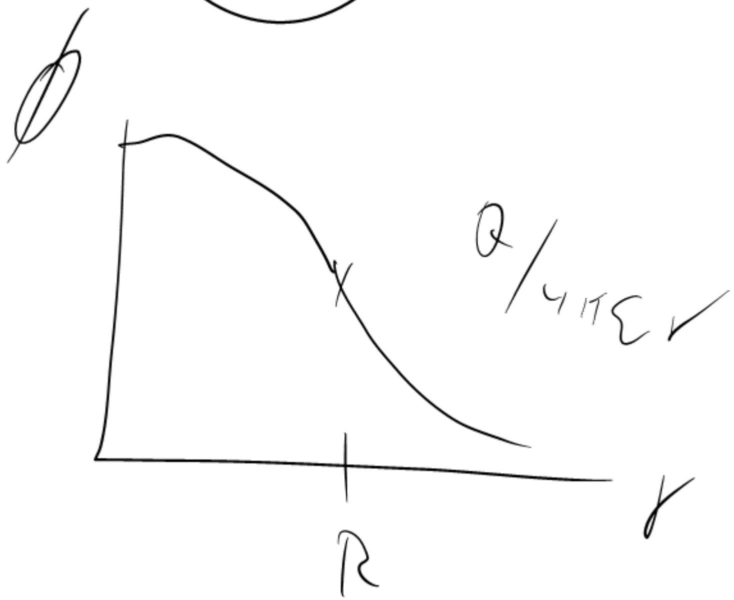
$$\rho = \frac{Q}{\frac{4\pi}{3} R^3}$$

int. const.

$$\phi(r) = \begin{cases} B - \dfrac{\rho}{6\varepsilon} r^2 & r \leq R \\[2ex] \dfrac{Q}{4\pi\varepsilon r} & r > R \end{cases}$$



$B = ?$

$$\frac{\rho r^2}{6\varepsilon} = \frac{Q r^2}{6 \cdot \frac{4\pi}{3} \varepsilon R^3} = \frac{Q r^2}{8\pi\varepsilon R^3} = \frac{Q}{8\pi\varepsilon R} \frac{r^2}{R^2}$$

$$\phi(R) = B - \frac{Q}{8\pi\varepsilon R} \overset{!}{=} \frac{Q}{4\pi\varepsilon R} \Rightarrow B = \frac{3Q}{8\pi\varepsilon R}$$

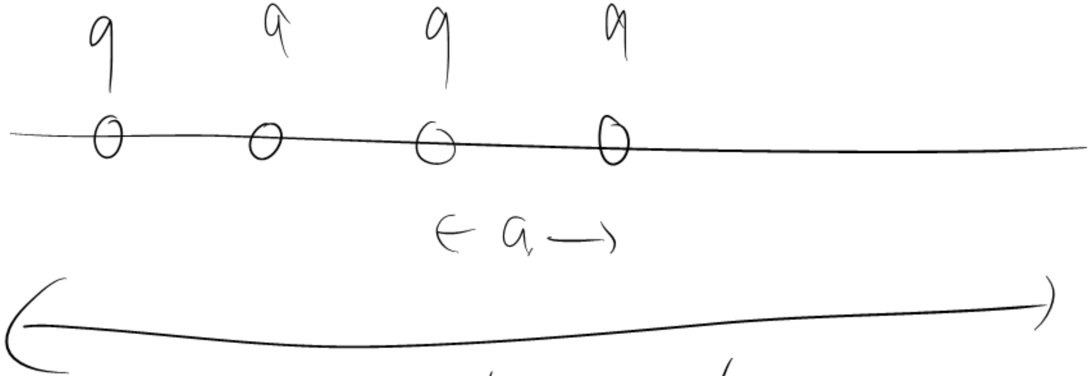
$$\boxed{\phi(r) = \frac{Q}{8\pi\varepsilon R}\left(3 - \frac{r^2}{R^2}\right) \qquad r \leq R}$$

$$U = \frac{1}{2}\int d^3\vec{r}\,\rho\,\phi = \frac{\rho}{2}\,4\pi\int_0^R r^2 dr\,\frac{Q}{8\pi\varepsilon R}\left(3-\frac{r^2}{R^2}\right)$$

