The conjectures have been proved  
in certain settings.

Incomplete list of credits:

Birkhoff, Hopf, Anosov, Sinai, Brin, Pesin, Grayson, Pugh,  
Shub, Didier, Dolgopyat, Wilkinson, Burns, Rodriguez-Hertz,  
Rodriguez-Hertz, Ures, Talitskaya, Tahzibi, Avila, Crovisier, ...

The conjectures are true when

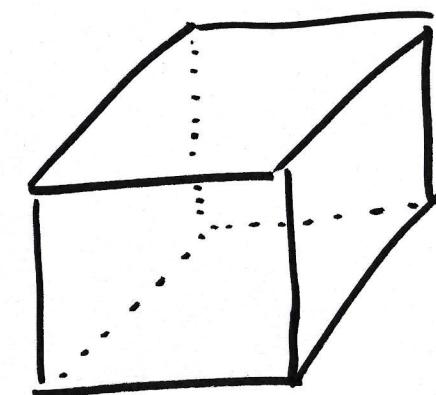
$E^c$  is one-dimensional.

What are the non-accessible  
or non-ergodic systems?

Can we classify them?

the 3-torus

$$\mathbb{T}^3 = \mathbb{R}^3 / \mathbb{Z}^3$$



My PhD Thesis with Charles (2009):

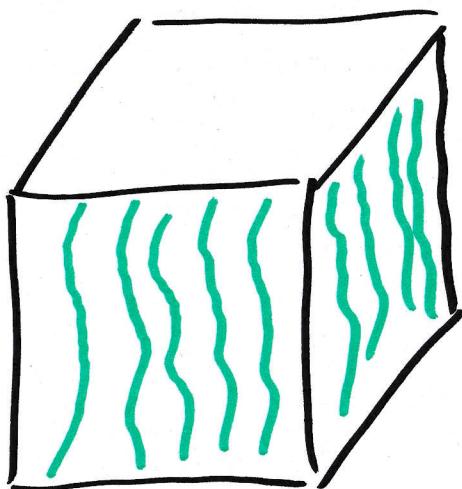
Any absolutely partially hyperbolic diffeo  
 $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$

is leaf conjugate to a linear system.

Later, w/ R. Potrie : can replace "absolutely"  
with "volume preserving"

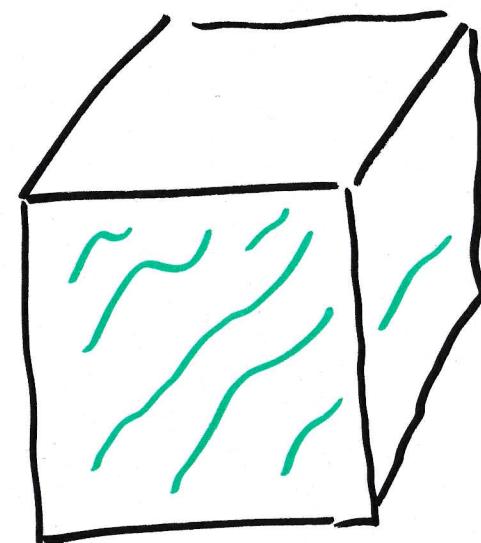
There are two types of p.h. systems on  $\mathbb{T}^3$

(topological) skew products



center  
circles

Derived - from - Anosov  
systems

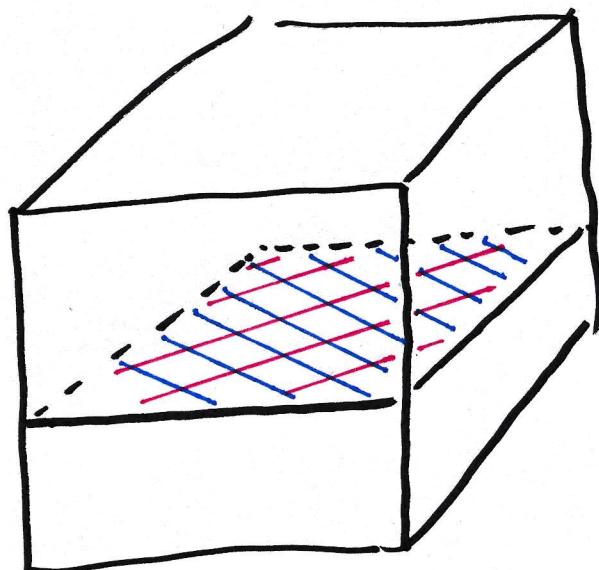


center  
lines

non-accessible  
examples

for skew products on  $\mathbb{T}^3$

## Example #1

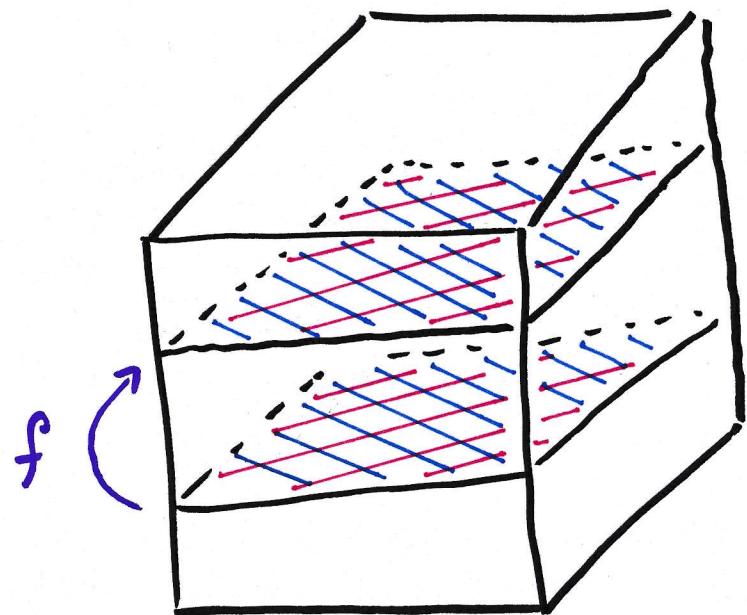


A  $\times$  id  
cat map  
on  $\Pi^2$       identity  
                  on  $S'$

