Influence Diagnostics for Multivariate GARCH Processes

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Abstract: This paper presents diagnostics for identifying influential observations when estimating multivariate GARCH models. We derive influence diagnostics by introducing minor perturbations to the conditional variances and covariances. The derived diagnostics are applied to a bivariate GARCH model of daily returns of the S&P500 and IBM. We find that univariate diagnostic procedures may be unable to identify the influential observations in a multivariate model. Importantly, the proposed curvature-based diagnostic identified influential observations where the correlation between the two series had a major change. These observations were not identified as influential using the univariate diagnostics for each asset separately. When estimating the bivariate GARCH model allowing for weights at influential observations, we found that the time-varying correlations behaved differently from that implied by the model ignoring influential observations. The application therefore highlights the importance of extending univariate diagnostic procedures to multivariate settings.

KEYWORDS: Curvature-based diagnostic; Modified likelihood displacement; Perturbation; Slope-based diagnostic; Time-varying beta; Time-varying correlation.

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1 INTRODUCTION

During the last 20 years we have seen a voluminous literature documenting the presence of volatility clustering and excess kurtosis in financial markets. The autoregressive conditional heteroscedasticity (ARCH) and generalized ARCH (GARCH) models have generally proven successful in capturing these features. See Bollerslev, Chou and Kroner (1992) and Bauwens, Laurent and Rombouts (2006) for reviews. However, many empirical studies have revealed that the usual estimation methods such as the quasi maximum likelihood estimate (QMLE), are not robust to influential observations commonly observed in many data sets. The detection of influential observations in GARCH models is important, because these observations may significantly affect parameter estimation and substantially worsen forecasting performance (see for example, Bera and Higgins, 1993; Zhang and King, 2005). Moreover, failing to allow for influential observations may also seriously affect the asymptotic size and power of the Lagrange multiplier test for the ARCH effect (van Dijk, Franses and Lucas, 1999; Franses and Ghysels, 1999; Zhang, 2004; among others). Hence it is very important to conduct diagnostic tests that assess the impact of influential observations when fitting GARCH models.

Many of the diagnostics that seek to identify influential observations are based on the deletion of single observations. Such diagnostics are called deletion diagnostics and are well documented in Cook and Weisberg (1982). However, in some examples, the importance of observations is not evident unless several observations are deleted at once (Atkinson, 1986). In such circumstances, multiple influential observations cannot be detected by single deletion diagnostics. This is because the effect of an individual observation is likely to be hidden by the effect of the other influential observations nearby. This phenomenon is called the masking effect (see for example, Atkinson, 1986, 1994; Cook, 1986; Lawrance, 1995).

To overcome the masking effect when assessing influential observations, Cook (1986) proposed a diagnostic that assesses influence through a vector of minor perturbations of the same dimension as the vector of observations. Cook (1986) indicated that statistical models usually involve a certain degree of approximation and therefore diagnostics are often necessary to assess inadequacies in a postulated model. The assessment of the influence of individual observations on some key
results in a postulated model through a vector of minor perturbations has subsequently received a lot of attention. To derive a diagnostic for assessing influence, a vector of minor perturbations, denoted as $\omega = (\omega_1, \omega_2, \ldots, \omega_T)'$, is often introduced into a postulated model, where $T$ is the length of the sample used for estimating the model. A simple perturbation scheme is to add a minor perturbation to each observation of the response variable, and such a perturbation scheme is often referred to as the additive perturbation. Influential observations are those observations that are identified by influence diagnostics as being most sensitive to such perturbations. If minor perturbations are found to have no serious effects on the specified key results, the postulated model may be tentatively accepted. Otherwise the appropriateness of the postulated model is in doubt.

A diagnostic that assesses influence through minor perturbations is clearly different from the deletion diagnostic because the vector of perturbations is of the same dimension as the vector of observations. The influence of the perturbations associated with each of the observations can therefore be examined simultaneously. As a consequence, influence diagnostics obtained through a vector of minor perturbations are immune to masking.

As discussed by Zhang (2004), the influence diagnostic method seems more appropriate than deletion diagnostics when assessing influential observations in GARCH models. A comprehensive literature investigates detecting influential observations for the univariate GARCH model (see, for example, Atkinson, Koopman and Shephard, 1997; Hotta and Tsay, 1998; van Dijk, Franses and Lucas, 1999; Charles and Darné, 2005). However, the procedures employed in these papers are likely to suffer from the masking effect. Zhang (2004) and Zhang and King (2005) overcame this limitation by employing influence diagnostics in the univariate GARCH model. To date, influence diagnostics have not been used to assess influence in multivariate GARCH models.

A common misconception is that when assessing influence in a multivariate GARCH model, the existing influence diagnostics for the univariate GARCH model can be applied to each series. However, in the multivariate GARCH model, influential observations may be detected when the correlations between the series experience a major change. For example, consider two series that are highly positively correlated. If one series experiences a major change while the other series remains virtually unchanged, it is conceivable that this observation may be identified as influential
via a multivariate diagnostic procedure, but may be undetected through univariate diagnostic procedures. Therefore, it might be misleading to justify the appropriateness of a multivariate GARCH model by only examining influence via univariate GARCH models for each series.

There are many alternative specifications for the multivariate GARCH model. In their review of the multivariate GARCH literature, Bauwens, Laurent and Rombouts (2006) categorize the models into three “nonmutually exclusive” approaches; i) direct generalizations of the univariate GARCH model, ii) linear combinations of univariate GARCH models, and iii) nonlinear combinations of univariate GARCH models. Whilst the approach proposed in this paper is general, the computation and derivation of the multivariate diagnostic is model specific. Given the large number of alternative parameterizations, we illustrate the procedure via two popular specifications. Both are from the third class of models, namely the dynamic conditional correlation GARCH (DCC-GARCH) model of Tse and Tsui (2002) and the BEKK GARCH model of Engle and Kroner (1995). The derivation of the relevant derivatives for the proposed diagnostic for the other multivariate GARCH models is left to the interested reader.

We illustrate the diagnostic using the multivariate versions of these models for a number of reasons. First, the models are relatively parsimonious and easily interpretable. Second, the DCC and BEKK models allow for time-varying correlations which have been shown to be important empirically (see for example, Tse and Tsui, 2002; and Engle, 2002). Third, the DCC-GARCH model nests the constant-correlation GARCH model of Bollerslev (1990). Fourth, the positive definiteness requirement is either guaranteed (BEKK) or easily imposed (DCC-GARCH).

The derived diagnostics are applied to a bivariate GARCH model of daily returns of the S&P500 and IBM. We find that influential observations in the bivariate GARCH model generally occur at locations where the correlation between the two series has a major change. Importantly, influential observations were identified in the bivariate model that were not identified through the univariate procedures and vice versa. The results therefore illustrate the need to implement multivariate diagnostic procedures when fitting multivariate GARCH models.

The rest of the paper is organized as follows. In Section 2 we briefly describe influence diagnostics and discuss the slope- and curvature-based diagnostic for the multivariate GARCH
models. Section 3 presents an empirical example that illustrates the importance of the diagnostic procedure. Section 4 concludes the paper.

2 INFLUENCE DIAGNOSTICS FOR THE MULTIVARIATE GARCH MODEL

2.1 Influence Diagnostics

Assume that there are \( T \) observations for a postulated model, and let \( \theta \) denote the vector of parameters to be estimated by maximizing the log-likelihood \( L(\theta) \). Suppose that a perturbation vector denoted by \( \omega = (\omega_1, \omega_2, \ldots, \omega_T)' \) is introduced to the model and is expressed as

\[
\omega = \omega_0 + a l,
\]

where \( \omega_0 \) is the point of null perturbation, and \( a \) measures the magnitude of the perturbation in the direction \( l \).