$$= \frac{1}{2}\,\frac{Q}{\frac{4\pi}{3}R^3}\,4\pi\,\frac{Q}{8\pi\varepsilon R}\left(3\,\frac{R^3}{3}-\frac{1}{R^2}\,\frac{R^5}{5}\right) =$$

$$= \frac{3}{2}\,\frac{Q^2}{8\pi\varepsilon R}\underbrace{\left(1-\frac{1}{5}\right)}_{4/5} = \boxed{\frac{3}{5}\,\frac{Q^2}{4\pi\varepsilon R}}$$

long charged rod: shish-kebap model



R total extension

\# charges: $N+1$

\# "bonds": $N = \frac{R}{a}$

$$U = \frac{q^2}{4\pi\varepsilon}\left\{ \frac{1}{a}N + \frac{1}{2a}(N-1) + \frac{1}{3a}(N-2) + \dots + \frac{1}{Na}\cdot 1 \right\} = \frac{q^2}{4\pi\varepsilon a}\sum_{k=1}^{N}\frac{1}{k}(N-k+1)$$

$$= \frac{q^2}{4\pi\varepsilon a} \left\{ (N+1) \underbrace{\sum_{k=1}^{N} \frac{1}{k}}_{\hookrightarrow H_N} - N \right\}$$

H_N (Nth harmonic number)

Wolfram Alpha

large N

$$\hookrightarrow \sim \ln N + \gamma + \frac{1}{2N} - \frac{1}{12}\frac{1}{N^2} \dots$$

Euler-Mascheroni constant

$\gamma = 0.577$

$$U = \frac{q^2}{4\pi\varepsilon a} \left\{ \underbrace{(N+1)\ln N}_{N \ln N} + \underbrace{(N+1)\gamma}_{N\gamma} + \frac{1}{2} + O\left(\frac{1}{N}\right) - N \right\}$$

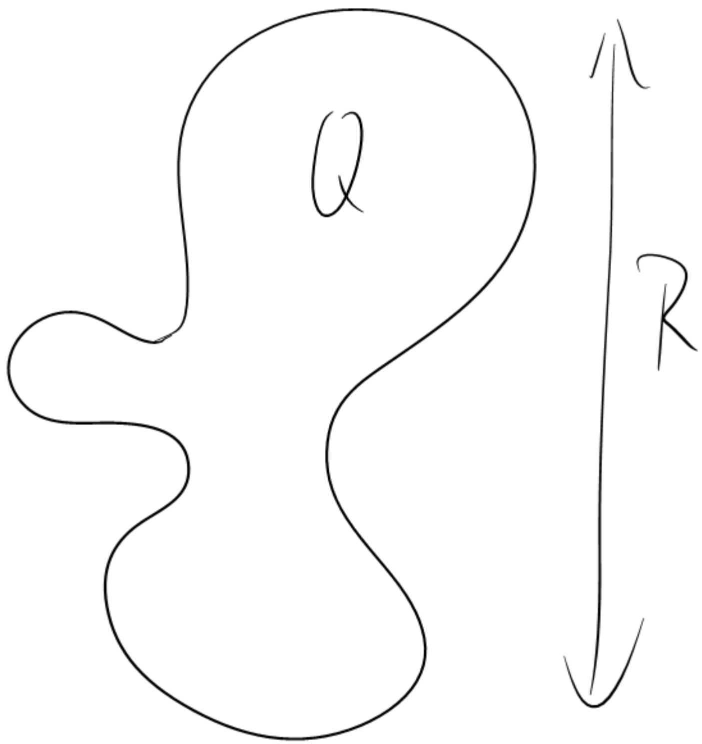
$+ \; O(1)$

$$U = \frac{q^2 N}{4\pi \varepsilon a}\left[\ln N + \gamma - 1\right] \qquad q \equiv \frac{Q}{N}$$

$$U = \frac{Q^2}{4\pi\varepsilon a N}\left(\ln N + \gamma - 1\right) = \frac{Q^2}{4\pi\varepsilon R}\left[\ln \frac{R}{a} + \gamma - 1\right]$$

Some arbitrary (non-fractal) object ($d_f = 3$)

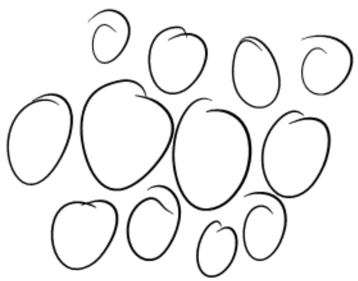
fractal dimension d_f: $\underbrace{M}_{\text{mass}} \propto \underbrace{R}_{\text{size}}^{d_f}$



$$U = \frac{Q^2}{\varepsilon R} f(\text{shape})$$

Shape of a charged droplet?

