Towards a general 3+1 continuous time formulation of the Regge Calculus

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Abstract

A completely general continuous time 3+1 formulation of the Regge Calculus will be presented. The theory will be based on an integral action obtained as the limiting form of the discrete action. The Regge field equations will be obtained by extremizing the action and will be seen to be a set of coupled ordinary differential equations. It will also be argued that a cubic lattice is to be preferred over a tetrahedral lattice primarily for two reasons, cubes are far better suited to adaptive schemes and secondly, cubic lattices provide a natural correspondence with the continuum theory.

1. Motivation

In the original formulation of the Regge Calculus, spacetime was represented by a large collection of 4-dimensional simplicial blocks. In each pair of adjacent blocks the metric was chosen to be flat and Lorentzian with the curvature localized to the 2-dimensional interfaces. All aspects of the geometry of the spacetime were encoded in the leg lengths of each block and the pattern in which the blocks were assembled to form the spacetime manifold.

This structure is particularly appealing for many reasons. It is well suited to numerical studies, it explicitly employs the equivalence principle, it describes gravity as a finite dimensional system and it employs concepts which any high school student could understand (after
a brief tutorial on Lorentzian geometry). It has been applied both in numerical relativity
and as a possible formulation of a theory of quantum gravity.

The purpose of this paper is to present a variation of Regge’s theory in which the time
parameter is continuous and the 3-dimensional blocks of the Cauchy surfaces are cubes rather
than tetrahedra. These two simple variations lead to a formulation with some particularly
important advantages over the traditional Regge Calculus.

In the fully discrete theory the field equations are a coupled system of non-linear alge-
braic equations, whereas in the continuous time approach the field equations are a set of
quasi-linear ordinary differential equations. This will be established in section (...). For
applications in numerical relativity this shift from non-linear to linear equations must surely
be a significant advantage. The use of cubes rather than tetrahedra as the fundamental block
may seem alien to the spirit of the Regge Calculus. In particular one might fear that the
resulting theory would be less general than that based on tetrahedra. There is some truth to
this statement but in fact it is not a real point of concern. It has been known for some time
that theories based on tetrahedra have more degrees of freedom than what one would expect
from Einstein’s theory. Choosing a cube as the basic building block avoids this problem and
leads to the correct number of degrees of freedom. The use of cubes will also impose some
degree of regularity in the datastructures. This could be used to advantage when designing
computer programs. It should also be of some advantage should one choose to use adaptive
algorithms. For example, one may wish to use a multigrid method when solving the initial
value problem or one may wish to add more cubes in regions where the curvature is increas-
ing (eg. during the formation of black holes). Without a regular datastructure it may be
quite difficult to implement these techniques.

Continuous time approaches to the Regge Calculus are not new (...) but it can be argued
that all previous approaches have their limitations. The first such theory was that of Porter.
In his theory the basic building block was a truncated 4-simplex (ie. a 4-simplex with the
top cut off). The main objection with that choice was that the timelike faces were planar
and thus it was hard to imagine how a spacelike leg could twist as it evolved forward in
time. Some years later Piran and Williams, and Friedmann and Jack attempted to form a
true Hamiltonian theory for the Regge Caculus. Their philosophies were similar in that they
attempted to construct a Regge action by making an informed guess as to how each term
of the continuum action should be represented in the Regge action. The difficulty here was
that this procedure was ambiguous in that there were many seemingly reasonable ways to
carry over terms from the continuum to the discrete. In the formulation due to Brewin (...) the
continuous time action was constructed from the fully discrete Regge action by reducing
the time step between successive Cauchy surfaces to zero. Though this led to a well defined
theory it later became apparent that some of the assumptions made were too restrictive and
the theory was only valid for a limited class of spacetimes. As it is from this theory that
the current theory has been developed it is instructive to examine exactly where the theory
fails. The starting point was the Regge action written as

\[ I = \sum_i (\alpha A)_i + \sum_j (\beta B)_j \]

where \( \alpha_i \) was the defect on a timelike bone with area \( A_i \) and \( \beta_j \) and \( B_j \) were their counterparts on the spacelike bones. This action included \( n + 1 \) Cauchy surfaces starting from \( t_{\text{begin}} = t_0 \) and ending at \( t_{\text{end}} = t_n \) with a time step of \( \delta t = (t_{\text{end}} - t_{\text{begin}})/n \) between each pair of slices. In order that the limit of \( n \to \infty \) would be well defined it was necessary to establish that the terms (or groups of terms) in the sums behaved as \( O(\delta t) \). For the timelike bones this was easy since the areas \( A_i \) could be seen to vary as \( A_i = O(\delta t) \). However for the spacelike bones \( B_j = O(1) \). It was therefore assumed, and it seemed reasonable at the time to do so, that each \( \beta_j = O(\delta t) \). It was then a straightforward matter to compute the continuous time action and the associated field equations. However, since then it has become apparent, by direct numerical calculation, that the \( \beta_j \) do not each vary as \( O(\delta t) \) but rather only their sum between successive Cauchy surfaces varies as \( O(\delta t) \). The simple metric

\[ ds^2 = \exp(2\gamma t)(-dt^2 + \gamma dt(ydx - xdy) + dx^2 + dy^2 + dz^2) \]

displays this behaviour for any non-zero constant \( \gamma \). The aim of this paper is to overcome all of the above limitations and to present a truly general continuous time formulation of the Regge Calculus.

2. 2+1 Kinematics

In order to simplify the discussion a Euclidian 2+1 continuous time theory will first be derived. This will then be generalized to a Lorentzian 3+1 theory.

Consider a pair of two dimensional Cauchy surfaces embedded in some three dimensional spacetime. Consider some Cartesian like coordinate system \((t, x, y)\) with the time coordinate \( t \) chosen to be constant on each Cauchy surface. This coordinate system induces a natural sub-division of the three dimensional space bounded by the Cauchy surfaces into a set of cubes, ie. a cubic lattice. The coordinate system serves no purpose other than to help us visualize this cubic lattice.

