ETF2700/ETF5970 Mathematics for Business

Lecture 7

Monash Business School, Monash University, Australia

Outline

Last week:

- Functions of multiple variables
- Partial differentiation
- Slope of an iso curve

This week:

- Unconstrained optimisation: Two variables
- Lagrange method for constrained maximisation

Derivative: Functions of one variable

$$f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}$$

- Rewrite the expression without Δ in the denominator
- Plug in $\Delta = 0$

Partial Derivative: Functions of two variables

$$f_x(x,y) = \lim_{\Delta \to 0} \frac{f(x+\Delta, y) - f(x,y)}{\Delta}$$
$$f_y(x,y) = \lim_{\Delta \to 0} \frac{f(x, y+\Delta) - f(x,y)}{\Delta}$$

- Treat y as constant: $f(x,y) = g(x)$, $f_x(x,y) = g'(x)$.
- Treat x as constant: $f(x,y) = h(y)$, $f_y(x,y) = h'(y)$.

Partial Derivative: Functions of three Variables

$$f'_x(x,y,\lambda) = \lim_{\Delta\to 0} \frac{f(x+\Delta,y,\lambda) - f(x,y,\lambda)}{\Delta}$$

$$f'_y(x,y,\lambda) = \lim_{\Delta\to 0} \frac{f(x,y+\Delta,\lambda) - f(x,y,\lambda)}{\Delta}$$

$$f'_\lambda(x,y,\lambda) = \lim_{\Delta\to 0} \frac{f(x,y,\lambda+\Delta) - f(x,y,\lambda)}{\Delta}$$

Partial derivative w.r.t. x

- Treat y and λ as constants:

$$f(x,y,\lambda) = g(x), \quad f_x(x,y,\lambda) = g'(x)$$

- Similar for $f_y(x,y,\lambda)$ and $f_\lambda(x,y,\lambda)$

Stationary point: Functions of one variable

First-order condition: Point x is a stationary point of $f(x)$, $x \in (a, b)$, if $f'(x) = 0$

- The optimal point of f, if exists, is the stationary point.
- Convex/concave function f:

$$\text{stationarity} = \text{minimum/maximum}$$

- Quadratic function $f(x) = ax^2 + bx + c$ $(a \neq 0)$

$$f'(x) = 2ax + b = 0 \quad \Rightarrow \quad x = -\frac{b}{2a}$$

Stationary point: Functions of two variables

First-order condition: (x, y) is a stationary point of $f(x, y)$, for $(x, y) \in D$, if

$$f_x(x, y) = 0, \quad f_y(x, y) = 0$$

- Let D be an open set, and end points are excluded
- Assume that the optimal value of f exists

$$\text{Optimality} \Rightarrow \text{Stationarity}$$

Example

The function f is defined for $x, y \in (-\infty, \infty)$ by

$$f(x,y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$$

Use the power rule and sum/subtraction rule

$$f_x(x,y) = -4x - 2y - 0 + 36 + 0 - 0 = -4x - 2y + 36$$
$$f_y(x,y) = -0 - 2x - 4y + 0 + 42 - 0 = -2x - 4y + 42$$

Solve the system

$$\begin{matrix} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} 4x + 2y = 36 \\ 2x + 4y = 42 \end{matrix}$$

Example

The function $f(x, y)$ is defined for $x, y \in (-\infty, \infty)$ by

$$f(x, y) = -2x^2 - 2xy - 2y^2 + 36x + 42y - 158$$

Solve the system by Cramer's Rule

$$\begin{array}{l} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{l} 4x + 2y = 36 \\ 2x + 4y = 42 \end{array}$$

we have $x = 5$ and $y = 8$.

The stationary point of $f(x, y)$ is $(5, 8)$.

If $f(x, y)$ has a maximum value, then $(5, 8)$ is the maximum point.

