ETF2700/ETF5970 Mathematics for Business

Lecture 3

Monash Business School, Monash University, Australia

Outline

Last week:

- Vectors
- Linear dependence and independence
- Orthogonal vectors and Orthonormal basis

This week:

- Matrix
- Matrix operation, and inverse matrix
- Eigenvalues and eigenvectors
- Linear programming

Matrix Algebra

General notation of a matrix

A rectangular array of numbers is called a *matrix*.

A $n \times m$ matrix:

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ a_{2,1} & \cdots & a_{2,m} \\ \vdots & \ldots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix}$$

where all a's are some real values.

Examples of matrices

2×2 matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

4×5 matrix:

$$\begin{bmatrix} 11 & 12 & 13 & 14 & 15 \\ 21 & 22 & 23 & 24 & 25 \\ 31 & 32 & 33 & 34 & 35 \\ 41 & 42 & 43 & 44 & 45 \end{bmatrix}$$

Special cases of a matrix: Vector and scalar

When $m = 1$, the general $n \times m$ matrix becomes a column of n elements:

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

which is a column vector.

When $n = 1$, the general $n \times m$ matrix becomes a row of m elements:

$$\begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}$$

which is a row vector.

When $n = m = 1$, the general $n \times m$ matrix becomes one element:

$$[a], \text{ or often just written as } a,$$

which is a scalar.

Note that a real number is simply a special form of a matrix.

Matrix addition: Example

2×2 matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1+0 & 2+2 \\ 3+3 & 4+5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$$

2×3 matrices:

$$\begin{bmatrix} 0 & 1 & 2 \\ 9 & 8 & 7 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0+6 & 1+5 & 2+4 \\ 9+3 & 8+4 & 7+5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 6 \\ 12 & 12 & 12 \end{bmatrix}$$

Addition: from number to matrix

Only happens for matrices of the same size!

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ a_{2,1} & \cdots & a_{2,m} \\ \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} {\color{red}+} \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ b_{2,1} & \cdots & b_{2,m} \\ \vdots & \cdots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1}{\color{red}+}b_{1,1} & \cdots & a_{1,m}{\color{red}+}b_{1,m} \\ a_{2,1}{\color{red}+}b_{2,1} & \cdots & a_{2,m}{\color{red}+}b_{2,m} \\ \vdots & \cdots & \vdots \\ a_{n,1}{\color{red}+}b_{n,1} & \cdots & a_{n,m}{\color{red}+}b_{n,m} \end{bmatrix}$$

Be careful

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = ?$$

$$2 + \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = ?$$

$$\begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} + 2 = ?$$

Matrix Subtraction: Examples

2×2 matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1-0 & 2-2 \\ 3-3 & 4-5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

2×3 matrices:

$$\begin{bmatrix} 0 & 1 & 2 \\ 9 & 8 & 7 \end{bmatrix} - \begin{bmatrix} 6 & 5 & 4 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 0-6 & 1-5 & 2-4 \\ 9-3 & 8-4 & 7-5 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -4 & -2 \\ 6 & 4 & 2 \end{bmatrix}$$

Subtraction: Same rule as addition

Only happens for matrices of the same size!

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ a_{2,1} & \cdots & a_{2,m} \\ \vdots & \cdots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix} - \begin{bmatrix} b_{1,1} & \cdots & b_{1,m} \\ b_{2,1} & \cdots & b_{2,m} \\ \vdots & \cdots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{bmatrix}$$

$$= \begin{bmatrix} a_{1,1} - b_{1,1} & \cdots & a_{1,m} - b_{1,m} \\ a_{2,1} - b_{2,1} & \cdots & a_{2,m} - b_{2,m} \\ \vdots & \cdots & \vdots \\ a_{n,1} - b_{n,1} & \cdots & a_{n,m} - b_{n,m} \end{bmatrix}$$

Be careful

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = ?$$

$$2 - \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = ?$$

$$\begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} - 2 = ?$$

Multiplication: Scalar × Matrix

Suppose c is a scalar and A is a $n \times m$ matrix

$$c\times A = cA = \begin{bmatrix} c\times a_{1,1} & \cdots & c\times a_{1,m} \\ c\times a_{2,1} & \cdots & c\times a_{2,m} \\ \vdots & \cdots & \vdots \\ c\times a_{n,1} & \cdots & c\times a_{n,m} \end{bmatrix}$$