Suppose one now specifies the leg lengths. The lattice is clearly not rigid in that the upper Cauchy surface can be displaced sideways without changing any of the leg lengths. To obtain a rigid structure one must add extra information. One option is to subdivide each cube into a set of tetrahedra. There are many ways in which this can be done. However, a simpler option is to just introduce one diagonal brace for each two dimensional face of the cubes. The surface of each cube now appears as a triangulation of \( S^2 \). It is therefore not hard to see that the lattice is now rigid (Cauchy (...)). There remains a small degree of freedom in that there are two choices for each diagonal. The procedure that we shall adopt goes as follows.
Choose a cube and locate its front lower left vertex. Then add three diagonals so as to form a tetrahedron containing this vertex. This introduces three diagonals on the surface of this cube, leaving the three remaining faces un-subdivided. However, by repeating this process for each cube one will find that each face does obtain a diagonal brace. The construction so far is shown in figure (1a,b).

One can imagine this construction being repeated throughout the whole spacetime. On each Cauchy surface there would stand a tetrahedron at each grid point. The metric in each cube would be fully determined by nine pieces of data, the six leg lengths of a tetrahedron on the lower Cauchy surface and the three leg lengths of the base triangle of the tetrahedron in the upper Cauchy surface. These nine pieces of data can in turn be viewed as point values of the six leg lengths of a time dependent tetrahedron (ie. six $L_{ij}(t)$ and three $L_{ij}(t + \delta t)$). This later view will be of use when we re-cast the discrete theory as a continuous time theory.

Some similarity with the continuum may now be apparent. The three legs of the tetrahedra that form the edges of the cube play the role of $g_{xx}, g_{yy}$ and $g_{tt}$. Whereas the three diagonals are simply related to $g_{xt}, g_{yt}$ and $g_{xy}$.

An important question now arises. Does this structure have the same number of degrees of freedom as the continuum? In the later case we know that the essential data are the three components of the intrinsic metric $h_{ij}$, one lapse function $N$, two shift functions $\beta_i$ and the three components of the extrinsic curvature $K_{ij}$. In all, a total of nine functions. For our discrete structure we have nine pieces of data and so it would seem that our discrete structure is capable of representing any continuum metric.

In the traditional approach to the Regge Calculus one would fully subdivide each cube into tetrahedra. This can be easily done in our structure by adding one extra body diagonal as depicted in figure 1c. As the metric in the cube is required to be flat we would need to choose the length of this leg so that the defect on this leg was zero. One is free however not to make this choice in which case the metric interior to each cube would no longer be flat. One would also have one extra degree of freedom, ie. 10 pieces of data rather than 9. The point here is that the generic Regge Calculus has more degrees of freedom than is necessary to represent an arbitrary smooth metric. This is one reason why the constraint that the interior metric be flat was chosen.

For computational purposes it is very convenient to retain the traditional view that the spacetime is fully sub-divided into tetrahedra. Some of these tetrahedra will be the original tetrahedra introduced to rigidify the cubic lattice. Such tetrahedra will be referred to as the primary tetrahedra. The remaining set of tetrahedra are those that arose after the introduction of the body diagonals and will be referred to as secondary tetrahedra. On each primary tetrahedron their will be six bones which we will refer to as the primary bones. However on each secondary tetrahedra there will be a set of primary bones and a set of secondary bones (eg. the body diagonal). Our stated aim of having a cubic lattice in which each cube is flat is equivalent to requiring the defect on each secondary bone to vanish.
These constraints can be dealt with in a very simple manner. Consider the generic Regge action written as

\[ I = \sum_i \theta_i A_i + \sum_i \theta_i' A_i' \]

where the two sum correspond, respectively, to the primary and secondary bones. The measures of the bones have been written as \( A_i \) and \( A_i' \) just to make it easier to extend the argument to 3+1 dimensions. Taking variations and using Regge’s identity one has

\[ \delta I = \sum_i \theta_i \delta A_i + \sum_i \theta_i' \delta A_i' \]

The variations must observe the constraints, which are just \( \theta_i' = 0 \). and so the second sum vanishes. Next notice that

\[ \delta A_i = \sum_j \frac{\partial A_i}{\partial L_j} \delta L_j + \frac{\partial A_i}{\partial L_j'} \delta L_j' \]

where \( L_i \) and \( L_i' \) are the independent and dependent legs lengths and the \( \delta L_j' \) are the variations required by the contraints. But the \( A_i \) do not depend on the \( L_i' \) (ie. in 2+1 dimensions \( A_i = L_i \)). Thus the full variation of the action which observes the constraints is just

\[ \delta I = \sum_i \theta_i \frac{\partial A_i}{\partial L_j} \delta L_j \]

This variation bears no trace of the constraints and is formally identical with what one would have obtained by ignoring the contributions from the secondary bones and treating the action as if it were unconstrained. The result is that the constraints play only a passive role in that they serve to define the values for the \( L_i' \) but they do not provide any explicit contribution to the field equations.

The introduction of the body diagonal can be seen to sub-divide the cube into a pair of cylinders each with a triangular cross-section. For much of the following discusion we shall need only consider one of the two cylinders. Each such cylinder is fully characterized by twelve leg lengths (nine from the parent cube and three from the neigbouring cubes).

Consider now a sequence of Cauchy surfaces, \( S_n \), \( n = 0, 1, 2, \cdots N \). Assign a coordinate time to each surface, \( t_n = n \delta t \), where \( \delta t = \Delta t/N \) is the time step and \( \Delta t \) is some fixed number. Between each pair of Cauchy surfaces we have a well defined triangular cylinder. Thus all of the defects between \( S_0 \) and \( S_N \) can be calculated and consequently the action can be expressed solely in terms of the leg lengths. Our aim is to examine the form of the action as \( N \rightarrow \infty \) ie. as \( \delta t \rightarrow 0 \). The only assumption that we will require is that in the limit as \( \delta t \rightarrow 0 \) each leg length becomes a smooth function of the time coordinate.

Some definitions at this stage will be helpful. The vertices in the lower Cauchy surface will be denoted by Latin indices. The corresponding vertices in the upper Cauchy surface will be denoted by the same Latin index but with a prime. Greek indices will be used whenever the
distinction is not important. Various objects will be denoted by the sequence of vertices that define that object. In some cases the order of the vertices will be important, for example the vector joining \((i)\) to \((j)\) will be denoted by \(e_{ij}\) with \(e_{ij} = -e_{ji}\). It is useful to distinguish the three types of legs in each triangular cylinder. A strut will be taken to mean a leg of the form \((ii')\). A diagonal (also called a brace) is a leg of the form \((ij')\) where \(i \neq j\). Finally, a leg will (usually) mean a leg in either Cauchy surface ie. \((ij)\) or \((i'j')\). The leg length between any pair of vertices \((\mu)\) and \((\nu)\) will be denoted by \(L_{\mu\nu}\). There are twelve such leg lengths, three \(L_{ij}\), three \(L_{i'j'}\), three \(L_{ii'}\) and three \(L_{ij'}\).