An economic example: Production

Consider a Cobb-Douglas production function

$$f(L, K) = K^{1/2} L^{1/4}, \quad L, K > 0$$

- L is the labor input
- K is the capital input
- $f(L, K)$ is the total production

Net Profit

- Price per unit of output is 12 thousand dollars
- Cost (or rental) per unit of capital is 1.2 thousand dollars
- Wage rate (cost per unit of labor) is 0.6 thousand dollars

Net profit function of L and K is

$$\begin{aligned}\pi(L, K) =& 12 \cdot f(L, K) - 0.6L - 1.2K \\ =& 12K^{1/2}L^{1/4} - 0.6L - 1.2K\end{aligned}$$

Net profit function of L and K

$$\pi(L, K) = 12K^{1/2}L^{1/4} - 0.6L - 1.2K$$

The partial derivatives are

$$\pi_L(L, K) = 12K^{\frac{1}{2}} \cdot \frac{1}{4}L^{\frac{1}{4}-1} - 0.6 - 0 = 3K^{\frac{1}{2}}L^{-\frac{3}{4}} - 0.6$$
$$\pi_K(L, K) = 12 \cdot \frac{1}{2}K^{\frac{1}{2}-1}L^{\frac{1}{4}} - 0 - 1.2 = 6K^{-\frac{1}{2}}L^{\frac{1}{4}} - 1.2$$

Solve the system

$$\begin{matrix} \pi_L(L, K) = 0 \\ \pi_K(L, K) = 0 \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} 3K^{\frac{1}{2}}L^{-\frac{3}{4}} = 0.6 \\ 6K^{-\frac{1}{2}}L^{\frac{1}{4}} = 1.2 \end{matrix}$$

Solve the linear equations: $1st$ Equation $\div$ $2nd$ Equation

$$\frac{3K^{\frac{1}{2}}L^{-\frac{3}{4}}}{6K^{-\frac{1}{2}}L^{\frac{1}{4}}} = \frac{0.6}{1.2} \Leftrightarrow \frac{K}{2L} = \frac{1}{2} \Leftrightarrow K = L$$

Substitute $K = L$ back into the first partial derivative equ:

$$3L^{1/2}L^{-3/4} = 0.6 \Leftrightarrow 3L^{-1/4} = 0.6$$

Therefore, $L = (0.6/3)^{-4} = 5^4 = 625$ and then $K = L = 625$.

- assume the maximum point exists
- the maximum point is the stationary point
- we have only one stationary point $(L, K) = (625, 625)$

Maximum point

Therefore, the maximum point is $(L, K) = (625, 625)$, and the maximum net profit is $\pi(625, 625) = 375$ thousand dollars.

Budget constraint

At the maximum point $(L, K) = (625, 625)$, the total cost equals to

$$0.6L + 1.2K = 0.6 \times 625 + 1.2 \times 625 = 1125$$

What if the budget is not enough to cover the cost?

Linear constraint

We assume $0.6L + 1.2K \leq b$, where b is the budget (in \$K).

Let us say, we only have a budget of $b = 900$ thousand dollars.

Optimisation with a budget constraint

Maximise $\pi(L, K) = 12K^{1/2}L^{1/4} - 0.6L - 1.2K$
subject to $0.6L + 1.2K \leq 900$, for $L, K > 0$.

- It is NOT a linear programming problem.
- Let us assume there is an optimal solution.