Scalar × Matrix: An example

$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$$

$$2\times A = \begin{bmatrix} 2\times 4 & 2\times 0 & 2\times 5 \\ 2\times(-1) & 2\times 3 & 2\times 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 10 \\ -2 & 6 & 4 \end{bmatrix}$$

$$-1\times A = \begin{bmatrix} -1\times 4 & -1\times 0 & -1\times 5 \\ -1\times(-1) & -1\times 3 & -1\times 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & -5 \\ 1 & -3 & -2 \end{bmatrix}$$

Multiplication: Matrix $\times$ Vector

(3 $\times$ 2 matrix) $\times$ (2 $\times$ 1 vector) = (3 $\times$ 1 vector)

Rule: Number of columns of the 1st matrix = Number of rows of the 2nd matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\mathbf{A} \times \mathbf{b} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Calculate the first entry using the first *Row* of **A** to multiply the column vector ***b***:

$$\mathbf{A} \times \mathbf{b} = \begin{bmatrix} 2 \times 4 + 3 \times 1 \\ ? \\ ? \end{bmatrix}$$

Multiplication: Matrix × Vector

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Calculate the second entry using the second *Row* of **A**:

$$\mathbf{A} \times \mathbf{b} = \begin{bmatrix} 11 \\ 1 \times 4 + (-5) \times 1 \\ ? \end{bmatrix}$$

Calculate the 3rd entry using the 3rd *Row* of **A**:

$$\mathbf{A} \times \mathbf{b} = \begin{bmatrix} 11 \\ -1 \\ 1 \times 4 + 1 \times 1 \end{bmatrix}$$

Matrix × Vector: Solution

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\mathbf{A} \times \mathbf{b} = \begin{bmatrix} 2\times 4 + 3\times 1 \\ 1\times 4 + (-5)\times 1 \\ 1\times 4 + 1\times 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ 5 \end{bmatrix}$$

Multiplication: Matrix × Vector

$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix}_{n\times m}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

$$\mathbf{A} \times \mathbf{b} = \mathbf{Ab} = \begin{bmatrix} a_{1,1}b_1 + a_{1,2}b_2 + \cdots + a_{1,m}b_m \\ \vdots \\ a_{n,1}b_1 + a_{n,2}b_2 + \cdots + a_{n,m}b_m \end{bmatrix}_{n\times 1}$$

Multiplication: Matrix × Matrix

(3 × 2 matrix) × (2 × 2 matrix) = (3 × 2 matrix)

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{A} \times \mathbf{B} = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}$$

Matrix as a collection of vectors

We can write

$$\mathbf{B} = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$$

where

$$\mathbf{b}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

We define

$$\mathbf{A} \times \mathbf{B} = \begin{bmatrix} \mathbf{A} \times \mathbf{b}_1 & \mathbf{A} \times \mathbf{b}_2 \end{bmatrix}$$

which combines two side-by-side vectors.

Matrix Multiplication

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$$

$$\mathbf{A} \times \mathbf{b}_1 = \begin{bmatrix} 2\times 4 + 3\times 1 \\ 1\times 4 + (-5)\times 1 \\ 1\times 4 + 1\times 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ 5 \end{bmatrix}$$

$$\mathbf{A} \times \mathbf{b}_2 = \begin{bmatrix} 2\times 3 + 3\times (-2) \\ 1\times 3 + (-5)\times (-2) \\ 1\times 3 + 1\times (-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \\ 1 \end{bmatrix}$$

Matrix Multiplication: Solution

$$\mathbf{A} \times \mathbf{B} = \begin{bmatrix} \mathbf{A} \times \mathbf{b}_1 & \mathbf{A} \times \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 11 & 0 \\ -1 & 13 \\ 5 & 1 \end{bmatrix}$$

Matrix Multiplication: General

Let $\mathbf{A}$ be a $n \times m$ matrix and $\mathbf{B}$ be a $m \times k$ matrix:

$$\mathbf{A} \times \mathbf{B} \text{ is a } n \times k \text{ matrix}$$

such that

$$\mathbf{A} \times \mathbf{B} = \mathbf{AB} = \begin{bmatrix} \mathbf{A} \times \mathbf{b}_1 & \cdots & \mathbf{A} \times \mathbf{b}_k \end{bmatrix}$$

where

$$\begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{bmatrix} = \mathbf{B}$$

Be careful

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \times \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = ?$$

$$2 \times \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} = ?$$

$$\begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} \times 2 = ?$$

Example of "work-out" pants from the 1st week (P56)