Clearly as \(\delta t \to 0\) we must have the following

\[
\begin{align*}
L_{ii'} & \to 0 \\
L_{i'j'} & \to L_{ij} \\
L_{ij'} & \to L_{ij}
\end{align*}
\]

In which case the \(L_{\mu\nu}\) fail to adequately describe the geometry when \(\delta t = 0\). This minor problem can be avoided by introducing quantities that are defined on the lower Cauchy surface in such a way that no information is lost as the limit is approached. To this end, define a lapse \(N_i\) on each vertex according to \(L_{ii'} = N_i \delta t\). Define \(\phi_{ij}\) to be the angle between the legs \((ij)\) and \((jj')\). These quantities are depicted in figure (2). Note that, in general, \(\phi_{ij} \neq \phi_{ji}\). These angles specify the direction of the struts relative to the lower Cauchy surface. The lapse \(N_i\) dictates where the future vertex \((i')\) should be located on the worldline of vertex \((i)\). Each future vertex can thus be located and hence all of the \(L_{ij'}\) and \(L_{i'j'}\) computed. What this means is that the twelve \(L_{\mu\nu}\) can be replaced by the three \(L_{ij}\), the three \(N_i\) and the six \(\phi_{ij}\). This choice of data remains well behaved as \(\delta t \to 0\).

In taking the continuous time limit one must accept a slight change of perspective. We have just seen how the twelve numbers \(L_{ij}\), \(N_i\) and \(\phi_{ij}\) fully describe the geometry for one cube lying between a pair of adjacent Cauchy surfaces. This procedure can be repeated for any future number of cubes lying above the original cube. It follows that the geometry of the series of cubes is fully specified by the \(N_i\) and \(\phi_{ij}\) on each Cauchy surface and the \(L_{ij}\) on the original Cauchy surface. In the continuous time limit the independent variables will therefore be the nine functions \(N_i(t), \phi_{ij}(t)\) and the three values \(L_{ij}(0)\). This will issue will only be of importance when we come to develop the action integral.

The typical 2+1 action will be of the form

\[
I = \sum_i (\alpha A)_i + \sum_i (\beta B)_i + \sum_i (\beta' B')_i
\]

where \(\alpha\) and \(A\) are the defect and length of a typical strut, \(\beta\) and \(B\) are the corresponding quantities for a typical leg and finally, \(\beta'\) and \(B'\) refer to the diagonal legs. Our primary goal in this section is to determine the contribution to the action from one triangular cylinder. In particular we need to be able to accurately calculate the dihedral angles subtended at each
of the bones (which in this 2+1 example are just the legs). The calculations are tedious but straightforward (see Appendix A). The important results are as follows.

\[
\cos \theta_i = \frac{\cos \rho_i - \cos \phi_{ji} \cos \phi_{ki}}{\sin \phi_{ji} \sin \phi_{ki}} \\
\cos \theta_{ij} = \frac{\cos \phi_{kj} - \cos \phi_{ij} \cos \rho_j}{\sin \phi_{ij} \sin \rho_j} \\
\cos \theta_{ij'} = \frac{\cos \phi_{k'i'} - \cos \phi_{ij'} \cos \rho_{i'}}{\sin \phi_{ij'} \sin \rho_{i'}} \\
\gamma_{ij'} = \left( \frac{\delta t}{2A_{ijk}} \right) \left\{ -L_{ij}(N_i \sin \phi_{ji} \sin \theta_{ji} + N_j \sin \phi_{ij} \sin \theta_{ij}) + L_{ik}(N_i \sin \phi_{ki} \sin \theta_{ki} + N_k \sin \phi_{ik} \sin \theta_{ik}) \cos \rho_i + L_{jk}(N_j \sin \phi_{kj} \sin \theta_{kj} + N_k \sin \phi_{jk} \sin \theta_{jk}) \cos \rho_j \right\} + O(\delta t^2)
\]

\[
\theta_{ij'} = 2\pi - \theta_{ij} - \theta_{i'j'} - \gamma_{i'j'} + O(\delta t^2) \\
L_{i'j'} = L_{ij} + \delta t(N_i \cos \phi_{ji} + N_j \cos \phi_{ij}) + O(\delta t^2) \\
L_{ij'} = L_{ij} + \delta t(N_j \cos \phi_{ij}) + O(\delta t^2)
\]

2.1. The Continuous Time Action

Consider three successive Cauchy surfaces starting with \(S_n\). Consider a pair of triangles \((ijk)\) and \((ijl)\) in \(S_n\). The images of these triangles in the later Cauchy surfaces will be denoted by \((i'j'k')\) and \((i'j'l')\) in \(S_{n+1}\) and \((i''j''k'')\) and \((i''j''l'')\) in \(S_{n+2}\).

The triangles in \(S_n\) share a common leg \((ij)\) and therefore the cells share a common 2-dimensional face generated by that leg. Consider now the contribution to action from the bones on the two dimensional face generated by \((ij)\). This contains three spacelike bones \((ij)\), \((ij')\) and \((i'j')\) (assuming \(\delta t\) is sufficiently small). Let \(J_{ij}^n\) be the contribution to the action from these bones in this pair of cells based on \(S_n\). Then

\[
J_{ij}^n = \left\{ L_{ij}(\pi - \theta_{ij}^+) + L_{ij'}(2\pi - \theta_{ij'}^+) + L_{i'j'}(\pi - \theta_{i'j'}^+) \right\} + O(\delta t^2)
\]

where (...) has been used and the notation \(x^+\) denotes the sum of the terms from the two cells ie. \(x(ijk) + x(ijl)\) and the arrows are included just as a reminder for which Cauchy surface the quantity applies (the arrows point into the region defined by the Cauchy surfaces). Notice the use of \(\pi\) rather than \(2\pi\) in the contributions from the legs \((ij)\) and \((i'j')\). This reflects the fact that those legs lie within a Cauchy surface and the above expression only accounts for the contributions from the cells on one side of the Cauchy surface. A similar
contribution arise from the cells on the other side of the Cauchy surface. In this form the contributions are exactly equal to the extrinsic curvatures for these legs.

Notice that $J^n_{ij}$ vanishes as $\delta t \to 0$.