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Accessibility classes are of the form  $\pi^2 \times \{z\}$  for  $z \in S'$ .

## Example #2



$A \times R_\theta$   
cat map  
on  $\mathbb{T}^2$       irrational  
                    rotation on  $S^1$

$$R_\theta(z) = z + \theta \pmod{1}$$

This is ergodic, but  
not stably ergodic and  
not mixing.

Example #3

Start with  $A \times \text{id}$  on  $\pi^2 \times S^1$ .

Choose an open subset  $U \subset S^1$ .

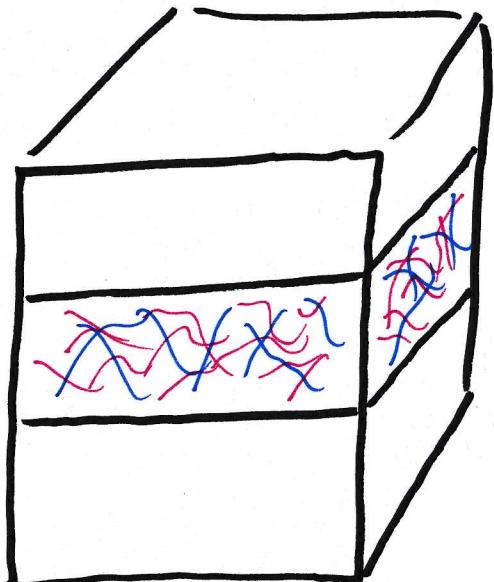
For each connected component

$$I \subset U$$



perturb on  $\pi^2 \times I$   
to make it an accessibility class.

$$\pi^2 \times I$$



$\pi^3$

$S^1$

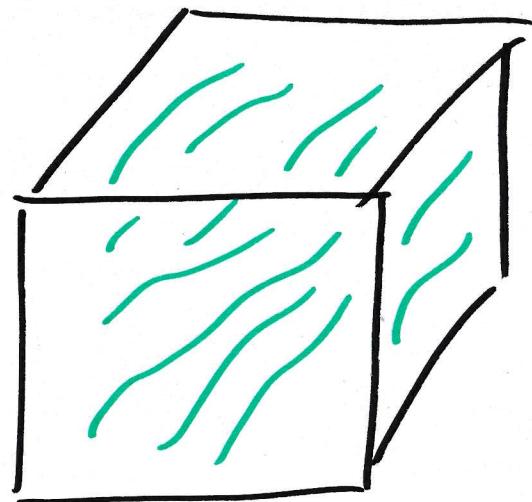
Thm [H, 2017] If  $f: \mathbb{P}^3\mathcal{S}$  is a  $C^2$  vol. pres. top. skew product,  
then one of the following holds:

- 1)  $f$  is accessible and stably ergodic
- 2)  $f$  is top. conjugate to  $\underline{A} \times \underline{R_\theta}$   
Anosov rotation
- 3) there is a  $C^1$  surjection  $p: \mathbb{P}^3 \rightarrow S'$   
and  $U \subset S'$  open such that  
the accessibility classes of  $f$  are
  - $p^{-1}(I)$  where  $I$  is a conn comp of  $U$
  - $p^{-1}(\{z\})$  where  $z \in S' \setminus U$ .

slightly more complicated

results hold for non-volume  
preserving skew-products

Derived - from - Anosov  
systems on  $\mathbb{T}^3$



Thm [H, Ures 2014] Suppose  $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$  is vol. pres., p.h.,  
and homotopic to a linear Anosov diffeo  $A: \mathbb{T}^3 \rightarrow \mathbb{T}^3$

Then either 1)  $f$  is acc. and erg. or

2) there is a topological conjugacy  
 $h$  between  $f$  and  $A$ , and  
 $h$  maps the  
stable, center, and unstable  
foliations of  $f$  to those of  $A$ .

Thm [Gan-Shi, 2020] A vol. pres.  $C^2$  p.h. diffeo  $f: \mathbb{T}^3 \rightarrow \mathbb{T}^3$   
is accessible or Anosov or both.

Thm [H-Shi, 2021] Same result in the non-volume  
preserving setting.

# Other 3-manifolds

Nil

accessible

[Rodríguez-Hertz  $\times 2$ , Ures]

Sol

same as skew products on  $\mathbb{T}^3$

[H, Potrie], [H]

Seifert Manifolds

acc. proved in  
certain isotopy classes

[H, Potrie, Shannon]

[H, Jana R-H, Ures]

Hyperbolic 3-Manifolds

accessible

[Barthélémy, Frankel  
Fenley, Potrie]

[Fenley, Potrie]

Thank you for all of your help,  
Charles,  
and Happy Birthday!