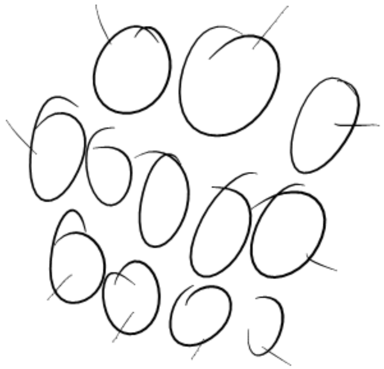
Interactions: – short-range cohesive interactions



dominate

take into account by <u>constraint</u> $V = \text{const.}$

- lack of neighbors at the surface



↳ surface tension σ

↳ energy $\sigma \cdot A$ → surface

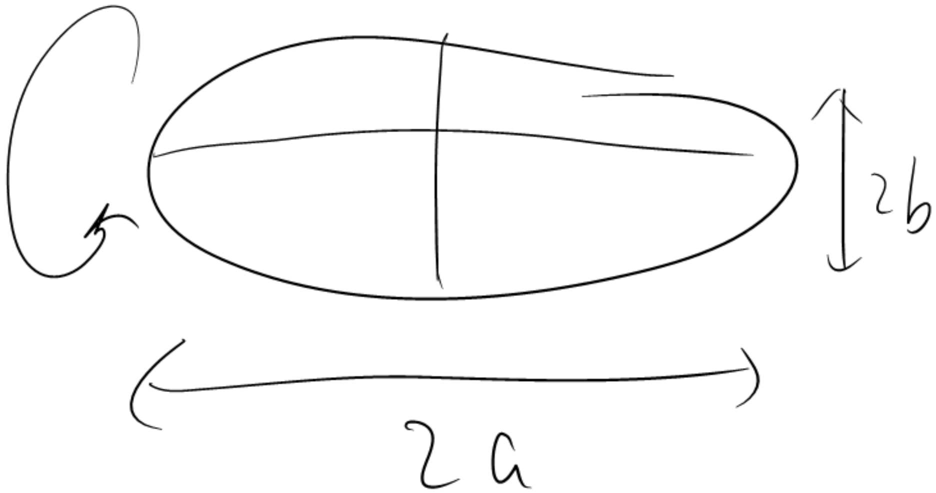
- electrostatics

↳ <u>optimization</u>: balance electrostatics against surface energy!

$$U = \frac{Q^2}{\varepsilon V^{1/3}} \underbrace{f_1(\text{shape}) + \sigma V^{2/3} f_2(\text{shape})}_{\text{in general difficult}}$$

study a "cigar" i.e. an ellipsoid

- large half-axis a
- small " b



aspect ratio $x = \frac{a}{b} \geq 1$

$$V = \frac{4\pi}{3} a b^2 \qquad A = 4\pi a b \qquad x = \frac{a}{b}$$

independent parameters: V, x

$$a = \left(\frac{3x^2}{4\pi}\right)^{1/3} V^{1/3} \qquad b = \left(\frac{3}{4\pi x}\right)^{1/3} V^{1/3}$$

$$A = (36\pi x)^{1/3} V^{2/3}$$

electrostatic energy of the "cigar"

(encyclopedia of physics)

inverse function of cosh

$$U = \frac{3}{5}\,\frac{Q^2}{4\pi\varepsilon a}\,\underbrace{\frac{\cosh^{-1}x}{\sqrt{1-x^{-2}}}}_{f(x)}$$

Wolfram Alpha

$$f(x) \simeq 1 + \frac{2}{3}(x-1) - \frac{1}{5}(x-1)^2 + \ldots \qquad \underline{x \text{ near } 1}$$

$$f(x) \simeq \ln x + \ln 2 + O\left(\frac{1}{x^2}\right) \qquad \text{large } x$$

in total:

$$U = \frac{3Q^2}{20\pi\varepsilon}\left(\frac{4\pi}{3Vx^2}\right)^{1/3}\frac{\cosh^{-1}x}{\sqrt{1-x^{-2}}} + \sigma\left(36\pi V^2 x\right)^{1/3}$$