Stationarity $\neq$ Optimality

Note that we can rewrite our problem as:
maximise the net profit function

$$\pi(L, K) = 12K^{1/2}L^{1/4} - 0.6L - 1.2K, \qquad \text{for } (L, K) \in D$$

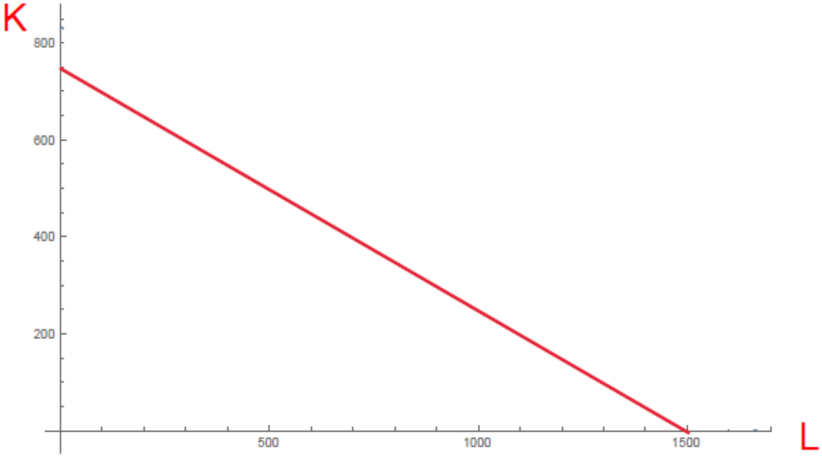
where $D = \{(L, K) : 0.6L + 1.2K \leq 900, L, K > 0\}$

- Stationary point $(L, K) = (625, 625)$ is outside the domain
- The optimal point is not stationary: D is not an open set

The domain D is not an open set

Any point on the straight line satisfies the constraint.

$$D = \{(L, K) : 0.6L + 1.2K \leq 900, L, K > 0\}$$



An easier formulation

- Assume that the optimal solution exists
- At the maximum point (L, K), the entire budget will be exhausted. Why?
- The physical constraint $L, K > 0$ can be ignored.

Maximise $\pi(L, K) = 12K^{1/2}L^{1/4} - 0.6L - 1.2K$
subject to $0.6L + 1.2K = 900$, for $L, K > 0$.

Why is the maximum point on the line of entire budget

IF there is an optimal point (L^*, K^*) that is below the entire budget: $0.6L^* + 1.2K^* < 900$. Let's define the interior of the feasible region:

$$D^\circ = \{(L, K) : 0.6L + 1.2K < 900, L, K > 0\},$$

which is an open set. It is obvious that

$$\pi(L^*, K^*) = \max_{(L,K)\in D^\circ} \pi(L, K).$$

There comes a contradiction due to the former "IF"

- As D° is an open set, the optimal point (L^*, K^*) must be a stationary point.
- However, the only stationary point is $(L, K) = (625, 625)$, which is outside the domain D.
- Thus, the optimal point (L^*, K^*) has to be on the boundary.

Add a variable: Lagrange multiplier λ

- Initially, we want to maximise

$$\pi(L, K) = 12K^{1/2}L^{1/4} - 0.6L - 1.2K,$$

subject to $0.6L + 1.2K = 900,\ \text{for } L, K > 0.$

- The **Lagrange** multiplier method introduces a new variable λ and defines a new function:

$$f(L, K, \lambda) = \pi(L, K) + \lambda(900 - 0.6L - 1.2K),$$

which is called the Lagrangian.

- Maximisation of $\pi(L, K)$ subject to a budget constraint becomes a maximisation of $f(L, K, \lambda)$ on an extended domain: $L, K > 0$ and $\lambda \in (-\infty, \infty)$.

Lagrange method

Maximise

$$f(L,K,\lambda) = 12K^{1/2}L^{1/4} - 0.6L - 1.2K + \lambda(900 - 0.6L - 1.2K), \quad (L,K,\lambda) \in D'$$

where $D' = \{(L,K,\lambda) : L,K > 0, \lambda \in (-\infty,\infty)\}$.

