Exercise #1
Mini-course on the Classification of Anosov Systems
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Disclaimer: While I have taken care in preparing questions, I have not solved all of these problems in complete detail. As such there may be errors or omissions in the problems as stated. These could be considered as part of the exercise. Please, contact me if you have any questions.

1. Show that any homeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ with $f_* : H_1(\mathbb{T}^2) \to H_1(\mathbb{T}^2)$ hyperbolic has a fixed point. For f_* not hyperbolic, give a counterexample.

2. (a) For any Anosov diffeomorphism $f : M \to M$, show that if x and $f^n(x)$ lie on the same unstable leaf, then there is a periodic point y also on that leaf. Find a formula bounding the distance $d(x, y)$ in terms of $d(x, f^n(x))$ and the constants given in the definition of an Anosov diffeomorphism.

   (b) Fix $r > 0$ small. Show that if $d(x, f^n(x))$ is sufficiently small $(n > 0)$ and $W^s_r(x)$ is the connected component of $W^s(x)$ in the r-ball about x, then for every $y \in f^n(W^s_r(x))$ there is $z \in W^s_r(x)$ such that y and z are connected by a short unstable path.

   (c) Prove for every $\epsilon > 0$, there is $\delta > 0$ such that if $d(x, f^n(x)) < \delta$, there is a periodic point $y \in M$ where $d(x, y) < \epsilon$.

   (d) Show that any Anosov diffeomorphism is Axiom A.

3. For a diffeomorphism $f : M \to M$, show that $NW(f) = M$ if and only if $NW(f^n) = M$. If $\tilde{M}$ is a finite cover for which there is a lift $\tilde{f} : \tilde{M} \to \tilde{M}$, show that $NW(f) = M$ if and only if $NW(\tilde{f}) = \tilde{M}$.

4. Using graph transform techniques (as in, say the book of Hirsch-Pugh-Shub), show that any Anosov map on $\mathbb{T}^2$ has C^1 stable and unstable foliations.

5. Show that the Arnol'd cat map is ergodic by using the Fourier transform on $\mathbb{T}^2$ to show that any invariant L^1 function is constant almost everywhere.

6. For a homeomorphism $f : X \to X$ on a second countable Baire space, show that f has a dense orbit if and only if for all non-empty open sets $U, V \subset X$, there is $n \in Z$ such that $U \cap f^n(V) \neq \varnothing$. Show in fact that if f has a dense orbit, then the generic point has dense orbit.