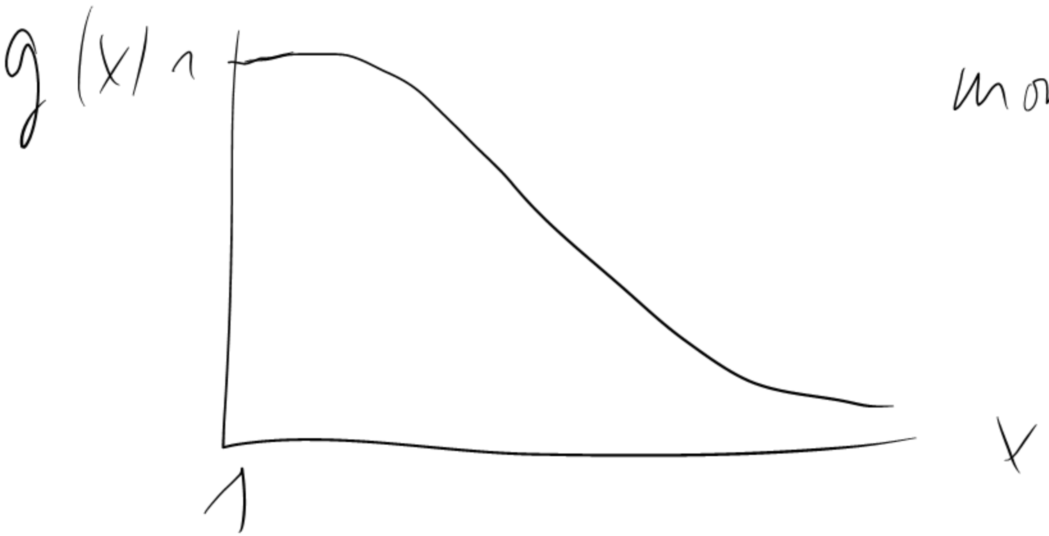
$$u := \frac{U}{\sigma\left(36\pi V^2\right)^{1/3}}$$ dimensionless energy

$$q := \frac{1}{\sigma\left(36\pi V^2\right)^{1/3}}\frac{3Q^2}{20\pi\varepsilon}\left(\frac{4\pi}{3V}\right)^{1/3}$$ dimensionless charging parameter

$$\boxed{u = 9\,\frac{1}{x^{2/3}}\,\frac{\cosh^{-1} x}{\sqrt{1 - x^{-2}}} + x^{1/3}}$$

$$g(x) = \frac{1}{x^{2/3}}\, f(x) \underset{\uparrow}{=} 1 - \frac{4}{45}(x-1)^2 + \frac{244}{2835}(x-1)^3 - + \ldots$$