- D' is open: Optimal point is the stationary point
- Derive stationary point via the 1st-order conditions:

$$f_L(L,K,\lambda) = 0, \quad f_K(L,K,\lambda) = 0, \quad f_\lambda(L,K,\lambda) = 0$$

Solve first-order conditions

$$f(L,K,\lambda) = \pi(L,K) + \lambda(900 - 0.6L - 1.2K)$$

$$f_L(L,K,\lambda) = \pi_L(L,K) - 0.6\lambda = 3K^{\frac{1}{2}}L^{-\frac{3}{4}} - 0.6 - 0.6\lambda$$

$$f_K(L,K,\lambda) = \pi_K(L,K) - 1.2\lambda = 6K^{-\frac{1}{2}}L^{\frac{1}{4}} - 1.2 - 1.2\lambda$$

$$f_\lambda(L,K,\lambda) = 900 - 0.6L - 1.2K$$

Solutions to first-order conditions

$$\begin{cases} f_L(L,K,\lambda)=0 \\ f_K(L,K,\lambda)=0 \\ f_\lambda(L,K,\lambda)=0 \end{cases} \Leftrightarrow \begin{cases} 3K^{\frac{1}{2}}L^{-\frac{3}{4}} = 0.6(1+\lambda) \\ 6K^{-\frac{1}{2}}L^{\frac{1}{4}} = 1.2(1+\lambda) \\ 0.6L+1.2K=900 \end{cases}$$

- First equation ÷ Second equation

$$\frac{3K^{\frac{1}{2}}L^{-\frac{3}{4}}}{6K^{-\frac{1}{2}}L^{\frac{1}{4}}} = \frac{0.6(1+\lambda)}{1.2(1+\lambda)} \Leftrightarrow \frac{K}{2L} = \frac{1}{2} \Leftrightarrow K = L$$

- Substitute $K = L$ into the third equation

$$0.6L+1.2L=900, \quad L=\frac{900}{1.8}=500$$

- Therefore, $K = L = 500$

Optimal solution with budget $b = 900$

The maximum point is $(L,K) = (500,500)$ and the maximum net profit is $\pi(500,500) = 12\cdot(500)^{3/4} - 900 \approx 368.85$ \$K.

With any other budget $b \in (0, 1125)$

It can be shown that the maximum point is $(L, K) = \left(\frac{5}{9}b, \frac{5}{9}b\right)$ and the maximum net profit is

$$\pi_*(b) = \pi\left(\frac{5}{9}b, \frac{5}{9}b\right) = 12 \cdot \left(\frac{5}{9}b\right)^{3/4} - b$$

Increase budget

- For one unit increase of the budget, the increase of net profit will be approximated by the derivative of $\pi_*(b)$:

$$\pi_*'(b) = 12 \cdot \frac{3}{4}\left(\frac{5}{9}b\right)^{3/4-1} \cdot \frac{5}{9} - 1 = 5 \cdot \left(\frac{5}{9}b\right)^{-1/4} - 1$$

- So $\pi_*'(900) = 5 \cdot (500)^{-1/4} - 1 \approx 0.0574$
- If the budget increases from 900 by 1 unit, the max net profit will approximately increase by 0.0574 units.

Lagrange multiplier as the derivative of net profit

- Recall our first-order conditions (with $b = 900$):

$$\begin{cases} f_L(L, K, \lambda) = 0 \\ f_K(L, K, \lambda) = 0 \\ f_\lambda(L, K, \lambda) = 0 \end{cases} \Leftrightarrow \begin{cases} 3K^{\frac{1}{2}}L^{-\frac{3}{4}} = 0.6(1+\lambda) \\ 6K^{-\frac{1}{2}}L^{\frac{1}{4}} = 1.2(1+\lambda) \\ 0.6L + 1.2K = 900 \end{cases}$$

- We have solved that $K = L = 500$.
- Substitute $K = L = 500$ into the 1st or 2nd equation and get

$$\lambda = 5 \cdot (500)^{-1/4} - 1 = \pi_*'(900) \approx 0.0574$$

- If the budget increases from 900 by 1 unit, the max net profit will increase approximately by $\lambda \approx 0.0574$ units.

Lagrange method: General

Optimise $\pi(x, y)$, $(x, y) \in D$, subject to $px + wy = b$.