The total contribution from all of the legs of the form $(ij)$ will be denoted by $J_N$ and is given by

$$J_N = \sum_{i>j} \sum_{n=0}^N J^n_{ij}$$

The restriction $i > j$ is just to ensure that each leg is only counted once in the sum. This equation can be simplified using (...) and

$$L_{ij'} = L_{ij} + N_j \cos \phi_{ij} \delta t + O(\delta t^2)$$
$$L_{i'j'} = L_{ij} + (N_i \cos \phi_{ji} + N_j \cos \phi_{ij}) \delta t + O(\delta t^2)$$
$$\gamma_{i'j'} = \gamma_{ij'} \delta t + O(\delta t^2)$$
$$2\pi = \lim_{\delta t \to 0} (\theta^{\uparrow}_{ij} + \theta^{\uparrow}_{j'i'})$$
$$0 = \lim_{\delta t \to 0} (\theta^{\downarrow}_{ji} - \theta^{\downarrow}_{ij})$$

leading to

$$J_N = \sum_{i>j} \sum_{n=0}^N L_{ij} \dot{\gamma}_{ij} \delta t$$

Noting that $\frac{dL_{ij}}{dt} = N_i \cos \phi_{ji} + N_j \cos \phi_{ij}$

this can be further simplified to

$$J = \sum_{i>j} \int \left( L_{ij} \dot{\gamma}_{ij} + (N_i \cos \phi_{ji} + N_j \cos \phi_{ij})(\pi - \theta^{\downarrow}_{ij}) \right) dt$$

Noting that

$$\frac{dL_{ij}}{dt} = N_i \cos \phi_{ji} + N_j \cos \phi_{ij}$$

this can be further simplified to

$$J = \sum_{i>j} \int \left( L_{ij} \dot{\gamma}_{ij} + \dot{L}_{ij}(\pi - \theta^{\downarrow}_{ij}) \right) dt$$
It is worth noting that after an integration by parts this expression can be re-written (upon neglecting boundary terms) as

\[ J = \sum_{i>j} \int L_{ij} \left( \dot{\gamma}_{ij} + \dot{\theta}^i_{ij} \right) dt \]

There is a simple interpretation for this integral. As a result of going to a continuous time formulation the defect on a worldtube of a leg is now smoothly distributed over that worldtube. The contribution to the defect from a short segment of the worldtube is just \((\dot{\gamma}_{ij} + \dot{\theta}^i_{ij})\delta t\) for leg \((ij)\) and small \(\delta t\). Though this form of the action is illuminating we shall retain the previous form since it contains the proper boundary terms and is better suited to our aim of developing a 3+1 action.

What remains to be done is a similar calculation for timelike bones, i.e. legs of the form \((ii')\). In this case the calculations are relatively simple. On each such leg the defect will be of the form \(2\pi - \sum_k \theta_k\) and the leg length will be \(N_i\delta t\). Each \(\theta_k\) could be calculated by a formula of the form \(\ldots\). Define the defect on \((ii')\) to be \(\alpha_i\), that is

\[ \alpha_i = \left(2\pi - \sum_k \theta_k\right)_i \]

The contribution from all of the timelike legs between the Cauchy surfaces \(S_0\) and \(S_N\) will be denoted by \(K_N\) and is simply

\[ K_N = \sum_i \sum_{n=0}^{N-1} \alpha_i N_i \delta t \]

and the limiting form is easily seen to be

\[ K = \lim_{N \to \infty} K_N = \sum_i \int \alpha_i N_i \, dt \]

The complete action for the 2+1 spacetime can now be written as \(I = J + K\) and is given by

\[ I = \sum_i \int \alpha_i N_i \, dt \]

\[ + \sum_{i>j} \int L_{ij} \dot{\gamma}_{ij} + \dot{L}_{ij} (\pi - \theta^i_{ij}) \, dt \]

This action depends on twelve variables, three \(L_{ij}\), six \(\phi_{ij}\) and \(N_i\) all of which should now be considered as smooth functions of \(t\). It is clear that not all of these functions can be viewed as being truly independent. As noted in section \(\S\ldots\), the geometry of the each cylinder is described by the nine functions \(\phi_{ij}(t)\) and \(N_i(t)\) and the three point values of
However the above action depend upon the functions $L_{ij}(t)$ and $\dot{L}_{ij}(t)$ which in turn depend upon $N_i(t)$ and $\phi_{ij}(t)$ via the differential equations

$$\frac{dL_{ij}}{dt} = N_i \cos \phi_{ji} + N_j \cos \phi_{ij}$$

These constraints can be incorporated into the action by way of the Lagrange multipliers $\lambda_{ij} = \lambda_{ji}$, one for each leg, leading to the new action

$$I = \sum_i \int \alpha_i N_i \, dt$$
$$+ \sum_{i>j} \int L_{ij} \gamma_{ij} + \dot{L}_{ij}(\pi - \theta_{ij}^\downarrow)$$
$$+ \lambda_{ij} \left(\dot{L}_{ij} - N_i \cos \phi_{ji} - N_j \cos \phi_{ij}\right) \, dt$$

This is the action from which the full field equations will be derived. Formally the field equations are obtained by an extremization with respect to the dynamical variables. In Regge’s discrete action one has the simple identity

$$0 = \sum_i (A \delta \alpha)_i + \sum_j (B \delta \beta)_j$$

for any variations (excluding variations on the boundary). It is not hard to see that for our action a similar identity holds. One starts with this discrete identity and proceeds to the continuous time from by the same steps used to obtain the continuous time action. The result is

$$0 = \sum_i \int N_i \delta \alpha_i \, dt$$
$$+ \sum_{i>j} \int L_{ij} \delta \gamma_{ij} + \dot{L}_{ij}(\pi - \delta \theta_{ij}^\downarrow) \, dt$$

The field equations for the action are trivial to form and after some simple algebra one arrives at the following

$$0 = \alpha_i$$
$$0 = \lambda_{ij}$$
$$0 = \dot{\gamma}_{ij} + \dot{\theta}_{ij}^\downarrow$$

The first condition states that the defects on the timelike worldlines vanish. The third, in view of the discussion after (...), shows that the defects on the worldtubes for each leg vanishes. That is the 2+1 space is everywhere flat which is exactly what we would expect. The second condition arises for the simple reason that in this simple 2+1 model the $\phi_{ij}$ did not appear explicitly in the action. This will not be the case in a 3+1 formulation.
Notice that our action is additive, i.e. for three successive Cauchy surfaces $S_{t_0}, S_{t_1}$ and $S_{t_2}$ the actions are related by $I(S_{t_0}, S_{t_2}) = I(S_{t_0}, S_{t_1}) + I(S_{t_1}, S_{t_2})$. This follows as a simple consequence of the construction of $J^n$.