Wolfram Alpha



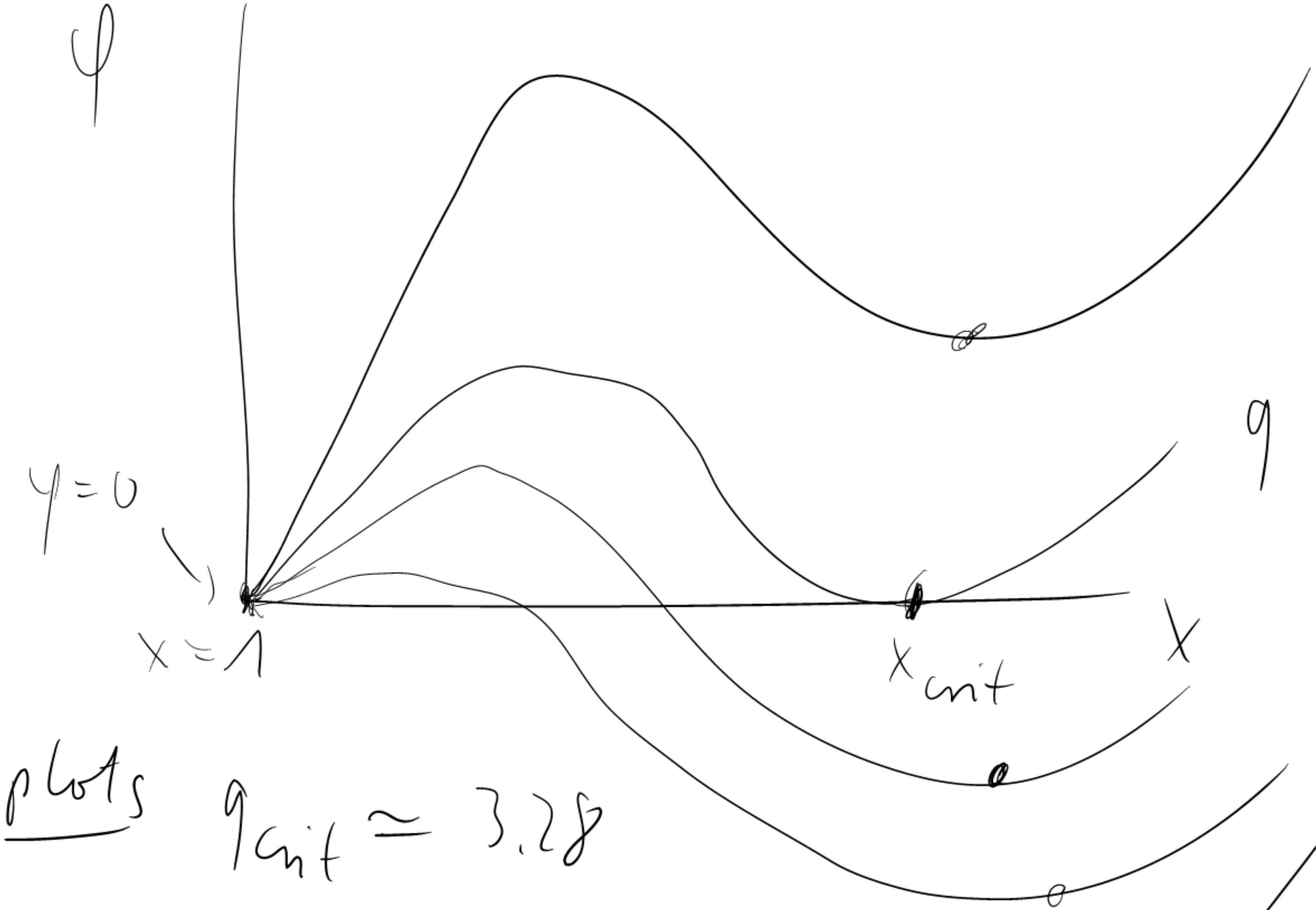
monotonous decrease

$$\frac{1}{x^{2/3}} = \frac{1}{[1 + (x-1)]^{2/3}}$$

monotonous increase

study $\varphi(q, x) = q\left[g(x) - 1\right] + \left(x^{1/3} - 1\right)$

(energy relative to the sphere)



Small q
$q < q_{crit}$
sphere is stable

$q = q_{crit}$
(phase transition from sphere to cigar)

large q
$q > q_{crit}$
cigar is stable

1st order transition

plots

$q_{crit} \simeq 3.28$

$x_{crit} \simeq 10$

Consider two droplets of the same size,

infinitely far away from each other

$$V = \frac{V}{2} + \frac{V}{2} \qquad Q = \frac{Q}{2} + \frac{Q}{2}$$

energy of the split state.

$$U = 2\left\{\frac{3}{20\pi\varepsilon}\left(\frac{Q}{2}\right)^2\left[\frac{4\pi}{3\frac{V}{2}}\right]^{1/3} + \sigma\left[36\pi\left(\frac{V}{2}\right)^2\right]^{1/3}\right\}$$

$$= \frac{3Q^2}{20\pi\varepsilon}\left(\frac{4\pi}{3V}\right)^{1/3} 2^{-2/3} + \sigma\left[36\pi V^2\right]^{1/3} 2^{+1/3}$$

$$u = q\, 2^{-2/3} + 2^{1/3}$$

normalized energy of the split state

same definition as before!

Split state is stable (more stable than the single sphere) if $q\, 2^{-2/3} + 2^{1/3} < q + 1$

$$2^{1/3} - 1 < q\left(1 - 2^{-2/3}\right)$$

$$q > \frac{2^{1/3} - 1}{1 - 2^{-2/3}} = 0.7024 = q_{crit}$$

"Rayleigh instability"

Split-up pre-empts the cigar formation!