- D is an open set, for example, $x, y > 0$.
- p, w, b are all known values.
- assume the optimal solution exists.

It is equivalent to optimising the Lagrangian function

$$f(x, y, \lambda) = \pi(x, y) + \lambda(b - px - wy),$$

defined for $(x, y) \in D$ and $\lambda \in (-\infty, \infty)$

- Find the optimal point(s) among stationary points:

$$\begin{cases} f_L(L, K, \lambda) = 0 \\ f_K(L, K, \lambda) = 0 \\ f_\lambda(L, K, \lambda) = 0 \end{cases} \Leftrightarrow \begin{cases} \pi_x(x, y) = p\lambda \\ \pi_y(x, y) = w\lambda \\ px + wy = b \end{cases}$$

There are two possible cases: $\lambda = 0$ and $\lambda \neq 0$

Case 1: $\lambda = 0$

- Check whether the first order conditions

$$\pi_x(x, y) = 0$$
$$\pi_y(x, y) = 0$$
$$px + wy = b$$

  have a solution.
- This is to check whether the stationary point of $\pi(x, y)$ satisfies the budget constraint.

Case 2: $\lambda \neq 0$

- First Equation $\div$ Second Equation (on page 21)

$$\frac{\pi_x(x,y)}{\pi_y(x,y)} = \frac{p}{w}, \quad \Rightarrow \quad y = g(x)$$

- Substitute $y = g(x)$ into the budget constraint:

$$px + wg(x) = b, \quad \Rightarrow x = x^*$$

- Solution: $x = x^*$, $y^* = g(x^*)$, and the optimal value $\pi(x^*, y^*)$.

First order conditions: Normal cases

Once we obtain $x = x^*$ and $y^* = g(x^*)$, we substitute them back to the 1st or 2nd equation to solve

$$\lambda^* = \frac{1}{p}\pi_x(x^*, y^*) \quad \text{or} \quad \lambda^* = \frac{1}{w}\pi_y(x^*, y^*)$$

Interpret the Lagrange multiplier

If the budget b is a variable, and solving the first order conditions gives the optimal point with

$$x = x^*(b), \quad y = y^*(b), \quad \lambda = \lambda^*(b)$$

and optimal value $\pi_*(b) = \pi(x^*(b), y^*(b))$, then

$$\lambda^*(b) = \pi_*'(b)$$

Interpretation: If the budget increases from b by one unit, the optimal value will change approximately by $\lambda^*(b)$ units.

Summary

For optimisation of a function of multiple variables

- In this unit, we always assume the optimal solution exists.

Optimise $f(x, y)$ over an open set D

- Find the optimal point(s) among stationary point(s).

Optimise $\pi(x, y)$ subject to a budget constraint $px + wy = b$

- Optimise the Lagrangian function $f(x, y, \lambda)$
- Find the optimal point(s) among stationary point(s)
- Interpret the Lagrange multiplier λ

Further examples using Lagrangian method

Example 1

Suppose $z = xy$, which we want to maximise subject to $x + y \leq 100$.
Solution: This constrained maximisation problem is to be solved using the Lagrangian given by

$$f(x, y, \lambda) = xy + \lambda(100 - x - y).$$

Compute the 1st-order partial derivatives and let them be 0:

$$\begin{aligned} f_x(x, y, \lambda) &= y - \lambda = 0 \\ f_y(x, y, \lambda) &= x - \lambda = 0 \\ f_\lambda(x, y, \lambda) &= 100 - x - y = 0 \end{aligned}$$

The first two equations show that $x = y$, which is then substituted into the 3rd equation, and then we obtain that $x = 50$ and $y = 50$.