3. Appendix

The purpose of this section is to provide the gory details behind the calculations of the future leg lengths, the various dihedral angles and their limiting behaviour for a typical triangular tube in terms of the basic data for the tube.

The structure of the typical triangular tube is fully determined by the three legs of the base $L_{ij}$, the three struts $L_{i'j'}$ and the six angles $\phi_{ij}$. From this information alone we wish to compute such things as the leg lengths on the upper surface $L_{i'j'}$ and the angles between various two dimensional faces. It seems reasonable to expect that some of these results may depend upon the choice we make for the primary diagonals. So before we begin we shall establish some useful equalities and inequalities between the limiting forms of some of these quantities.

Define the unit vector pointing from vertex $(i)$ to vertex $(j)$ to be $e_{ij}$. The vector $e_{i'j'}$ is defined in a similar manner. For the strut $(i'i')$ the unit vector pointing from $(i)$ to $(i')$ will be defined as $m_i$. For the short segment of the worldtube generated by the leg $(ij)$ one has

\[
L_{i'j'} e_{i'j'} = L_{ij} e_{ij} + \delta t (N_j m_j - N_i m_i)
\]
\[
L_{ij} e_{ij} = L_{ij} e_{ij} + \delta t (N_j m_j)
\]

Thus, trivially,

\[
0 = \lim_{\delta t \to 0} (e_{i'j'} - e_{ij})
\]
\[
0 = \lim_{\delta t \to 0} (e_{ij} - e_{ij})
\]

Since $\cos \phi_{ji} = m_i \cdot e_{ij}$ and $\cos \phi_{j'i'} = -m_i \cdot e_{i'j'}$ we obtain

\[
L_{i'j'} \cos \phi_{j'i'} = -L_{ij} \cos \phi_{ji} + O(\delta t)
\]

and therefore

\[
\pi = \lim_{\delta t \to 0} (\phi_{j'i'} + \phi_{ji})
\]

One also obtains, by dotting out each of the equations (...) with themselves,

\[
L_{i'j'}^2 = L_{ij}^2 + 2L_{ij} \delta t (N_j \cos \phi_{ij} + N_i \cos \phi_{ji})
\]
\[
L_{ij}^2 = L_{ij}^2 + 2L_{ij} \delta t (N_j \cos \phi_{ij})
\]
which naturally leads to
\[ \dot{L}_{ij} = N_j \cos \phi_{ij} + N_i \cos \phi_{ji} \]
\[ \dot{L}_{ij'} = N_j \cos \phi_{ij} \]

Consider now the various unit normals to the triangular faces of the the tube. Let \( n_{ijk} \) be the outward pointing unit normal to the base \((ijk)\). Similarly let \( n_{ij} \) be the outward unit normal to the the time like face \((ijj')\). Note that, in general, \( n_{ij} \neq n_{ji} \). The normals for the other faces, eg \((i'j'k')\), are defined in similar terms. Each of these normals can be computed by forming the appropriate cross products amongst the basic edge vectors \( e_{ij} \) and \( m_i \). It follows that
\[
\begin{align*}
n_{ijk} &= \frac{e_{ik} \times e_{ij}}{\sin \rho_i} \\
n_{i'j'k'} &= \frac{e_{i'j'} \times e_{i'k'}}{\sin \rho_{i'}} \\
n_{ij} &= \frac{e_{ij} \times m_j}{\sin \phi_{ij}} \\
n_{i'j'} &= \frac{e_{i'j'} \times m_{ij}'}{\sin \phi_{ij'}}
\end{align*}
\]

and therefore
\[
\begin{align*}
0 &= \lim_{\delta t \to 0} (n_{i'j'k'} + n_{ijk}) \\
0 &= \lim_{\delta t \to 0} (n_{i'j'} - n_{ij})
\end{align*}
\]

From these observations we can easily establish a relation between \( \theta_{j'i'} \) and \( \theta_{ij} \). These angles can be computed from
\[
\begin{align*}
\cos \theta_{ij} &= -n_{ijk} \cdot n_{ij} \\
\cos \theta_{j'i'} &= -n_{i'j'k'} \cdot n_{i'j'}
\end{align*}
\]

which, in view of the above limits, one sees that
\[
\pi = \lim_{\delta t \to 0} (\theta_{ij} + \theta_{j'i'})
\]

However, in general,
\[
\pi \neq \lim_{\delta t \to 0} (\theta_{ij} + \theta_{i'j'})
\]

We shall now turn our attention to obtaining explicit formula for the various dihedral angles. We begin by considering a generic tetrahedra and asking how might one compute the angles between its faces given just the angles between its legs. Consider the diagram (3). Let \( n_{ij} \)
be the unit outward normal to the face spanned by the unit vectors \( e_i \) and \( e_j \). Then

\[
\begin{align*}
n_{12} &= \frac{1}{\sin \alpha_3} e_1 \times e_2 \\
n_{13} &= \frac{1}{\sin \alpha_2} e_3 \times e_1 \\
n_{23} &= \frac{1}{\sin \alpha_1} e_2 \times e_3
\end{align*}
\]

and the angle \( \beta_1 \) can be computed by forming a scalar product

\[
\cos \beta_1 = -n_{12} \cdot n_{13} = \frac{(e_1 \times e_2) \cdot (e_3 \times e_1)}{\sin \alpha_2 \sin \alpha_3} = \frac{(e_1 \cdot e_3)(e_2 \cdot e_1) - (e_1 \cdot e_1)(e_2 \cdot e_3)}{\sin \alpha_2 \sin \alpha_3}
\]

Thus one arrives at the simple formula

\[
\cos \beta_1 = \frac{\cos \alpha_1 - \cos \alpha_2 \cos \alpha_3}{\sin \alpha_2 \sin \alpha_3}
\]

This basic result will crop up many times in the following discussion.