Let \( L(\theta|\omega) \) denote the quasi log-likelihood for the perturbed model. According to Billor and Loynes (1993), the modified (quasi) likelihood displacement for assessing influence is defined as

\[
LD(\omega) = -2[L(\theta) - L(\theta|\omega)].
\]

Influence of the perturbation vector on the modified likelihood displacement can be examined on a graph spanned by \( \omega \) and \( LD(\omega) \) and is expressed as

\[
\alpha(\omega) = (\omega_1, \omega_2, \ldots, \omega_n, LD(\omega))'.
\]

Influence is assessed through the normal curvature, which measures the local behaviour of \( LD(\omega) \) at \( \omega = \omega_0 \) on \( \alpha(\omega) \). The curvature-based diagnostic is the directional vector, along which the normal curvature achieves the maximum value at \( \omega = \omega_0 \). This method is known as the curvature-based diagnostic or second-order approach to influence (see for example, Wu and Luo, 1993).

For computing simplicity, we may also examine influence through the directional vector, along which the slope of \( LD(\omega) \) with respect to \( \omega \) achieves a maximum value at \( \omega = \omega_0 \). This method
is known as the slope-based diagnostic or first-order approach to influence (see, for example, Lawrance, 1988; Billor and Loynes, 1993; Wu and Luo, 1993). Wu and Luo (1993) showed that the curvature-based diagnostic can provide information that the slope-based diagnostic may fail to provide. This provided the motivation for Zhang and King (2005) to investigate the curvature-based diagnostic through the modified likelihood displacement defined by (2).

According to Wu and Luo (1993) and Zhang and King (2005), the normal curvature of \( LD(\omega) \) at \( \omega = \omega_0 \) is

\[
C_l = \frac{\bar{l} \dddot{l}}{(1 + \dddot{F})^{1/2} \nu \left( I + \dddot{F} \dddot{l} \right)}
\]

where

\[
\dddot{F} = 2 \left( \frac{\partial^2 L(\theta|\omega)}{\partial \omega \partial \omega'} - \left( \frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \omega'} \right)^\prime \left( \frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \theta'} \right)^{-1} \frac{\partial^2 L(\theta|\omega)}{\partial \theta \partial \omega'} \right)
\]

Both \( \dddot{F} \) and \( \dddot{F} \) should be computed at \( \omega = \omega_0 \) or equivalently \( a = 0 \), and at the QMLE of \( \theta \) under the unperturbed model.\(^2\) The curvature-based diagnostic is the directional vector, along which \( C_l \) achieves its maximum at \( \omega = \omega_0 \) or \( a = 0 \) and is denoted as

\[
C_{\text{max}} = \max_l C_l|_{a=0}.
\]

Let \( A = \dddot{F} \) and \( B = (1 + \dddot{F})^{1/2}(I + \dddot{F} \dddot{l}) \). It follows that the maximum curvature denoted by \( C_{\text{max}} \), is the largest eigenvalue of the characteristic equation

\[
| A - \lambda B | = 0.
\]

The eigenvector associated with \( C_{\text{max}} \) is the curvature-based diagnostic vector denoted as \( l_{\text{max}} \). If some components of \( l_{\text{max}} \) are relatively large in magnitude, those observations at corresponding locations are considered as influential observations. A plot of the components of \( l_{\text{max}} \) is often useful for locating observations that are influential on \( LD(\omega) \). The slope-based diagnostic vector \( \dot{F} \), is a by-product of the curvature-based diagnostic. Both the slope- and curvature-based diagnostic vectors are standardized with a unit length.

\(^2\)According to the computation of derivatives presented in Wu and Luo (1993), it would be more natural to take the derivatives with respect to \( a \) and evaluate them at \( a = 0 \) rather than taking derivatives with respect to \( \omega \) and evaluating them at \( \omega = \omega_0 \). We thank the referee for pointing out this.
When we examine influence in a multivariate GARCH model, there is little guidance on how to determine whether the curvature (or slope) is significant. Observations at locations where the components of the diagnostic vector are large in magnitude are regarded as influential. However, for a given large curvature (or slope) value and its associated directional vector, we often have difficulty in deriving a threshold for locating influential observations. In empirical studies, visual inspection of the plot of the diagnostic vector might be a practically plausible method for locating influential observations (see for example, Atkinson, 1985).

When assessing influence in the univariate GARCH model, Zhang and King (2005) employed Monte Carlo simulation to determine threshold values for locating influential observations. These threshold values were heavily dependent on the sample size and not the number of parameters. For this reason we use these threshold values to locate the influential observations in our study.

The procedure of locating influential observations is therefore as follows. First, we obtain the QMLEs of parameters from the unperturbed model. Second, we solve the characteristic equation given by (7) and use the eigenvector associated with the largest eigenvalue (in magnitude) to locate influential observations.

### 2.2 Influence diagnostics for the Multivariate GARCH Model

We consider the derivation of influence diagnostics for a multivariate GARCH model of $d$ asset return series. Let $y_t = (y_{1,t}, y_{2,t}, \ldots, y_{d,t})'$ denote a $d \times 1$ vector of asset returns at time $t$, for $t = 1, 2, \ldots, T$. Assuming conditional normality, $y_t \sim N(0, H_t)$, where $H_t$ is the conditional variance-covariance matrix of $y_t$. Let $\theta$ denote the parameter vector of the multivariate GARCH model. Further, let $h_{i,t}$ denote the conditional variance of Asset $i$ and $h_{ij,t}$ denote the conditional covariance between Asset $i$ and Asset $j$. The quasi log-likelihood of $(y_1, y_2, \ldots, y_T)'$ for a given $\theta$ is (ignoring the constant term)

$$L_m(\theta) = -\frac{1}{2} \sum_{t=1}^{T} \left( \ln |H_t| + y_t' H_t^{-1} y_t \right).$$

To derive influence diagnostics for the multivariate GARCH model, we consider two types of perturbation schemes. The first scheme perturbs the time-varying conditional variance of one of the $d$ asset returns, keeping the conditional variances of the other $(d-1)$ asset returns unperturbed.
Assume that a perturbation vector \( \omega = (\omega_1, \omega_2, \ldots, \omega_T)' \) is introduced via the conditional variance of the first asset return:

\[
h_{1,t}(\omega) = h_{1,t}/\omega_t,
\]

for \( t = 1, 2, \ldots, T \), where the point of null perturbation is a \( T \times 1 \) vector of ones. Let \( L_m(\theta|\omega) \) denote the quasi log-likelihood under this perturbation scheme. To calculate \( \dot{F} \) and \( \ddot{F} \), we require \( \partial L_m(\theta|\omega)/\partial \omega \), \( \partial^2 L_m(\theta|\omega)/\partial \theta \partial \omega' \) and \( \partial^2 L_m(\theta|\omega)/\partial \omega \partial \omega' \), where all derivatives are calculated at the point of null perturbation. See Comte and Lieberman (2003) and Hafner and Herwartz (2008) for analytical expressions for the gradient and Hessian for a number of multivariate GARCH models.

The second perturbation scheme perturbs the time-varying conditional covariance between one pair of asset returns. Say for example, the conditional covariance between the \( i \)th asset return and the \( j \)th asset return is perturbed as

\[
h_{ij,t}(\omega) = h_{ij,t}/\omega_t,
\]

for \( t = 1, 2, \ldots, T \). Let \( L_{ij}(\theta|\omega) \) denote the quasi log-likelihood under this perturbation scheme. The computation of \( \dot{F} \) and \( \ddot{F} \) requires \( \partial L_{ij}(\theta|\omega)/\partial \omega \), \( \partial^2 L_{ij}(\theta|\omega)/\partial \theta \partial \omega' \) and \( \partial^2 L_{ij}(\theta|\omega)/\partial \omega \partial \omega' \).

It should be noted that in terms of perturbations to conditional variances and covariances, it is only possible to perturb one conditional variance or covariance series. It is not computationally possible to jointly perturb all conditional variances and covariances, because the length of the perturbation vector should be the same as the length of the data.

Further, perturbations to the conditional variance of one series will also have an effect on the conditional covariances with the other series. Perturbations to the conditional variance may therefore identify observations as influential when there are obvious changes in the covariance between two series. For this reason, influential observations identified using this procedure may differ from those identified using univariate diagnostic procedures.