- Market price of the work-out brand pants: P
- Sales volume of the produced pants: Q
- Market supply: $Q_s = P - 10$
- Market demand: $Q_d = -2P + 200$
- Market clearing: $Q_s = Q_d = 3Q$

Linear equations

$$\begin{aligned} 2P + 3Q &= 200 \\ P - 3Q &= 10 \end{aligned} \qquad \begin{bmatrix} 2 & 3 & 200 \\ 1 & -3 & 10 \end{bmatrix}$$

The system of linear equations in P and Q is written as

$$\begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 200 \\ 10 \end{bmatrix}$$

System of Linear Equation: General expression

A system of linear equations in $\boldsymbol{x} = (x_1, \ldots, x_n)$ is

$$\mathbf{A}\boldsymbol{x} = \mathbf{b}$$

where $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{b}$ is a $m \times 1$ vector.
In the above example ($m = n = 2$):

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} P \\ Q \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 200 \\ 10 \end{bmatrix}$$

How to solve the linear system $\mathbf{A}\boldsymbol{x} = \mathbf{b}$?

- Elimination method discussed in the 1st week
- Can we just “divide” both sides of $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ by $\mathbf{A}$?
- Yes, but it is not always possible.
- When is it possible? How can we conduct the “division”?

How to divide both sides of $\mathbf{A}x = \mathbf{b}$ by $\mathbf{A}$

Square matrix

The first requirement is that A is a *square matrix*, that is, $m = n$. In other words, we require that

$$\text{Number of equations} = \text{Number of variables!}$$

If $m \neq n$, we need to use elimination to solve the system.

Inverse Matrix

- The 2nd requirement is to derive the "Inverse Matrix" of **A**. Let **A** be a $n \times n$ square matrix.
- Dividing both sides of $\mathbf{A}x = \mathbf{b}$ by **A** is equivalent to multiplying both sides by the inverse of **A**, known as $\mathbf{A}^{-1}$

Inverse of Scalar $(n = 1)$

A=[a], or **A**=a with the real number $a \neq 0$, then

$$\mathbf{A}^{-1} = \frac{1}{a}$$

Inverse of a matrix

The inverse of a square matrix $\mathbf{A}$, sometimes called a reciprocal matrix, is a matrix $\mathbf{A}^{-1}$ such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = I_n$$

where I_n is the $n \times n$ *identity matrix*

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

- A *square* matrix $\mathbf{A}$ is invertible if $\mathbf{A}^{-1}$ exists.
- $\mathbf{A}^{-1}$ does not always exist. For example, $\mathbf{A}$ has only zero entries.
- If the determinant of a given square matrix is non-zero, then this matrix is invertible.

Determinant of a matrix

- The determinant is a scalar value that is a function of the entries of a square matrix.
- The determinant of a 2×2 square matrix is as follows.
- Suppose that

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(\mathbf{A}) = ad - bc$$

- We can also write the determinant as

$$\det(\boldsymbol{A}) = |\boldsymbol{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Determinant of a matrix

The inverse of **A** is

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example of the 'work-out' brand pants: $\mathbf{A}\boldsymbol{x} = \mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix}$$

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} = 2 \times (-3) - 3 \times 1 = -9$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} -3 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{-9} \begin{bmatrix} -3 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 1/9 & -2/9 \end{bmatrix}$$

Solve the system: $\mathbf{A}x = \mathbf{b}$

Consider a system of linear equations in x and y:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

Multiply both sides by $\mathbf{A}^{-1}$ *on the left*:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \times \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \times \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Example of Pant-making Firm: $\mathbf{A}\boldsymbol{x} = \mathbf{b}$

$$\begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 200 \\ 10 \end{bmatrix}$$

Multiply both sides by $\mathbf{A}^{-1}$ *on the left*:

$$\begin{aligned} \begin{bmatrix} P \\ Q \end{bmatrix} &= \begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix}^{-1} \times \begin{bmatrix} 200 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{9} & -\frac{2}{9} \end{bmatrix} \times \begin{bmatrix} 200 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} \times 200 + \frac{1}{3} \times 10 \\ \frac{1}{9} \times 200 - \frac{2}{9} \times 10 \end{bmatrix} = \begin{bmatrix} 70 \\ 20 \end{bmatrix} \end{aligned}$$

An Explicit Formula: Cramer's Rule

Recall the solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \times \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Work it out:

$$x = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Alternative: Cramer's Rule

$$\begin{bmatrix} 2 & 3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 200 \\ 10 \end{bmatrix}$$

$$P = \frac{\begin{vmatrix} 200 & 3 \\ 10 & -3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix}} = \frac{200 \times (-3) - 3 \times 10}{2 \times (-3) - 3 \times 1} = 70$$

$$Q = \frac{\begin{vmatrix} 2 & 200 \\ 1 & 10 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix}} = \frac{2 \times 10 - 200 \times 1}{2 \times (-3) - 3 \times 1} = 20$$

Summary: Matrix Operations

Summation and Subtraction: all matrices have same size

Multiplication

- Two rules: Scalar $\times$ Matrix, Matrix $\times$ Matrix
- Matrix $\times$ Matrix: match the size!