Consider the typical triangular tube depicted in figure (4a,b). The dihedral angle on the strut \((ii')\) can be calculated using the above formula applied to the tetrahedron \((ii'j'k)\). In the limit as \( \delta t \to 0 \), one obtains

\[
\lim_{\delta t \to 0} \cos \theta_i = \frac{\cos \rho_i - \cos \phi_{ji} \cos \phi_{ki}}{\sin \phi_{ji} \sin \phi_{ki}}
\]

An important point to note here is that this result does not depend on the choice of diagonals. That is the same result would be obtained if one had started with the tetrahedron \((ii'jk)\).

Our next target will be the dihedral angle on the leg \((ij)\). For the set of diagonals indicated in figure (5) the appropriate tetrahedron is \((ijj'k)\). Then applying \((...)\) to this tetrahedron leads directly to

\[
\lim_{\delta t \to 0} \cos \theta_{ij} = \frac{\cos \phi_{kj} - \cos \phi_{ij} \cos \rho_j}{\sin \phi_{ij} \sin \rho_j}
\]

For the future leg \((i'j')\) the appropriate tetrahedron is \((ii'j'k')\) which leads to

\[
\lim_{\delta t \to 0} \cos \theta_{i'j'} = \frac{-\cos \phi_{ki} + \cos \phi_{ji} \cos \rho_i}{\sin \phi_{ji} \sin \rho_i}
\]
where we have used \( \pi = \lim_{\delta t \to 0} (\phi_{k_t} + \phi_{k_i}) \). It should be noted that for both of the legs \((ij)\) and \((i'j')\) the choice of diagonals is significant. For example, using the tetrahedron \((ii'jk)\) for the leg \((ij)\) leads to

\[
\lim_{\delta t \to 0} \cos\theta_{ji} = \frac{\cos\phi_{ki} - \cos\phi_{ji} \cos\rho_i}{\sin\phi_{ji} \sin\rho_i}
\]

clearly showing that \(\theta_{ji} \neq \theta_{ij}\). Another way of seeing this is to just recognize that the timelike faces generated by the spacelike legs \((ij)\) need not, in general, be planar.

Notice also that the limit \(\delta t \to 0\) is not strictly necessary for these two legs. However for a leg such as \((ik)\) there are two tetrahedra meeting on this leg. There are therefore two angles which contribute to the dihedral angle, one of which vanishes in the limit \(\delta t \to 0\). The limit is included in the above formula so that it may be applied to any of the space like legs.

So far the calculations have been rather straightforward. Now the fun begins. Consider the diagonal \((ij')\). There are two tetrahedra on this diagonal and so there are two angles which contribute to the dihedral angle. However, unlike the previous case, neither of these angles vanishes in the limit \(\delta t \to 0\). Let \(\theta_{ij}^{(1)}\) be the angle arising from \((ijj'k)\) and \(\theta_{ij}^{(2)}\) that from \((ii'j'k)\). Using the standard formula \((\ldots)\) leads to

\[
\theta_{ij'} = \theta_{ij}^{(1)} + \theta_{ij}^{(2)}
\]

\[
\lim_{\delta t \to 0} \cos\theta_{ij}^{(1)} = \frac{-\cos\phi_{kj} + \cos\phi_{ji} \cos\rho_j}{\sin\phi_{ij} \sin\rho_j}
\]

\[
\lim_{\delta t \to 0} \cos\theta_{ij}^{(2)} = \frac{\cos\phi_{ki} - \cos\phi_{ji} \cos\rho_i}{\sin\phi_{ji} \sin\rho_i}
\]

By comparing with \((\ldots)\) it follows that

\[
2\pi = \lim_{\delta t \to 0} (\theta_{ij} + \theta_{ij'} + \theta_{i'j'})
\]

This result can be used to calculate \(\lim_{\delta t \to 0} \theta_{ij'}\). Unfortunately this is not sufficient for our purposes as we require, at least, the first order expansion of \(\theta_{ij'}\) in powers of \(\delta t\) (see section \((\ldots)\)). One option would be to re-work the calculations which lead to \((\ldots)\) but on this occasion retaining the higher order terms. This approach has not been followed through. Instead an application of Stoke’s theorem leads to a second order accurate approximation. The details are somewhat lengthy but that’s what appendices are for.

Consider a flat two-dimensional cross section of the tube as indicated in figure (6a). Our approach will be to use the angles \(\bar{\theta}_{ij'}\) and \(\bar{\theta}_{i'j'}\) on the dashed path as an approximation to \(\theta_{ij'}\) and \(\theta_{i'j'}\). The first point to be established is that this approximation is second accurate in \(\delta t\).

Consider now the vectors \(a, b, c, d, e\) and \(f\) as indicated in figure (6b). Vectors \(a\) and \(c\) are tangent to \((ii'j')\), while \(b\) and \(d\) are tangent to \((ijj')\). The vector \(e = c \times d\) is tangent to the
diagonal while \( f = a \times b \) coincides with \( e \) only as \( \delta t \to 0 \). Let the angle between \( a \) and \( e \) be \( \lambda_a \) and that between \( b \) and \( e \) be \( \lambda_b \). As \( \delta t \to 0 \) we must have \( a \to c , b \to d \) and hence \( \lambda_a \) and \( \lambda_b \) tend to \( \pi/2 \) as \( \delta t \to 0 \). Now form a tetrahedron based on the three edges \( a, b \) and \( e \). This tetrahedron has no geometrical importance except to simplify the following calculation. The angle \( \theta_{ij'} \) is then the angle between the faces generated by \( a, e \) and \( b, e \). Whereas the angle \( \tilde{\phi}_{ij'} \) corresponds to the angle between \( a \) and \( b \) (see figure (6b)). Thus applying, once again, the standard formula (...) we obtain

\[
\cos \phi_{ij'} = \frac{\cos \tilde{\phi}_{ij'} - \cos \lambda_a \cos \lambda_b}{\sin \lambda_a \sin \lambda_b}
\]

Since both \( \lambda_a \) and \( \lambda_b \) vary as \( \pi/2 + O(\delta t) \) it follows that

\[
\theta_{ij'} = \tilde{\theta}_{ij'} + O(\delta t^2)
\]

thus completing the proof. The same style of argument can be used in proving that \( \tilde{\theta}_{ij'} \) is a second order approximation to \( \theta_{ij'} \). (Leo, is this last statement really true?). The question now is, how do we calculate \( \tilde{\theta}_{ij'} \)? To start, consider, once again, figure (2). The sum of the angles at the vertices of the dashed path must equal \( 3\pi \) (the polygon can be divided into three triangles). Thus we have

\[
2\pi = \gamma_{i'j'} + \tilde{\theta}_{i'j'} + \tilde{\theta}_{ij} + \theta_{ij}
\]

Consequently

\[
\theta_{ij'} = 2\pi - \gamma_{i'j'} - \theta_{i'j'} - \theta_{ij} + O(\delta t^2)
\]

and so our attention now shifts to obtaining an accurate calculation of \( \gamma_{i'j'} \). This is where we will make use of Stoke’s theorem.