We now consider the diagnostics for two popular multivariate GARCH models, the DCC-GARCH model of Tse and Tsui (2002) and the BEKK model of Engle and Kroner (1995).
2.3 Multivariate DCC-GARCH Model

The approach of Tse and Tsui (2002) decomposes $H_t$ as

$$H_t = D_t \Gamma_t D_t,$$

(10)

where $D_t$ is the $d \times d$ diagonal matrix with its $i$th diagonal element being $\sqrt{h_{i,t}}$, and $\Gamma_t$ is the $d \times d$ conditional correlation matrix of $y_t$, for $t = 1, 2, \cdots, T$. The conditional variances are specified as

$$h_{i,t} = \gamma_i + \alpha_i h_{i,t-1} + \beta_i y_{i,t-1}^2,$$

(11)

where $\gamma_i$, $\alpha_i$ and $\beta_i$ are nonnegative, and $\alpha_i + \beta_i < 1$, for $i = 1, 2, \cdots, d$. In addition, the conditional correlation matrix $\Gamma_t$ is generated from the following recursion:

$$\Gamma_t = (1 - \pi_1 - \pi_2) \Gamma + \pi_1 \Gamma_{t-1} + \pi_2 \Psi_{t-1},$$

(12)

where $\Gamma = \{\rho_{ij}\}$ is a time-invariant $d \times d$ positive parameter matrix with unit diagonal elements, and the elements of $\Psi_{t-1}$ are specified as

$$\psi_{ij,t-1} = \frac{\sum_{k=1}^{m} z_{i,t-k} z_{j,t-k}}{\sqrt{\sum_{k=1}^{m} z_{i,t-k}^2 \sum_{k=1}^{m} z_{j,t-k}^2}},$$

for $1 \leq i < j \leq d$, where $z_{i,t} = y_{i,t}/\sqrt{h_{i,t}}$, for $i = 1, 2, \cdots, d$. A necessary condition for $\Psi_{t-1}$ to be a well-defined correlation matrix is $m \geq d$.

Let $\theta = (\gamma_1, \cdots, \gamma_d, \alpha_1, \cdots, \alpha_d, \beta_1, \cdots, \beta_d, \rho_{1,2}, \cdots, \rho_{d-1,d}, \pi_1, \pi_2)$ denote the parameter vector of the DCC MGARCH model. The quasi log-likelihood of $(y_1, y_2, \cdots, y_T)'$ for a given $\theta$ is (ignoring the constant term)

$$L_v(\theta) = -\frac{1}{2} \sum_{t=1}^{T} \ln |\Gamma_t| - \frac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{d} \ln h_{i,t} - \frac{1}{2} \sum_{t=1}^{T} y_t D_t^{-1} \Gamma_t^{-1} D_t^{-1} y_t.$$

(13)

The first perturbation scheme perturbs the conditional variance of one of the $d$ asset return series as discussed in Section 2.2. For the second scheme, we propose perturbing the conditional correlation between the $i$th asset return and the $j$th asset return as

$$\rho_{ij,t}/\omega_t = (1 - \pi_1 - \pi_2) \rho_{ij} + \pi_1 \rho_{ij,t-1}/\omega_{t-1} + \pi_2 \psi_{ij,t-1},$$

(14)

for $t = 1, 2, \cdots, T$. 

8
2.4 Multivariate BEKK GARCH Model

The conditional variance-covariance matrix of the BEKK MGARCH(1,1) model is defined as

\[ H_t = C'C + A'y_t y_t' A + G'H_{t-1} G, \]  

(15)

where \( C, A \) and \( G \) are \( d \times d \) matrix and \( C \) is upper triangular. The quadratic specification ensures positive definiteness. The number of parameters in the model is \( d(5d+1)/2 \). To reduce the number of parameters (and hence the generality), \( A \) and \( G \) can be diagonalized. See Bauwens, Laurent and Rombouts (2006) for further discussion. The perturbation scheme can be introduced to perturb one element of \( H_t \) at a time. This can be either the conditional variance of one of the \( d \) asset returns or the conditional covariance between any two asset returns. To derive the derivatives required for the computation of \( \dot{F} \) and \( \ddot{F} \), the perturbed conditional variance-covariance matrix is then substituted into (8).

3 EMPIRICAL RESULTS

In this section, we illustrate the proposed diagnostics via a bivariate GARCH model of daily returns of the S&P500 index and IBM. The sample contains 1215 paired observations from 03/01/2005 to 30/10/2009 excluding weekends and holidays. This section consists of two parts. In Section 3.1 we apply the proposed diagnostics in order to identify influential observations in the bivariate GARCH model. For comparison purposes, we also apply the univariate diagnostic procedure of Zhang and King (2005) to identify influential observations in univariate GARCH models for each series. In Section 3.2, we re-estimate the bivariate GARCH model allowing for weights at the identified influential observations.

Time-series plots of the two return series are presented in Figures 1(a) and (b). Both series exhibit heightened levels of volatility around the time of the Global Financial Crisis. The correlation between the two series is 0.7213, and the Lagrange multiplier test of Engle (1993) indicates that both series exhibit ARCH effects.

The choice between using a constant-correlation model and one that allows for time-varying
correlations is important, because a misspecified model may lead to wrongly identified influential observations. The hypothesis of constant correlation is therefore tested using the Lagrange multiplier test of Tse (2000) and the information matrix test of Bera and Kim (2002). The test statistics are respectively, 22.49 and 199.56, which reject the null hypothesis of constant correlation at the 1% level of significance.

In this section we therefore illustrate the proposed multivariate diagnostic via the DCC-GARCH model of Tse and Tsui (2002). The derivation of the derivatives can be found in Appendices A.1 to A.4. It should be noted however that the diagnostic procedure may be implemented via the use of numerical derivatives. If conditional densities other than normal are employed, the additional complexity may mean that numerical derivatives are more suitable.

3.1 Identification of Influential Observations

We considered five perturbation schemes. The first two schemes perform the innovative model-perturbation scheme of Zhang and King (2005) to univariate GARCH(1,1) models of each series. The other three consist of the perturbation schemes to the bivariate model, namely perturbations to the conditional correlation and conditional variances of the S&P500 returns and the IBM returns. Under each perturbation scheme, we calculated the curvature-based diagnostic vector, whose components are plotted in Figures 2 and 3, respectively. For the purpose of locating influential observations, we set the threshold value at 0.2185. This was obtained through interpolation between threshold values for sample sizes of 1000 and 1500 given in Table 3 of Zhang and King (2005). The significance of the identified influential observations is then determined in Section 3.2, when the model is re-estimated allowing for weights at the influential observations.

Under the univariate perturbation scheme to the conditional variance of the S&P500 returns, the 540th (27/02/2007) and 940th (29/09/2008) observations were identified as influential (when using the curvature-based diagnostic). On applying the univariate perturbation scheme to the conditional variance of IBM returns, the 71st (15/04/2005) and 1060th (23/03/2009) observations were identified.

Using the multivariate diagnostic procedures proposed in this paper, we found that the 942nd
(01/10/2008) and 1018th (21/01/2009) observations were influential with respect to perturbations to the dynamic correlations. When perturbing the conditional variance of S&P500 returns, we found that the 961st (28/10/2008) and 1018th (21/01/2009) observations were identified as influential. The 940th (29/09/2008) and 961st (28/10/2008) observations were identified as influential observations under the perturbation scheme to the conditional variance of IBM returns. Thus, the bivariate diagnostic procedures also identified four influential observations. Only one of these however was identified as influential under the univariate diagnostic procedures.

Table 1 presents the locations of the identified influential observations, as well as the corresponding return, previous return and the change in return for both return series. The univariate procedures identify influential observations at locations where the returns are generally large in magnitude. To illustrate, for the S&P500 returns, the 540th and 940th observations are influential and their returns are respectively, $-3.53\%$ and $-9.22\%$; for IBM returns, the 71st and 1060th observations are influential and their returns are respectively, $-8.67\%$ and $6.49\%$.