Inverse

- Only defined for square matrix
- $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$
- Formula for the inverse of 2×2 matrix

Summary: Solve linear equations

Consider a system of linear equations in x and y: $\mathbf{A}\boldsymbol{x} = \mathbf{b}$

1) Solve by elimination method (first week lecture)
2) Solve by using inverse matrix
3) Solve by Cramer's rule (2×2 matrix)

Eigenvalues and Eigenvectors

- Definition: An eigenvector of a square matrix $\boldsymbol{A}$ is a non-zero vector x such that when $\boldsymbol{A}$ is multiplied by x, the result is a scalar multiple of x, that is,

$$\boldsymbol{A}x = \lambda x,$$

  where λ is a scalar called the eigenvalue corresponding to the eigenvector x.
- To solve eigenvalues, we can rewrite $\boldsymbol{A}x = \lambda x$ as

$$\boldsymbol{A}x = \lambda \boldsymbol{I}x, \quad \text{and then} \quad (\boldsymbol{A} - \lambda \boldsymbol{I})x = 0$$

  where I is an identity matrix with same dimension as $\boldsymbol{A}$
- If x is non-zero, this equation will only have a solution if

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

  which says the determinant of $\boldsymbol{A} - \lambda \boldsymbol{I}$ is zero.
- This equation is called the characteristic equation of $\boldsymbol{A}$

An Example of solving eigenvalues of a 2×2 matrix

Let $\boldsymbol{A} = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$. To solve the eigenvalues of $\boldsymbol{A}$, we have

$$\begin{aligned} |\boldsymbol{A} - \lambda I| &= \left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \left| \begin{bmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix} \right| \\ &= (-6-\lambda)(5-\lambda) - 3 \times 4 \\ &= \lambda^2 + \lambda - 42 = (\lambda+7)(\lambda-6) = 0 \end{aligned}$$

Therefore, the eigenvalues are $\lambda_1 = -7$ and $\lambda_2 = 6$.

Examples of solving eigenvalues of a 2×2 matrix

Plugging-in $\lambda = -7$ into $(\boldsymbol{A} - \lambda \boldsymbol{I})x = 0$, we have

$$\begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix} x = 0$$

This leads to $x_1 + 3x_2 = 0$ and $4x_1 + 12x_2 = 0$
Either equation will lead to $x_1 = -3x_2$. So, the eigenvector is any non-zero multiple of $(-3, 1)^\top$

Eigenvector corresponding to $\lambda = 6$

Plugging-in $\lambda = 6$ into $(\boldsymbol{A} - \lambda \boldsymbol{I})x = 0$, we have

$$\begin{bmatrix} -12 & 3 \\ 4 & -1 \end{bmatrix} x = 0$$

This leads to $-12x_1 + 3x_2 = 0$ and $4x_1 - x_2 = 0$
Either equation will lead to $x_2 = 4x_1$. So, the eigenvector is any non-zero multiple of $(1, 4)^\top$

Normallise Eigenvectors

- Corresponding to an eigenvalue, the eigenvector is obtained subject to a non-zero multiplicative scalar. As such, there are an infinite number of eigenvectors.
- One such vector that is particularly nice is that whose sum of squared elements equals to 1
- Normalisation of a vector is to divide it by its norm.
- Normalised eigenvectors in the above problem is: $(-3/\sqrt{10}, 1/\sqrt{10})^\top$ and $(1/\sqrt{17}, 4/\sqrt{17})^\top$

Rank

- The maximum number of linearly independent rows in a matrix $\boldsymbol{A}$ is called the row rank of $\boldsymbol{A}$.
- The maximum number of linearly independent columns in $\boldsymbol{A}$ is called the column rank of $\boldsymbol{A}$.
- If $\boldsymbol{A}$ is an $m \times n$ matrix, then it is obvious that "row rank of $\boldsymbol{A}$ is less than m, and column rank of $\boldsymbol{A}$ is less than n".
- What is not so obvious, however, is that for any matrix $\boldsymbol{A}$, "the row rank of $\boldsymbol{A}$ = the column rank of $\boldsymbol{A}$".