For any triangular tube Stokes’ theorem states that \( 0 = \sum \alpha n_\alpha A_\alpha \) where the sum includes all of the triangular faces of the tube. When written out explicitly one has

\[
0 = A_{i'j'k'}n_{i'j'k'} + A_{ijk}n_{ijk} + (A_{i'jj'}n_{i'jj'} + A_{j'ii'}n_{j'ii'})
+ (A_{kii'}n_{kii'} + A_{j'k'k}n_{j'k'k})
+ (A_{kkj}n_{kkj} + A_{j'k'k}n_{j'k'k})
\]

From this equation we shall derive an equation for \( \gamma_{i'j'} \) by taking an appropriate scalar product. Consider the unit vector \( w_{ij} \) which is tangent to the base triangle \((ijk)\), normal to the leg \((ij)\) and inward pointing. The vector \( w_{i'j'} \) on the upper triangle \((i'j'k')\) is defined in a similar way. Now it is not hard to see that

\[
\sin \gamma_{i'j'} = n_{i'j'k'} \cdot w_{ij}
\]
To follow this through we will need to know the scalar products of $w_{ij}$ with each of the unit normals. Two of these are trivial

\[ n_{ijk} \cdot w_{ij} = 0 \]
\[ n_{ijj'} \cdot w_{ij} = -\sin \theta_{ij} \]

For $n_{i'd'j'}$ we can use $w_{i'd'} = w_{ij} + O(\delta t)$ to obtain

\[ n_{i'd'i} \cdot w_{ij} = n_{i'd'i} \cdot w_{i'd'} + O(\delta t) \]
\[ = -\sin \theta_{i'j'} + O(\delta t) \]
\[ = -\sin \theta_{ji} + O(\delta t) \]

We shall also need expressions for $n_{kii'} \cdot w_{ij}$. The trick here is to recognize that $w_{ij}$ can be expressed as the linear combination

\[ w_{ij} = -\cos \rho_i w_{ik} + \sin \rho_i e_{ik} \]

and therefore

\[ n_{kii'} \cdot w_{ij} = \cos \rho_i \sin \theta_{ki} \]

Proceeding in a similar fashion one obtains

\[ n_{kjj'} \cdot w_{ij} = \cos \rho_j \sin \theta_{kj} \]
\[ n_{i'k'k} \cdot w_{ij} = \cos \rho_i \sin \theta_{ik} + O(\delta t) \]
\[ n_{j'k'k} \cdot w_{ij} = \cos \rho_j \sin \theta_{jk} + O(\delta t) \]

and consequently Stokes theorem takes the form

\[ -A_{i'j'k'} \sin \gamma_{i'j'} = (-A_{ijj'} \sin \theta_{ij} - A_{j'i'i} \sin \theta_{ji}) \]
\[ + (A_{kii'} \sin \theta_{ki} + A_{i'k'k} \sin \theta_{ik}) \cos \rho_i \]
\[ + (A_{kjj'} \sin \theta_{kj} + A_{j'k'k} \sin \theta_{jk}) \cos \rho_j + O(\delta t^2) \]

The last remaining step is to express the areas of the triangles in terms of the basic data. This is very easy to do, for example

\[ 2A_{ijj'} = \delta tL_{ij}N_j \sin \phi_{ij} \]
\[ 2A_{j'i'i} = \delta tL_{ij}N_i \sin \phi_{ji} + O(\delta t^2) \]

After some tedious algebra we have finally arrived at the following expression for $\gamma_{ij}$,

\[ \frac{2A_{ijk} \sin \gamma_{ij}}{\delta t} = L_{ij} \left( N_j \sin \phi_{ij} \sin \theta_{ij} + N_i \sin \phi_{ji} \sin \theta_{ji} \right) \]
\[ - L_{ik} \left( N_i \sin \phi_{ki} \sin \theta_{ki} + N_k \sin \phi_{ik} \sin \theta_{ik} \right) \cos \rho_i \]
\[ - L_{jk} \left( N_j \sin \phi_{kj} \sin \theta_{kj} + N_k \sin \phi_{jk} \sin \theta_{jk} \right) \cos \rho_j + O(\delta t) \]
The last point to be made here is that since the right hand side is well defined as \( \delta t \to 0 \) one can argue that 
\[
\gamma_{ij} = \dot{\gamma}_{ij} \delta t + O(\delta t)
\]
and therefore

\[
2A_{ijk} \dot{\gamma}_{ij} = L_{ij} (N_j \sin \phi_{ij} \sin \theta_{ij} + N_i \sin \phi_{ji} \sin \theta_{ji}) \\
- L_{ik} (N_i \sin \phi_{ki} \sin \theta_{ki} + N_k \sin \phi_{ik} \sin \theta_{ik}) \cos \rho_i \\
- L_{jk} (N_j \sin \phi_{kj} \sin \theta_{kj} + N_k \sin \phi_{jk} \sin \theta_{jk}) \cos \rho_j
\]

4. The 3+1 Action

This section is just for local consumption, ie. Warner and Ruth. I have yet to fully work out the details though I think you will agree that the jist of the following would seem reasonable.

My feeling is that the correct 3+1 action will be

\[
I = \sum_{i>j} \int \left( \frac{1}{2} L_{ij} (N_i \alpha_{ji} \sin \phi_{ji} + N_j \alpha_{ij} \sin \phi_{ij}) \right) dt \\
+ \sum_{i>j>k} \int \left( B_{ijk} \dot{\gamma}_{ijk} + \dot{B}_{ijk} (\pi - \theta^i_{ijk}) \right) \\
+ \lambda_{ij} \left( \dot{L}_{ij} - N_i \cos \phi_{ji} - N_j \cos \phi_{ij} \right) dt
\]

The changes just reflect that the defects in the discrete case are now localized to triangles rather than legs. Thus the contribution from a timelike bone of the form \((ijj')\) will be \(\alpha_{ij} A_{ijj'}\) and the area \(A_{ijj'}\) equals \(N_j \sin \phi_{ij}\). Now the less obvious adaption is the term involving \(\dot{B}\). This is time derivative of the area of a triangle lying in a Cauchy surface.