The bivariate perturbation schemes generally identified observations at locations where the correlation or covariance between the two series experienced an obvious change. This partly explains why three out of the four influential observations identified through the bivariate diagnostics were not identified through the univariate diagnostics. To illustrate, on 01/10/2008, the S&P500 return is $-0.46\%$, which is a decrease of $5.74\%$ from the previous day’s return of $5.28\%$. On the same day, the return of IBM is $-6.02\%$, which is a much larger decrease of $8.18\%$ from the previous day’s return of $2.16\%$.

We also found that the four influential observations identified through the bivariate diagnostic procedure are in the period of the recent financial turmoil. In contrast, only two out of four influential observations identified through the univariate diagnostics are in this period. These differences highlight the importance of extending the univariate diagnostic procedure to a multivariate model. This is because the univariate procedures may be unable to identify the influential observations in a multivariate model.
3.2 Estimation of Bivariate GARCH Model Allowing for Influential Observations

To further examine the importance of the influential observations identified by the proposed diagnostic, let Asset 1 represent the S&P500 index and let Asset 2 denote the IBM price. The following bivariate DCC-GARCH model was estimated via QMLE:

\[
y_{1,t} = \varepsilon_{1,t}, \quad y_{2,t} = \varepsilon_{2,t}, \quad (16)
\]
\[
h_{i,t} = \gamma_i + \alpha_i y_{i,t-1}^2 + \beta_i h_{i,t-1} \quad \text{for } i = 1, 2, \quad (18)
\]
\[
h_{12,t} = \rho_t \sqrt{h_{1,t} h_{2,t}}, \quad (19)
\]

where \(y_{1,t}\) and \(y_{2,t}\) represent the de-meaned returns, and

\[
\rho_t = (1 - \pi_1 - \pi_2) \rho + \pi_1 \rho_{t-1} + \pi_2 \psi_{t-1}, \quad (20)
\]

and

\[
\psi_{t-1} = \frac{\sum_{j=1}^{2} z_{1,t-j} z_{2,t-j}}{\sqrt{\sum_{j=1}^{2} z_{1,t-j}^2 \sum_{j=1}^{2} z_{2,t-j}^2}}, \quad (21)
\]

with \(z_{i,t} = \varepsilon_{i,t}/\sqrt{h_{i,t}}\), for \(i = 1, 2\). We introduce a weight denoted by \(d_t\), which is a parameter at the location of an influential observation, otherwise unity, for \(t = 1, 2, \cdots, T\). The weights enter the conditional covariance matrix in the following way: \(h_{i,t}/d_{i,t}\) for influential observations to the conditional variance of Asset \(i\), and \(\rho_t/d_{p,t}\) for influential observations to the conditional correlation.

To ensure that \(-1 < \rho_t < 1\), the logistic transformation was used when the conditional correlation weights were not equal to unity:

\[
\rho_t^* = LL + \frac{UL - LL}{1 + \exp(-\rho_t/d_{p,t})},
\]

where \(\rho_t\) is the estimated correlation from (20), \(LL\) and \(UL\) are respectively, the lower and upper limits to \(\rho_t^*\), which is the the correlation used to construct the variance covariance matrix for observation \(t\).
Let Model A denote the bivariate GARCH model given by (16) to (21), and Model B denote the bivariate GARCH model given by (16) to (21) allowing for the six identified influential observations (because the 961st and 1018th observations have been identified twice). Models C, D and E denote the bivariate GARCH models given by (16) to (21) when allowing for influential observations identified through the perturbations to the S&P500 variance (Model C), IBM variance (Model D) and the correlation (Model E).

The parameter estimates for each model are presented in Table 2. When introducing a weight for all of the six observations identified through the bivariate diagnostics, we found that the AIC and likelihood ratio test ($\chi^2_6 = 15.68$, $p$ value 0.0156) do not support the restrictions implied by Model A. When allowing for influential observations under each perturbation scheme separately (Models C, D and E), we found that Model E is the preferred model. Model E has the lowest AIC and the likelihood ratio test ($\chi^2_2 = 11.20$, $p$ value 0.0037) clearly rejects the restrictions implied by Model A. This finding indicates the importance of the perturbation scheme to the conditional correlation between the two series. The AIC and likelihood ratio test also support Model C over Model A ($\chi^2_2 = 10.00$, $p$ value 0.0067). Model D however is not favoured over Model A according to the AIC or the likelihood ratio. This indicates that such a perturbation scheme may not be appropriate for the bivariate model under investigation.

To examine the effects of the influential observations on the time varying correlations implied by Models A and E, Figure 4 plots the time varying correlations from both models. The results clearly demonstrate a difference in the correlations around the times of the identified influential observations — observations 942 (01/10/2008) and 1018 (21/01/2009), dates that are within the Global Financial Crisis. More specifically, there is a divergence in the correlations at the influential observations. The subsequent correlations from Model E then converge to the correlations from Model A at the exponential rate implied by the estimated model.

The model can also be used to estimate time-varying betas in accordance with the capital asset pricing model (see for example, Braun, Nelson and Sunier, 1995; Giannopoulos, 1995; McClain, Humphreys and Boscan, 1996; Gonzalez-Rivera, 1996). Figure 4 presents the time-varying beta estimates from Models A and E. Whilst the introduction of the weights has little effect on the

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3 The time varying beta is estimated as $h_{12,t}/h_{11,t}$, for $t = 1, 2, \cdots, T$. 
beta estimate for observation 942 (1/10/2008), it has a significant effect on the beta estimate at observation 1018 (21/1/2009). These findings clearly indicate the importance of applying the proposed influence diagnostic procedure when estimating multivariate GARCH models.

4 CONCLUSION

This paper extends the curvature-based diagnostic of Zhang and King (2005) to examine the influence of minor perturbations on the modified likelihood displacement in multivariate GARCH models. An application of the diagnostic to a bivariate GARCH model between daily returns of the S&P500 index and IBM, identified influential observations at locations where the conditional correlation between the two return series was disturbed. We find that after controlling for the effects of the influential observations, the implied correlations and time-varying betas differ substantially. The results also demonstrated that the identified influential observations in a multivariate model may not necessarily be influential in a univariate model for each series. The results therefore highlight the importance of assessing influence via the curvature-based diagnostic when estimating multivariate GARCH models.

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APPENDIX: Diagnostics for the Bivariate DCC-GARCH Model

In the bivariate DCC-GARCH model, the conditional variance between the return series is

\[ h_{12,t} = \rho_t \sqrt{h_{1,t} h_{2,t}}, \]  

(22)

where \( \rho_t \) is the conditional correlation between the two return series and is specified as

\[ \rho_t = (1 - \pi_1 - \pi_2) \rho + \pi_1 \rho_{t-1} + \pi_2 \psi_{t-1}, \]  

(23)

for \( t = 1, 2, \ldots, T \), where \( \rho_t = \rho \), for \( t = 1, 2 \). Note that \( 0 \leq \pi_1, \pi_2 \leq 1, \pi_1 + \pi_2 \leq 1, \) and

\[ \psi_{t-1} = \frac{\sum_{j=1}^{m} z_{1,t-j} z_{2,t-j}}{\sqrt{\sum_{j=1}^{m} z_{1,t-j}^2 \sum_{j=1}^{m} z_{2,t-j}^2}}, \]

with \( m \geq 2 \) for a bivariate model and \( z_{i,t} = y_{i,t}/\sqrt{h_{i,t}} \), for \( i = 1, 2 \). The quasi log-likelihood is

\[ L_v(\theta) = -\frac{1}{2} T \sum_{t=1}^{T} (\ln h_{1,t} + \ln h_{2,t}) - \frac{1}{2} \sum_{t=1}^{T} \ln(1 - \rho_t^2) - \frac{1}{2} \sum_{t=1}^{T} \frac{z_{1,t}^2 + z_{2,t}^2 - 2 \rho_t z_{1,t} z_{2,t}}{1 - \rho_t^2}. \]

(24)

To derive the influence diagnostics, we consider two types of perturbation schemes as follows.