Example 2

Maximise $u = 4x^2 + 3xy + 6y^2$, subject to $x + y = 56$.
Solution: This constrained maximisation problem can be solved using the Lagrangian given by

$$u(x, y, \lambda) = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y).$$

Take the 1st-order partial derivatives and set them to 0:

$$\begin{aligned} u_x(x, y, \lambda) &= 8x + 3y - \lambda = 0 \\ u_y(x, y, \lambda) &= 3x + 12y - \lambda = 0 \\ u_\lambda(x, y, \lambda) &= 56 - x - y = 0 \end{aligned}$$

The first two equations lead to: $8x + 3y = 3x + 12y$, by which we obtain $x = 1.8y$. Substituting it to the 3rd equation, we obtain that $2.8y = 56$, and thus, $y = 20$ and then $x = 36$. Substituting the values of x and y into the 1st or 2nd equation, we obtain that $\lambda = 348$.

Example 3: Cost minimisation

A firm produces two goods, in the quantity of x and y, respectively. Due to a government regulation, the firm's production must satisfy the constraint $x + y = 42$. The firm's cost function is $c(x, y) = 8x^2 - xy + 12y^2$, which we want to minimise subject to the above constraint.

Solution: The Lagrangian of the constrained minimisation is

$$L(x, y, \lambda) = 8x^2 - xy + 12y^2 + \lambda(42 - x - y).$$

Take the 1st-order partial derivatives and set them to 0:

$$\begin{aligned} L_x(x, y, \lambda) &= 16x - y - \lambda = 0 \\ L_y(x, y, \lambda) &= -x + 24y - \lambda = 0 \\ L_\lambda(x, y, \lambda) &= 42 - x - y = 0 \end{aligned}$$

The first two equations lead to: $16x - y = -x + 24y$, by which we obtain $17x = 25y$. Substituting it to the 3rd equation, we obtain that $x = 25$ and $y = 17$, and then $\lambda = 383$.

Example 4: Utility Maximisation

Consider a consumer with the utility function $u(x, y) = xy$, who faces a budget constraint of $b = p_x x + p_y y$, where b, p_x and p_y are the known budget and prices. The choice problem is to maximise $u(x, y) = xy$, subject to $p_x x + p_y y = b$.

Solution: The Lagrangian of the constrained maximisation is

$$L(x, y, \lambda) = xy + \lambda(b - p_x x - p_y y).$$

Take the 1st-order partial derivatives and set them to 0:

$$\begin{aligned} L_x(x, y, \lambda) &= y - \lambda p_x = 0 \\ L_y(x, y, \lambda) &= x - \lambda p_y = 0 \\ L_\lambda(x, y, \lambda) &= b - p_x x - p_y y = 0 \end{aligned}$$

Eliminating λ from the first 2 equations, we have $y/p_x = x/p_y$, by which we obtain $xp_x = yp_y$. Substituting it to the 3rd equation, we obtain that $x = b/(2p_x)$ and $y = b/(2p_y)$, and then $\lambda = b/(2p_x p_y)$.

Example 5: Minimisation of cost

Consider the same consumer with the utility function $u(x, y) = xy$, who has cost function $c(x, y) = p_x x + p_y y$, where p_x and p_y are known prices values. The choice problem is to minimise $c(x, y) = p_x x + p_y y$, subject to $u(x, y) = u_0$.

Solution: The Lagrangian of the constrained minimisation is

$$U(x, y, \lambda) = p_x x + p_y y + \lambda(u_0 - xy).$$

Take the 1st-order partial derivatives and set them to 0:

$$\begin{aligned} U_x(x, y, \lambda) &= p_x - \lambda y = 0 \\ U_y(x, y, \lambda) &= p_y - \lambda x = 0 \\ U_\lambda(x, y, \lambda) &= u_0 - xy = 0 \end{aligned}$$

Eliminating λ from the first 2 equations, we have $p_x/y = p_y/x$, by which we obtain $xp_x = yp_y$. Substituting it to the 3rd equation, we obtain that $x = (p_y u_0/p_x)^{1/2}$ and $y = (p_x u_0/p_y)^{1/2}$, and then $\lambda = (p_x p_y/u_0)^{1/2}$.