I believe that the above action looks correct. But to be sure I must follow through the same steps as I did for the 2+1 action and I’m sure I will arrive at the above action.

One difficulty that has just come to light is the issue of embedding the 4-cell in flat space. For the 3-cell this was not a problem. But for the 4-cell it turns out that there are 56 basic leg lengths (ie. no angles at this stage, just leg lengths) but only 54 dependent coordinates. One can not impose any constraints on the leg lengths since they are given to us be the primary 4-simplices each of which has ten independent leg lengths. We are only short by two pieces of information so it looks like we can almost embed the 4-cell in flat space. A number of options come to mind. One (which I don’t like) is to add one vertex and two legs to the 4-cell (ie. a body diagonal in the lower 3-cell, the vertex is placed somewhere on the leg and its total length determined by the requirement that the 3-cell be flat). The number counting works but it just seems too ad-hoc to me. Another option is that perhaps the 4-cell can be embedded in a non-flat space for which the curvature varies as \(\delta t^2\).
This option I like. We know that the base 3-cell has a flat intrinsic geometry. So any curvature arises from the variation of the unit normal throughout the 4-cell. As $\delta t \to 0$ this variation becomes purely spatial and so probably measures the spatial curvature which we know to be flat. So if this idea pans out it means that the 4-cell is embeddable in flat space only in the infinitesimal limit. This reminds me of the idea of a tangent space. A closely related option is to propose a form of the metric eg. $g_{\mu\nu} = \phi(t)\eta_{\mu\nu}$ and to see if the 4-cell can be embedded in this space and if so how does the curvature vary with $\delta t$. Note that such spaces can have less symmetry than flat space and so less degrees of freedom. This makes a significant change to the number counting used above. Yet another approach might be to show that the 4-cell can be embedded in a higher dimensional flat space (true) and that the induced curvature again varies as $\delta t^2$ (I hope).

This issue is really very important. I want to be able to fully specify the discrete spacetime by ten numbers per 4-cell and its natural to choose the edges of the primary 4-simplex for this purpose. If this did not produce a rigid 4-cell then extra legs would need to be introduced (as happened in the 2+1 case). We do not want these legs to be dynamical since we already have our ten pieces of data. So somehow we have to dispose of these legs. Hence the idea of embedding the 4-cell in flat space. But we've seen that this is not possible. It appears that we have over specified the data for the lattice. But this can not be true since in the continuum we have just ten pieces of data. So I am loathe to surrender the idea of employing a cubic lattice built from primary 4-simplices.

I tend to think that the computation of the defects will not be too difficult. I suspect that the formulas used for the 2+1 case will carry over to the 3+1 case. That is the formulas (...) can be applied to all of the three dimensional faces of the 4-cylinders. This would give us the angles between adjacent bones. Then the same formulas can be applied a second time but with edges replaced by triangles. This would then give us the angles between the 3-dimensional faces which is all we need to compute the defects. I could also attempt to derive an explicit formula for the dihedral angles but I suspect the algebra will be quite messy. For computational purposes I'm sure we can get by with this two-step recursion.

By an integration by parts it is easy to re-write the above action in the form

$$I = \sum_{i>j} \int \left( \frac{1}{2} L_{ij}(N_i \alpha_{ji} \sin \phi_{ji} + N_j \alpha_{ij} \sin \phi_{ij}) dtight.$$  

$$+ \sum_{i>j>k} \int B_{ijk} \left( \dot{\gamma}_{ijk} + \dot{\theta}_{ijk} \right)$$  

$$+ \lambda_{ij} \left( \dot{L}_{ij} - N_i \cos \phi_{ji} - N_j \cos \phi_{ij} \right) dt$$

Apart from some notational differences (and the use of a Euclidian signature), this action is
very similar to that which I proposed in my earlier paper. Define $\beta_{ijk}$ by

$$\beta_{ijk} = \dot{\gamma}_{ijk} + \dot{\theta}^i_{ijk}$$

then the equations of motion, for $N_i, \phi_{ij}, L_{ij}$ and $\lambda_{ij}$, are

$$0 = \sum_j L_{ij} \alpha_{ji} \sin \phi_{ji} - 2\lambda_{ij} \cos \phi_{ji} \quad (A)$$

$$0 = \alpha_{ij} L_{ij} N_j \cos \phi_{ij} + 2\lambda_{ij} N_j \sin \phi_{ij} \quad (B)$$

$$2\dot{\lambda}_{ij} = N_i \alpha_{ji} \sin \phi_{ji} + N_j \alpha_{ij} \sin \phi_{ij} + 2 \sum_k \beta_{ijk} \partial B_{ijk} / \partial L_{ij} \quad (C)$$

$$\dot{L}_{ij} = N_i \cos \phi_{ji} + N_j \cos \phi_{ij} \quad (D)$$

Two of the field equations are easily simplified to

$$0 = \sum_j L_{ij} \alpha_{ji} \sin \phi_{ji} \quad (A')$$

$$\alpha_{ji} \tan \phi_{ij} = \alpha_{ij} \tan \phi_{ji} \quad (B')$$

The main difficulty is, I believe, in the $\beta_{ijk}$ term. What a mess! In my previous paper I found a trick using the Biannchi identities which helped me get rid of this term. I have not had a chance to see if a similar trick can be used here. Another possibility, which I have not yet persued, is to use Stokes theorem to obtain an explicit expression for $\dot{\theta}^i_{ijk}$. In the appendix I have shown how (the 2+1 form of) $\dot{\gamma}_{ijk}$ can be computed using Stokes theorem. I’m sure it should carry over to the 3+1 formulation. I tend to think that an application of Stokes theorem along the lines used to calculate $\dot{\gamma}_{ij}$ could be used to obtain similar expressions for $\dot{\theta}_{ij}$. This would simplify the process of solving the full 3+1 equations. Again, this is something I have yet to follow through.

Notice that in equation (C) we see terms containing $\dot{\theta}^i_{ijk}$. These terms represent $\dot{K}_{\mu\nu}$ so equation (C) must be an evolution equation. Equation (D) is clearly an evolution equation whereas equations (A') and (B') must act as constraint equations. One must also remember that we have non-trivial constraints $\beta_{ijk} = 0$ for all of the secondary bones.