A.1 Perturbation to conditional correlation

Assume that perturbations are introduced to the model in the way that \( \rho_t(\omega) = \rho_t/\omega_t \), for \( t = 1, 2, \ldots, T \). The point of null perturbation is a \( T \times 1 \) vector of ones, and the perturbed conditional correlation is

\[ \rho_t/\omega_t = (1 - \pi_1 - \pi_2) \rho + \pi_1 \rho_{t-1}/\omega_{t-1} + \pi_2 \psi_{t-1}, \]  

(25)

for \( t = 1, 2, \ldots, T \). Under this perturbation scheme, the quasi log-likelihood is

\[ L_{v1}(\theta|\omega) = -\frac{1}{2} T \sum_{t=1}^{T} (\ln h_{1,t} + \ln h_{2,t}) - \frac{1}{2} \sum_{t=1}^{T} \ln(1 - \rho_t^2(\omega)) - \frac{1}{2} \sum_{t=1}^{T} \frac{z_{1,t}^2 + z_{2,t}^2 - 2 \rho_t(\omega) z_{1,t} z_{2,t}}{1 - \rho_t^2(\omega)}. \]

(26)

The calculation of \( \tilde{F} \) and \( \tilde{F} \) requires \( \partial L_{v1}(\theta|\omega)/\partial \omega, \partial^2 L_{v1}(\theta|\omega)/\partial \theta \partial \omega \) and \( \partial^2 L_{v1}(\theta|\omega)/\partial \omega \partial \omega \). The details are provided in Appendix A.3.
A.2 Perturbation to conditional variance

Assume that a perturbation vector \( \omega = (\omega_1, \omega_2, \cdots, \omega_T)' \) is introduced via the conditional variance of the first return series in the form of

\[
h_{1,t}(\omega) = h_{1,t}/\omega_t,
\]

for \( t = 1, 2, \cdots, T \), where the null-perturbation point is a \( T \times 1 \) vector of ones, and the conditional variance of the second return series is not perturbed. When assessing influence by perturbing the conditional variance of the second return series, we keep the conditional variance of the first return series unperturbed. Under this perturbation scheme, the log-likelihood function is

\[
L_{v2}(\theta|\omega) = -\frac{1}{2} \sum_{t=1}^{T} (\ln h_{1,t}(\omega) + \ln h_{2,t}) - \frac{1}{2} \sum_{t=1}^{T} \ln \left(1 - \rho_t^2(\omega)\right)
\]

\[
- \frac{1}{2} \sum_{t=1}^{T} \left( \frac{z_{1,t}^2(\omega) + z_{2,t}^2}{1 - \rho_t^2(\omega)} + 2 \rho_t(\omega) z_{1,t}(\omega) z_{2,t} \right),
\]

where \( z_{1,t}(\omega) = y_{1,t}/\sqrt{h_{1,t}(\omega)} \), for \( t = 1, 2, \cdots, T \). The derivatives required for computing \( \dot{F} \) and \( \ddot{F} \) are provided in Appendix A.4.

A.3 Derivatives under the perturbation to conditional correlation

The derivatives required for computing \( \dot{F} \) and \( \ddot{F} \) given in (4) are as follows.

\[
\frac{\partial L_{v1}(\theta|\omega)}{\partial \omega} = \sum_{t=1}^{T} \left( \frac{\rho_t}{1 - \rho_t^2} + \frac{z_{1,t} z_{2,t} (1 + \rho_t^2) - \rho_t (z_{1,t}^2 + z_{2,t}^2 - 2 \rho_t z_{1,t} z_{2,t})}{(1 - \rho_t^2)^2} \right) \frac{\partial \rho_t(\omega)}{\partial \omega},
\]

\[
\frac{\partial^2 L_{v1}(\theta|\omega)}{\partial \theta \partial \omega'} = \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} + \frac{2 \rho_t z_{1,t} z_{2,t} (3 + \rho_t^2) - (1 + 3 \rho_t^2)(z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^3} \right) \frac{\partial^2 \rho_t(\omega)}{\partial \theta \partial \omega'}
\]

\[
+ \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} \right) \frac{\partial \theta}{\partial \omega'} + \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} \right) \frac{\partial \rho_t(\omega)}{\partial \omega'} + \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} \right) \frac{\partial \rho_t(\omega)}{\partial \omega},
\]

\[
\frac{\partial^2 L_{v1}(\theta|\omega)}{\partial \omega \partial \omega'} = \sum_{t=1}^{T} \left( \frac{\rho_t}{1 - \rho_t^2} + \frac{z_{1,t} z_{2,t} (1 + \rho_t^2) - \rho_t (z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^2} \right) \frac{\partial^2 \rho_t(\omega)}{\partial \omega \partial \omega'}
\]

\[
+ \sum_{t=1}^{T} \left( \frac{1 - \rho_t^4 + 2 \rho_t (3 + \rho_t^2) z_{1,t} z_{2,t} - (1 + 3 \rho_t^2)(z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^3} \right) \frac{\partial \rho_t(\omega)}{\partial \omega} \frac{\partial \rho_t(\omega)}{\partial \omega'},
\]

\[16\]
The derivatives of $\partial z_{i,t}/\partial \theta$, $\partial \rho_t/\partial \theta$, $\partial \rho_t(\omega)/\partial \omega$, $\partial^2 \rho_t(\omega)/\partial \theta \partial \omega'$, and $\partial^2 \rho_t(\omega)/\partial \omega' \partial \omega$ are provided as follows.

Let $\theta = (\theta_1, \theta_2, \theta_3)'$ with $\theta_i = (\gamma_i, \alpha_i, \beta_i)'$ for $i = 1, 2$ and $\theta_3 = (\pi_1, \pi_2, \rho)'$. The derivative $\partial z_{i,t}/\partial \theta_j$ for $i = 1, 2$, are obtained as follows.

\[
\begin{align*}
\frac{\partial z_{i,t}}{\partial \theta_j} &= \begin{cases} 
-\frac{1}{2} \frac{\partial h_{i,t}}{\partial \theta_j} & \text{if } j = i, \text{ for } i = 1, 2, \text{ and } j = 1, 2, 3, \\
0 & \text{otherwise}
\end{cases},
\end{align*}
\]

where $\partial h_{1,t}/\partial \theta_1$ and $\partial h_{2,t}/\partial \theta_2$ obtained via the following recursive equations:

\[
\begin{align*}
\frac{\partial h_{i,t}}{\partial \gamma_i} &= 1 + \beta_i \frac{\partial h_{i,t-1}}{\partial \gamma_i}, \\
\frac{\partial h_{i,t}}{\partial \alpha_i} &= \beta_i \frac{\partial h_{i,t-1}}{\partial \alpha_i}, \\
\frac{\partial h_{i,t}}{\partial \beta_i} &= \beta_i \frac{\partial h_{i,t-1}}{\partial \beta_i},
\end{align*}
\]

with the initial values $\partial h_{1,1}/\partial \gamma_1 = 1$, $\partial h_{1,1}/\partial \alpha_1 = \gamma_{1,0}^2$ and $\partial h_{1,1}/\partial \beta_1 = h_{1,0}$, for $i = 1, 2$.