How to compute the rank: Gaussian Elimination

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{2R_1+R_2 \to R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{-3R_1+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-2R_2+R_1 \to R_1} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank

- The final matrix (in row echelon form) has two non-zero rows and thus the rank of matrix $\boldsymbol{A}$ is 2.
- Solve the ranks: $\boldsymbol{B} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 5 \\ 4 & 2 & 6 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -1 & -2 & -4 \end{bmatrix}$

Properties of rank

- The rank of an $m \times n$ matrix $\boldsymbol{A}$ is a non-negative integer and cannot be greater than either m or n. that means $\text{Rank}(\boldsymbol{A}) \leq \min(m, n)$
- Only a zero matrix has rank zero.
- If $\boldsymbol{A}$ is a square matrix (that is, $m = n$), then $\boldsymbol{A}$ is invertible if and only if $\boldsymbol{A}$ has rank n (that is, $\boldsymbol{A}$ has a full rank).
- If B is any $n \times k$ matrix, then

$$\text{Rank}(\boldsymbol{AB}) \leq \min(\text{Rank}(\boldsymbol{A}), \text{Rank}(\boldsymbol{B}))$$

- If B is an $n \times k$ matrix of rank n, then

$$\text{Rank}(\boldsymbol{AB}) = \text{Rank}(\boldsymbol{A})$$

- Rank of $\boldsymbol{A}$ is r if and only if there exist an invertible $m \times m$ matrix X and an invertible $n \times n$ matrix Y such that

$$X\boldsymbol{A}Y = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Properties of rank

- If $\boldsymbol{A}$ is a matrix over the real numbers, then we have

$$\mathrm{Rank}(\boldsymbol{A}) = \mathrm{Rank}(\boldsymbol{A}^\top \boldsymbol{A}) = \mathrm{Rank}(\boldsymbol{A}\boldsymbol{A}^\top) = \mathrm{Rank}(\boldsymbol{A}^\top)$$

Applications of the rank of matrices: Linear Equations

Suppose a system of linear equations is: $\boldsymbol{A}x = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix}$$

The augmented matrix is given by

$$(\boldsymbol{A}|\boldsymbol{b}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

Applications of the rank of matrices: Linear Equations

- The linear system is inconsistent if the rank of the augmented matrix is greater than the rank of the coefficient matrix.
- If the ranks of these two matrices are equal, then the system must have at least one solution.
- The solution is unique if and only if the rank equals the number of variables, which means

$$\mathrm{Rank}(\boldsymbol{A}) = \mathrm{Rank}(\boldsymbol{A}|\boldsymbol{b}) = n.$$

Linear Programming: An example

Example 2 in Chapter 17, Essential mathematics for economic analysis (2012): link here.

Consider a firm that

- produces two goods, A and B;
- has **two** factories that jointly produce the two goods; and
- receives an order: 300 units of A and 500 units of B
- What is the minimal cost to meet this order?

Fixed cost

- The costs of operating the two factories are 10 thousand dollars and 8 thousand dollars per hour.

Total cost (in thousand dollars)

$$C = 10u_1 + 8u_2$$

where u_1 and u_2 are the number of operating hours for the two factories, and physical constraints are $u_1 \geq 0$ and $u_2 \geq 0$.

Production constraints

The two factories jointly produce the two goods in the following quantities (per hour):

	Factory 1	Factory 2
Good A	10	20
Good B	25	25

- Total production of A: $10u_1 + 20u_2$
- Total production of B: $25u_1 + 25u_2$

Order constraints:

$$10u_1 + 20u_2 \geq 300, \quad 25u_1 + 25u_2 \geq 500$$

Formulating the problem

Minimize $10u_1 + 8u_2$

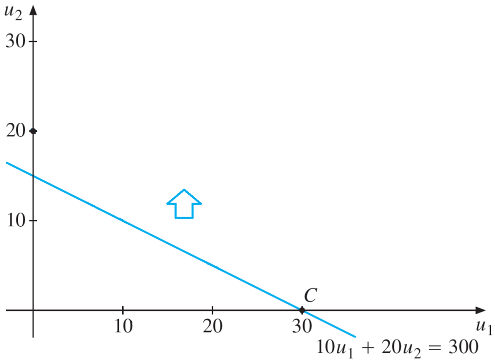
subject to $\begin{cases} 10u_1 + 20u_2 \geq 300 \\ 25u_1 + 25u_2 \geq 500 \end{cases}$, and $u_1, u_2 \geq 0$

