Exercise #2
Mini-course on the Classification of Anosov Systems

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Disclaimer: While I have taken care in preparing questions, I have not solved all of these problems in complete detail. As such, there may be errors or omissions in the problems as stated. These could be considered as part of the exercise. Please, contact me if you have any questions.

1. Suppose Λ is a finite subset of the unit circle in $\mathbb{C}$. Show that if

$$\lim_{n\to\infty} \prod_{\lambda\in\Lambda} |1-\lambda^n|$$

exists (and is non-zero), then $\Lambda = \varnothing$.

2. Suppose $f : M \to M$ is Axiom A and Ω_1 is a basic set such that $W^u(x) \subset \Omega_1$ for all $x \in \Omega_1$. Show that if $x \in \Omega_1$ is periodic, then $\overline{W^u(x)} = \Omega_1$.

3. Show that an Anosov diffeomorphism has Global Product Structure (or is splitting) if and only if the following holds: For every compact subset $K \subset \overline{M}$ of the universal cover, the set of points

$$\{x \in \overline{M} : W^s(x) \cap K \neq \varnothing \quad \text{and} \quad W^u(x) \cap K \neq \varnothing\}$$

is compact.

4. For a map $f : \mathbb{R}^n \to \mathbb{R}^n$, the index $I_x(f)$ of a fixed point x is given by the degree of the map

$$g(y) = \frac{y - f(y)}{\|y - f(y)\|}$$

defined on a circle around the point. For a map on a manifold $f : M \to M$, the index is given be choosing a coordinate chart $\mathbb{R}^n \to M$ about the fixed point.

   (a) Show that the index is independent of the choice of coordinate chart.

   (b) Show that if f is a diffeomorphism, then $I_x(f) = \pm 1$ where the sign is given by the sign of the determinant of $I - Df_x$.

   (c) Show that if x is a hyperbolic fixed point, then $I_x(f)$ is given by $(-1)^u\Delta$ where $u = \dim E^u_x$ and $\Delta = \pm 1$ depending on whether Df_x preserves or reverses the orientation of E^u_x.

   (d) Show for an Anosov diffeomorphism $f : M \to M$, after lifting to a finite cover, we have

$$\#\,\mathrm{Fix}(f^n) = |L(f^n)|$$

for all $n \neq 0$. Here $L(f^n)$ is the Lefschetz number.

5. (a) Show that any Anosov diffeomorphism has an infinite number of periodic points.

   (b) Show that there is no Anosov diffeomorphism on a sphere of any dimension.

6. An Anosov diffeomorphism $f : M \to M$ has *Global Product Structure* if for every two points x, y on the universal cover, the lifted unstable leaf through x and the lifted stable leaf through y intersect in a single point, denoted $[x, y]$. Say that f has *Polynomial Global Product Structure* if, in addition, there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that

$$d_u(x, [x, y]) + d_s(y, [x, y]) < p(d(x, y))$$

   where d_u and d_s denote distance measured along the foliations and d is distance on the universal cover. By adapting the work of Brin and Manning, show that any f with Polynomial Global Product Structure is topologically conjugate to an infranilmanifold automorphism.

7. A foliation W is *quasi-isometric* if, after lifting to the universal cover, there is a linear polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $d_W(x, y) \leq p(d(x, y))$ for all x, y on the same lifted leaf. Show that if the unstable and stable foliations of an Anosov diffeomorphism are quasi-isometric, the system has Polynomial Global Product Structure.

   See my preprint "Dynamics of quasi-isometric foliations" for more details.