The formulae for computing $\partial \rho_t/\partial \theta_i$, for $i = 1, 2, 3$, are as follows.

\[
\begin{align*}
\frac{\partial \rho_t}{\partial \theta_1} &= \pi_1 \frac{\partial \rho_{t-1}}{\partial \theta_1} - \pi_2 D \sum_{m=1}^{2} \left( z_{2,t-m} - GE^{-1} z_{1,t-m} \right) \frac{z_{1,t-m}}{2h_{1,t-m}} \frac{\partial h_{1,t-m}}{\partial \theta_1}, \\
\frac{\partial \rho_t}{\partial \theta_2} &= \pi_1 \frac{\partial \rho_{t-1}}{\partial \theta_2} - \pi_2 D \sum_{m=1}^{2} \left( z_{2,t-m} - GH z_{2,t-m} \right) \frac{z_{2,t-m}}{2h_{2,t-m}} \frac{\partial h_{2,t-m}}{\partial \theta_2}, \\
\frac{\partial \rho_t}{\partial \theta_3} &= (-\rho + \rho_{t-1} + \pi_1 \partial \rho_{t-1}/\partial \pi_1, -\rho + \psi_{t-1} + \pi_1 \partial \rho_{t-1}/\partial \psi_{t-1}, 1 - \pi_1 - \pi_2 + \pi_1 \partial \rho_{t-1}/\partial \rho),
\end{align*}
\]

where

\[
D = \left( \sum_{j=1}^{2} z_{1,t-j}^2 \sum_{j=1}^{2} z_{2,t-j}^2 \right)^{-1/2}, \quad E = \sum_{j=1}^{2} z_{1,t-j}^2, \quad G = \sum_{j=1}^{2} z_{1,t-j} z_{2,t-j}, \quad H = \left( \sum_{j=2}^{2} z_{2,t-j}^2 \right)^{-1}.
\]

The derivatives appeared in $\partial L(\theta|\omega)/\partial \omega$ are as follows.

\[
\begin{align*}
\frac{\partial \rho_t(\omega)}{\partial \omega_k} &= \begin{cases} 
-\rho_t & \text{if } k = t, \\
0 & \text{otherwise}
\end{cases},
\end{align*}
\]

for $t = 1, 2, \cdots, T$, and $k = 1, 2, \cdots, T$.

The derivatives appeared in $\partial^2 L(\theta|\omega)/\partial \theta \partial \omega'$ are as follows

\[
\begin{align*}
\frac{\partial^2 \rho_t(\omega)}{\partial \theta \partial \omega'_k} &= \begin{cases} 
-\partial \rho_t/\partial \theta & \text{if } k = t, \\
0 & \text{otherwise}
\end{cases},
\end{align*}
\]

for $t = 1, 2, \cdots, T$, and $k = 1, 2, \cdots, T$. 

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In order to obtain \( \partial^2 \rho_t(\omega)/\partial \omega \partial \omega' \)

\[
\frac{\partial^2 \rho_t(\omega)}{\partial \omega_j \partial \omega_k} = \begin{cases} 2 \rho_t & \text{if } j = k = t \\ 0 & \text{if } j \neq t \text{ or } k \neq t \end{cases}
\]

for \( t = 1, 2, \cdots, T \), \( j = 1, 2, \cdots, T \), and \( k = 1, 2, \cdots, T \).

### A.4 Derivatives under the perturbation to conditional variance

In order to obtain \( \hat{F} \) and \( \hat{F} \), we have

\[
\frac{\partial L_{v2}(\theta|\omega)}{\partial \omega} = \sum_{t=1}^{T} \left( \frac{\rho_t}{1 - \rho_t^2} + \frac{(1 + \rho_t^2)z_{1,t}z_{2,t} - \rho_t(z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^2} \right) \frac{\partial \rho_t(\omega)}{\partial \omega}
\]

\[
-\frac{1}{2} \sum_{t=1}^{T} \left( -\rho_t z_{1,t} \rho_t z_{2,t} \right) \frac{\partial h_{1,\ell}(\omega)}{\partial \omega}.
\]

To obtain \( \partial^2 L_{v2}(\theta|\omega)/\partial \theta \partial \omega' \), we have

\[
\frac{\partial^2 L_{v2}(\theta|\omega)}{\partial \theta_1 \partial \omega'} = \frac{1}{2} \sum_{t=1}^{T} \left( -\frac{1}{2} \rho_t z_{1,t}^2 - \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} \right) \frac{\partial \rho_t(\omega)}{\partial \theta_1}
\]

\[
+ \frac{1}{2} \sum_{t=1}^{T} \left( \frac{2z_{1,t}^2 - 3/2\rho_t z_{1,t} z_{2,t} - \rho_t z_{1,t}^2}{1 - \rho_t^2} \right) \frac{\partial h_{1,\ell}(\omega)}{\partial \theta_1}
\]

\[
- \frac{1}{2} \sum_{t=1}^{T} \left( -\frac{z_{1,t}^2 - \rho_t z_{1,t} z_{2,t}}{1 - \rho_t^2} \right) \frac{\partial^2 h_{1,\ell}(\omega)}{\partial \theta_1 \partial \omega'}
\]

\[
+ \sum_{t=1}^{T} \left( \frac{\rho_t}{1 - \rho_t^2} + \frac{(1 + \rho_t^2)z_{1,t} z_{2,t} - \rho_t(z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^2} \right) \frac{\partial^2 \rho_t(\omega)}{\partial \theta_1 \partial \omega'}
\]

\[
- \frac{1}{2} \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} + \frac{2\rho_t(3 + \rho_t^2)z_{1,t} z_{2,t} - (1 + 3\rho_t^2)(z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^3} \right) \frac{\partial \rho_t(\omega)}{\partial \theta_1}
\]

\[
+ \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} + \frac{2\rho_t(3 + \rho_t^2)z_{1,t} z_{2,t} - (1 + 3\rho_t^2)(z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^3} \right) \frac{\partial^2 \rho_t(\omega)}{\partial \theta_2 \partial \omega'}
\]

\[
- \frac{1}{2} \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} + \frac{2\rho_t(3 + \rho_t^2)z_{1,t} z_{2,t} - (1 + 3\rho_t^2)(z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^3} \right) \frac{\partial \rho_t(\omega)}{\partial \theta_2}
\]

\[
+ \frac{1}{4} \sum_{t=1}^{T} (h_{1,\ell} h_{2,\ell})^{-1} \rho_t z_{1,t} z_{2,t} \frac{\partial h_{2,\ell}}{\partial \omega} \frac{\partial h_{1,\ell}(\omega)}{\partial \omega'}
\]
\[
\begin{align*}
\frac{\partial^2 L_{\nu_2}(\theta|\omega)}{\partial \theta_3 \partial \omega'} &= \frac{1}{2} \sum_{t=1}^{T} h_{1,t}^{-1} \frac{2p_t z_{1,t}^2}{(1 - \rho_t^2)} \left( (1 + \rho_t^2) z_{1,t} z_{2,t} \frac{\partial p_t}{\partial \theta_3} \frac{\partial h_{1,t}(\omega)}{\partial \omega'} + \frac{(1 + \rho_t^2) z_{1,t} z_{2,t} - \rho_t (z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^2} \frac{\partial^2 p_t(\omega)}{\partial \theta_3^2} \right) \\
&+ \sum_{t=1}^{T} \left( \frac{\rho_t}{1 - \rho_t^2} + \frac{(1 + \rho_t^2) z_{1,t} z_{2,t} - \rho_t (z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^2} \right) \frac{\partial p_t(\omega)}{\partial \omega} \frac{\partial h_{1,t}(\omega)}{\partial \omega'} \\
&+ \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} + \frac{2p_t (3 + \rho_t^2) z_{1,t} z_{2,t} - (1 + 3\rho_t^2) (z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^3} \right) \frac{\partial^2 p_t(\omega)}{\partial \theta_3 \partial \omega'} \\
&= \frac{1}{2} \sum_{t=1}^{T} h_{1,t}^{-1} \frac{2p_t z_{1,t}^2}{(1 - \rho_t^2)} \left( (1 - \rho_t^2) \frac{\partial h_{1,t}(\omega)}{\partial \omega} \frac{\partial h_{1,t}(\omega)}{\partial \omega'} \right) \\
&- \frac{1}{2} \sum_{t=1}^{T} h_{1,t}^{-1} \left( \frac{z_{1,t}^2 - \rho_t z_{1,t} z_{2,t}}{1 - \rho_t^2} \right) \frac{\partial^2 h_{1,t}(\omega)}{\partial \omega \partial \omega'} \\
&+ \frac{1}{2} \sum_{t=1}^{T} h_{1,t}^{0} \left( \frac{2p_t z_{1,t}^2}{1 - \rho_t^2} \right) \frac{\partial h_{1,t}(\omega)}{\partial \omega} \frac{\partial h_{1,t}(\omega)}{\partial \omega'} \\
&+ \frac{1}{2} \sum_{t=1}^{T} h_{1,t}^{-2} \left( \frac{2z_{1,t}^2 - 3/2p_t z_{1,t} z_{2,t}}{1 - \rho_t^2} \right) \frac{\partial h_{1,t}(\omega)}{\partial \omega} \frac{\partial h_{1,t}(\omega)}{\partial \omega'}.
\end{align*}
\]