Feasible region

First, we need to determine the set of (u_1, u_2) that satisfies all the constraints from the picture:

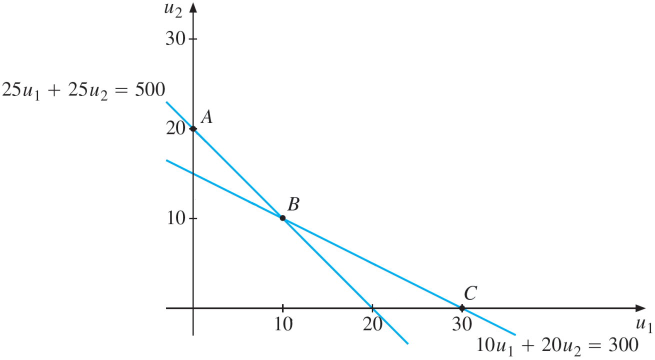
$$10u_1 + 20u_2 \geq 300$$
$$25u_1 + 25u_2 \geq 500$$
$$u_1, u_2 \geq 0$$

Graphical approach: First Constraint



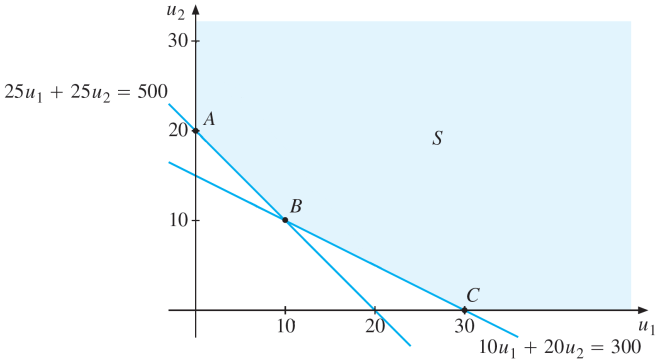
Adding another constraint

- Add the straight line $25u_1 + 25u_2 = 500$.
- $25u_1 + 25u_2 \geq 500$: the upper part of the line



Feasible region

- $u_1, u_2 \geq 0$: only look at the first quadrant
- Feasible Set S has three 'corner' points: A, B, C.



Extreme point theorem

If an optimal solution exists, it must be at one of the *corner points* of the feasible region. It is obvious that our problem has an optimal solution. We need to verify the cost at the corner points:

Determine the corner point

To determine the corner point B, we solve

$$\begin{aligned} 10u_1 + 20u_2 &= 300 \\ 25u_1 + 25u_2 &= 500 \end{aligned}$$

that is

$$\begin{bmatrix} 10 & 20 \\ 25 & 25 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 300 \\ 500 \end{bmatrix}$$

Solve by Cramer's Rule

$$\begin{bmatrix} 10 & 20 \\ 25 & 25 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 300 \\ 500 \end{bmatrix}$$

$$u_1 = \frac{\begin{vmatrix} 300 & 20 \\ 500 & 25 \end{vmatrix}}{\begin{vmatrix} 10 & 20 \\ 25 & 25 \end{vmatrix}} = \frac{300 \times 25 - 20 \times 500}{10 \times 25 - 20 \times 25} = 10$$

$$u_2 = \frac{\begin{vmatrix} 10 & 300 \\ 25 & 500 \end{vmatrix}}{\begin{vmatrix} 10 & 20 \\ 25 & 25 \end{vmatrix}} = \frac{10 \times 500 - 300 \times 25}{10 \times 25 - 20 \times 25} = 10$$

Determine the optimal solution

So we have all corner points now:

- A: $u_1 = 0$, $u_2 = 20$,

  giving total cost $10u_1 + 8u_2 = 160\$k$
- B: $u_1 = 10$, $u_2 = 10$

  giving total cost $10u_1 + 8u_2 = 180\$k$
- C: $u_1 = 30$, $u_2 = 0$

  giving total cost $10u_1 + 8u_2 = 300\$k$

Solution

The optimal solution is obtained at the corner point A, corresponding to $u_1 = 0$ and $u_2 = 20$. Hence,

The optimal solution is to operate factory 2 for 20 hours and not to use factory 1 at all, with minimum cost 160 thousand dollars.