In addition, we obtain
\[
\frac{\partial^2 L_{\nu_2}(\theta|\omega)}{\partial \omega \partial \omega'} = \sum_{t=1}^{T} \left( \frac{\rho_t}{1 - \rho_t^2} + \frac{(1 + \rho_t^2) z_{1,t} z_{2,t} - \rho_t (z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^2} \right) \frac{\partial^2 p_t(\omega)}{\partial \omega^2} \\
+ \sum_{t=1}^{T} \left( \frac{1 + \rho_t^2}{(1 - \rho_t^2)^2} + \frac{2p_t (3 + \rho_t^2) z_{1,t} z_{2,t} - (1 + 3\rho_t^2) (z_{1,t}^2 + z_{2,t}^2)}{(1 - \rho_t^2)^3} \right) \frac{\partial^2 p_t(\omega)}{\partial \theta_3 \partial \omega'} \\
- \frac{1}{2} \sum_{t=1}^{T} \frac{h_{1,t}^{-1} (1 + \rho_t^2) z_{1,t} z_{2,t} - 2p_t z_{1,t}^2}{(1 - \rho_t^2)^2} \frac{\partial^2 h_{1,t}(\omega)}{\partial \omega \partial \omega'} \\
- \frac{1}{2} \sum_{t=1}^{T} h_{1,t}^{-1} \left( \frac{z_{1,t}^2 - \rho_t z_{1,t} z_{2,t}}{1 - \rho_t^2} \right) \frac{\partial^2 h_{1,t}(\omega)}{\partial \omega \partial \omega'} \\
+ \frac{1}{2} \sum_{t=1}^{T} h_{1,t}^{0} \frac{2p_t z_{1,t}^2}{(1 - \rho_t^2)} \frac{\partial h_{1,t}(\omega)}{\partial \omega} \frac{\partial h_{1,t}(\omega)}{\partial \omega'} \\
+ \frac{1}{2} \sum_{t=1}^{T} h_{1,t}^{-2} \left( \frac{2z_{1,t}^2 - 3/2p_t z_{1,t} z_{2,t}}{1 - \rho_t^2} \right) \frac{\partial h_{1,t}(\omega)}{\partial \omega} \frac{\partial h_{1,t}(\omega)}{\partial \omega'}.
\]

The derivatives of \(\partial h_{1,t}(\omega)/\partial \omega\), \(\partial p_t(\omega)/\partial \omega\), \(\partial^2 h_{1,t}(\omega)/\partial \theta_1 \partial \omega\), \(\partial^2 p_t(\omega)/\partial \theta_1 \partial \omega\), \(\partial^2 p_t(\omega)/\partial \omega \partial \omega\) and \(\partial^2 h_{1,t}^2(\omega)/\partial \omega \partial \omega\) are as follows.

To obtain the derivative of \(\partial L(\theta|\omega)/\partial \omega\), we have
\[
\frac{\partial h_{1,t}(\omega)}{\partial \omega_k} = \begin{cases} 
-h_{1,t} & \text{if } k = t \\
0 & \text{otherwise}
\end{cases}
\]
for \(t = 1, 2, \cdots, T\), and \(k = 1, 2, \cdots, T\), and
\[
\frac{\partial p_t(\omega)}{\partial \omega} = \pi_1 \frac{\partial p_{t-1}(\omega)}{\partial \omega} + \pi_2 D \sum_{m=1}^{2} \left( z_{2,t-m} - GE^{-1} z_{2,t-m} \right) \frac{\partial z_{1,t-m}(\omega)}{\partial \omega},
\]
where
\[
\frac{\partial z_{1,t-m}(\omega)}{\partial \omega_k} = \begin{cases} 
z_{1,t-m}/2 & \text{if } k = t - m \\
0 & \text{otherwise}
\end{cases}
\]
for \(m = 1, 2, t = 1, 2, \cdots, T\), and \(k = 1, 2, \cdots, T\) with \(1 \leq t - m \leq T\).
where \( \partial h_{1,t} / \partial \theta \) is given in (29).

The derivative \( \partial^2 \rho_t(\omega) / \partial \theta \partial \omega' \) is obtained via \( \partial^2 \rho_t(\omega) / \partial \theta_1 \partial \omega' \), \( \partial^2 \rho_t(\omega) / \partial \theta_2 \partial \omega' \) and \( \partial^2 \rho_t(\omega) / \partial \theta_3 \partial \omega' \) as follows.

\[
\frac{\partial^2 \rho_t(\omega)}{\partial \theta_1 \partial \omega'} = \pi_1 \frac{\partial^2 \rho_{t-1}(\omega)}{\partial \theta_1 \partial \omega'} + \pi_2 D \sum_{m=1}^{2} \left( z_{2,t-m} - GE^{-1} z_{1,t-m} \right) \frac{\partial^2 z_{1,t-m}(\omega)}{\partial \theta_1 \partial \omega'} + 3\pi_2 D E^{-2} G \sum_{i=1}^{2} \sum_{m=1}^{2} \frac{z_{1,t-i} z_{1,t-m}}{\partial \theta_1} \frac{\partial z_{1,t-m}(\omega)}{\partial \omega'} - \pi_2 D E^{-1} \frac{\partial z_{1,t-1}}{\partial \theta_1} \left( (G + 2z_{1,t-1} z_{2,t-1}) \frac{\partial z_{1,t-1}(\omega)}{\partial \omega'} + K \frac{\partial z_{1,t-2}(\omega)}{\partial \omega'} \right) - \pi_2 D E^{-1} \frac{\partial z_{1,t-2}}{\partial \theta_1} \left( (G + 2z_{1,t-2} z_{2,t-2}) \frac{\partial z_{1,t-2}(\omega)}{\partial \omega'} + K \frac{\partial z_{1,t-1}(\omega)}{\partial \omega'} \right),
\]

with \( K = z_{1,t-1} z_{2,t-2} + z_{1,t-2} z_{2,t-1} \), and

\[
\frac{\partial z_{1,t-m}}{\partial \theta_1} = - \frac{1}{2} \frac{z_{1,t-m}}{h_{1,t-m}} \frac{\partial h_{1,t-m}}{\partial \theta_1},
\]

\[
\frac{\partial^2 z_{1,t-m}(\omega)}{\partial \theta_1 \partial \omega_k} = \begin{cases} (-z_{1,t-m}/(4h_{1,t-m}) \partial h_{1,t-m}/\partial \theta_1) & \text{if } k = t - m, \\ 0 & \text{otherwise} \end{cases},
\]

for \( m = 1, 2, t = 1, 2, \ldots, T \), and \( k = 1, 2, \ldots, T \) with \( 1 \leq t - m \leq T \).

\[
\frac{\partial^2 \rho_t(\omega)}{\partial \theta_2 \partial \omega'} = \pi_1 \frac{\partial \rho_{t-1}(\omega)}{\partial \theta_2 \partial \omega'} + \pi_2 D \sum_{m=1}^{2} \left( 1 - H z_{2,t-m}^2 \right) \frac{\partial z_{2,t-m}}{\partial \theta_2} \frac{\partial z_{1,t-m}(\omega)}{\partial \omega'} - \pi_2 D E^{-1} \sum_{m=1}^{2} \sum_{i=1}^{2} \left( z_{1,t-m} z_{1,t-i} - GH z_{2,t-m} z_{1,t-i} \right) \frac{\partial z_{2,t-m}}{\partial \theta_2} \frac{\partial z_{1,t-i}(\omega)}{\partial \omega'} - \pi_2 D H z_{2,t-1} z_{2,t-2} \left( \frac{\partial z_{2,t-1}}{\partial \theta_2} \frac{\partial z_{1,t-2}(\omega)}{\partial \omega'} + \frac{\partial z_{2,t-2}}{\partial \theta_2} \frac{\partial z_{1,t-1}(\omega)}{\partial \omega'} \right).
\]

The derivative \( \partial^2 \rho_t(\omega) / \partial \theta_3 \partial \omega' \) is partitioned as

\[
\frac{\partial^2 \rho_t(\omega)}{\partial \theta_1 \partial \omega'} = \frac{\partial \rho_{t-1}(\omega)}{\partial \omega'} + \pi_1 \frac{\partial^2 \rho_{t-1}(\omega)}{\partial \theta_1 \partial \omega'},
\]

\[
\frac{\partial^2 \rho_t(\omega)}{\partial \theta_2 \partial \omega'} = \frac{\partial \psi_{t-1}(\omega)}{\partial \omega'} + \pi_1 \frac{\partial^2 \rho_{t-1}(\omega)}{\partial \theta_2 \partial \omega'},
\]

\[
\frac{\partial^2 \rho_t(\omega)}{\partial \theta_3 \partial \omega'} = \pi_1 \frac{\partial^2 \rho_{t-1}(\omega)}{\partial \theta_3 \partial \omega'}.
\]

To obtain the derivative of \( \partial^2 L(\theta|\omega) / \partial \omega \partial \omega' \), we have

\[
\frac{\partial^2 h_{1,t}(\omega)}{\partial \omega_j \partial \omega_k} = \begin{cases} 2h_{1,t} & \text{if } j = k = t \\ 0 & \text{if } j \neq t \text{ or } k \neq t \end{cases}.
\]

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for \( t = 1, 2, \cdots, T, \ j = 1, 2, \cdots, T, \) and \( k = 1, 2, \cdots, T, \) and

\[
\frac{\partial^2 \rho_t(\omega)}{\partial \omega \partial \omega'} = \pi_1 \frac{\partial^2 \rho_{t-1}(\omega)}{\partial \omega \partial \omega'} + \pi_2 D \sum_{m=1}^{2} \left( z_{2,t-m} - GE^{-1}z_{1,t-m} \right) \frac{\partial^2 z_{1,t-m}(\omega)}{\partial \omega \partial \omega'} \\
- \pi_2 DE^{-1} \frac{\partial z_{1,t-1}(\omega)}{\partial \omega} \left( (G + 2z_{1,t-1}z_{2,t-1}) \frac{\partial z_{1,t-1}(\omega)}{\partial \omega'} + K \frac{\partial z_{1,t-2}(\omega)}{\partial \omega'} \right) \\
- \pi_2 DE^{-1} \frac{\partial z_{1,t-2}(\omega)}{\partial \omega} \left( (G + 2z_{1,t-2}z_{2,t-2}) \frac{\partial z_{1,t-2}(\omega)}{\partial \omega'} + K \frac{\partial z_{1,t-1}(\omega)}{\partial \omega'} \right) \\
+ 3\pi_2 DE^{-2}G \sum_{i=1}^{2} \sum_{m=1}^{2} z_{1,t-i}z_{1,t-m} \frac{\partial z_{1,t-i}(\omega)}{\partial \omega} \frac{\partial z_{1,t-m}(\omega)}{\partial \omega'}.
\]

References


Table 1: A summary of identified influential observations

<table>
<thead>
<tr>
<th>Perturbation scheme</th>
<th>Influential observation</th>
<th>S&amp;P500 returns</th>
<th>IBM returns</th>
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<tr>
<td>Serial No.</td>
<td>Date</td>
<td>Return</td>
<td>Previous return</td>
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<tr>
<td>Univariate diagnostics</td>
<td>Conditional variance of S&amp;P500 returns</td>
<td>540  27/02/2007</td>
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<td>940  29/09/2008</td>
<td>-9.22</td>
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<td>Conditional variance of IBM returns</td>
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<td>1060 23/03/2009</td>
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<td></td>
<td>1018 21/01/2009</td>
<td>4.26</td>
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<tr>
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<td></td>
<td>1018 21/01/2009</td>
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Table 2: QMLEs of parameters and $t$ statistics of Models A to E

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model A</th>
<th>Model B</th>
<th>Model C</th>
<th>Model D</th>
<th>Model E</th>
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<td>$\gamma_1$</td>
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<td>0.018</td>
<td>0.018</td>
<td>0.019</td>
<td>0.018</td>
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<td>(2.548)</td>
<td>(2.147)</td>
<td>(2.237)</td>
<td>(2.539)</td>
<td>(2.307)</td>
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<td>0.093</td>
<td>0.096</td>
<td>0.090</td>
<td>0.091</td>
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<td>(6.659)</td>
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<td>(7.146)</td>
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<td>$\beta_1$</td>
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<td>0.898</td>
<td>0.895</td>
<td>0.897</td>
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<td>(53.002)</td>
<td>(60.397)</td>
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<td>$\gamma_2$</td>
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<td>0.107</td>
<td>0.107</td>
<td>0.110</td>
<td>0.100</td>
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<td>(2.133)</td>
<td>(1.988)</td>
<td>(2.128)</td>
<td>(1.818)</td>
<td>(1.819)</td>
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<tr>
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<td>0.818</td>
<td>0.814</td>
<td>0.828</td>
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<tr>
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<td>0.715</td>
<td>0.684</td>
<td>0.760</td>
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<tr>
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<td>0.926</td>
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<td>(19.763)</td>
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<td>(29.368)</td>
<td>(17.536)</td>
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</tr>
<tr>
<td>$\pi_2$</td>
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<td>0.037</td>
<td>0.040</td>
<td>0.042</td>
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<td>(3.329)</td>
<td>(2.741)</td>
<td>(3.480)</td>
<td>(3.199)</td>
<td>(2.428)</td>
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</tr>
</tbody>
</table>

Weights

| $d_{1,961}$ | —       | 1.251   | 1.529   | —       | —       |
|             | (2.200) | (4.721) |         |         |         |
| $d_{1,1018}$| —       | 2.783   | 5.073   | —       | —       |
|             | (0.912) | (2.922) |         |         |         |
| $d_{2,940}$ | —       | 1.577   | —       | 0.662   | —       |
|             | (1.158) | (0.334) |         |         |         |
| $d_{2,961}$ | —       | 0.689   | —       | 0.607   | —       |
|             | (2.136) | (4.305) |         |         |         |
| $d_{\rho,942}$| —       | 0.567   | —       | —       | 0.519   |
|             | (1.959) |         |         | (1.962) |         |
| $d_{\rho,1018}$| —       | 0.598   | —       | —       | 0.837   |
|             | (1.544) |         |         | (2.796) |         |

Quasi log-likelihood

|                | -3545.07 | -3537.23 | -3540.07 | -3544.21 | -3539.47 |
|                | 5.8503   | 5.8473   | 5.8454   | 5.8522   | 5.8442   |

S&P500

|                | -0.498   | -0.487   | -0.457   | -0.499   | -0.501   |
|                | 1.769    | 1.831    | 1.925    | 1.779    | 1.803    |

IBM

|                | -0.230   | -0.254   | -0.235   | -0.235   | -0.248   |
|                | 3.784    | 3.846    | 3.844    | 3.838    | 3.930    |

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Figure 1: Time-series plots of (a) daily returns of S&P500 index; and (b) daily returns of IBM share price. In each graph, the x-axis is the date, and the y-axis is the percentage value.
Figure 2: Component plots of diagnostic vectors derived through (a) univariate GARCH(1,1) of S&P500 returns; and (b) univariate GARCH(1,1) of IBM returns. In each graph, the $x$-axis is the date, and the $y$-axis is the value of diagnostic components.
Figure 3: Component plots of bivariate diagnostic vectors derived through (a) perturbation to conditional correlation; (b) perturbation to conditional variance of S&P500 returns; and (c) perturbation to conditional variance of IBM returns. In each graph, the $x$-axis is the date, and the $y$-axis is the value of diagnostic components.
Figure 4: Time-varying conditional correlations and betas implied through Model A ignoring influential observations and Model E allowing for weights at influential observations.