Puzzle corner

Norman Do[*]

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to ndo@math.mcgill.ca or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 14 is 1 November 2009. The solutions to Puzzle Corner 14 will appear in the March 2010 issue of the *Gazette*.

Positive subsets

Let X be a set consisting of one hundred integers whose sum is 1. What is the maximum possible number of subsets of X which have positive sum?

Continued calculation

What is the sum of the following two expressions?

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{\cdots+\cfrac{1}{2009}}}}} \qquad \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{\cdots+\cfrac{1}{2009}}}}}}.$$

[*]Department of Mathematics and Statistics, McGill University, Montréal H3A 2K6, Québec, Canada. E-mail: ndo@math.mcgill.ca

Matching shoes

There are 15 left shoes and 15 right shoes jumbled up and placed in a row. Show that there must be 10 consecutive shoes consisting of 5 left shoes and 5 right shoes (in no particular order).

The itinerant queen

A subset of the squares of an 8×8 chessboard is called accessible if a queen can visit each of the squares, possibly more than once, without landing on any others. Someone decides to paint each square of a chessboard red or blue in a random manner. Prove that the set of red squares is accessible or the set of blue squares is accessible. (Recall that a queen moves along any horizontal, vertical or diagonal file of the chessboard without landing on squares that she passes over.)

Snail trail

(1) A snail crawls along flat ground with constant speed, turning through a right angle every 15 minutes. Prove that the snail can only return to its starting point after a whole number of hours.

(2) A snail crawls along a straight line for 10 hours, while several people are watching. It is known that each of these people watched the snail for exactly one hour, during which the snail crawled exactly one metre. It is also known that the snail was watched by at least one person at all times. What is the maximum distance that the snail could crawl during these 10 hours?

Solutions to Puzzle Corner 12

The $50 book voucher for the best submission to Puzzle Corner 12 is awarded to Michael Yastreboff.

Soccer stats

Solution by: David Angell

In order to obtain a contradiction, suppose that the player had never scored in precisely 80% of the matches. Then the player's percentage must have increased from less than 80% to more than 80% in a single game. Suppose that, prior to this game, the player had scored in a games out of a total of b games. Then we have

$$\frac{a}{b} < \frac{4}{5} \quad \text{and} \quad \frac{a+1}{b+1} > \frac{4}{5}.$$

But this implies $5a < 4b < 5a + 1$, which is impossible since a and b are integers. In fact, the same solution works if the figure of 80% is replaced by $\frac{n}{n+1}$, where n is a positive integer.

Loopy potatoes

Solution by: Alan Jones

If you imagine moving the two surfaces of the potatoes until they intersect, then the curve of intersection will form a loop on each potato. These loops will be congruent when considered as subsets of space.

Island tour

Solution by: Kevin McAvaney

First, we will show that the states can be properly coloured black and white — by this, we mean that any two states which share a border have different colours. This is clearly true if there is only one chord of the circle and we can now proceed by induction. So suppose that it is true whenever there are k chords and consider an island with $k + 1$ chords. Without loss of generality, let's assume that one of the chords is aligned north-south. Now remove that chord and use the induction hypothesis to properly colour the remaining configuration. Putting back the removed chord will divide each state that it intersects into two states. Change the colour of every state that is west of the chord. The result is a proper colouring so, by induction, the states can be properly coloured black and white.

Now for any tour, list the colours of the states in the order in which they are visited. Since no two borders are crossed simultaneously, the colours alternate between black and white and each change in colour represents a border crossing. The tour starts and ends in the same state so there is an even number of colour changes and therefore an even number of border crossings.

Table trouble

Solution by: James East

Suppose that rotating the table by k places clockwise produces a configuration such that a chair has a number which matches the number on the corresponding plate. Then that chair must be k places clockwise from the plate with the same number. So if the task is impossible, then there must be a chair that is k places clockwise from its corresponding plate for every value of k from 0 to 999.

Let d_k be the number of places that the chair numbered k is clockwise from the plate numbered k. Assuming that the task is impossible, we know that

$$d_1 + d_2 + d_3 + \cdots + d_{1000} \equiv 0 + 1 + 2 + \cdots + 999 \equiv 500 \pmod{1000}.$$

Number the positions of the table from 0 to 999 in a clockwise manner and let c_k be the position of the chair numbered k and let p_k be the position of the plate numbered k. Clearly, we have $c_k - p_k \equiv d_k \pmod{1000}$. Therefore,

$$d_1 + d_2 + d_3 + \cdots + d_{1000}$$
$$\equiv (c_1 - p_1) + (c_2 - p_2) + (c_3 - p_3) + \cdots + (c_{1000} - p_{1000})$$
$$\equiv (c_1 + c_2 + c_3 + \cdots + c_{1000}) - (p_1 + p_2 + p_3 + \cdots + p_{1000})$$
$$\equiv 0 \pmod{1000},$$

since $(c_1, c_2, c_3, \ldots, c_{1000})$ and $(p_1, p_2, p_3, \ldots, p_{1000})$ are both permutations of $(1, 2, 3, \ldots, 1000)$. This gives us the desired contradiction, so we can deduce that it is always possible to rotate the table so that no chair has a number which matches the number on the corresponding plate.

Height differences

Solution by: Michael Yastreboff

Let the boys' heights, in non-decreasing order, be $B_1, B_2, \ldots, B_{20}$ and the girls' heights, in non-decreasing order, be $G_1, G_2, \ldots, G_{20}$. We will show that, no matter how the boys and girls are paired, we can keep swapping partners while maintaining the height condition — that is, the difference in height between the boy and girl in each couple is no more than 10 centimetres — finally arriving at the configuration where the boy with height B_1 is paired with the girl with height G_1, the boy with height B_2 is paired with the girl with height G_2, and so on.

Suppose that the shortest boy is originally paired with the girl with height G_m while the shortest girl is originally paired with the boy with height B_n. If $m = n = 1$, then the shortest boy and the shortest girl are paired with each other, as desired. Otherwise, we have the string of inequalities

$$B_1 - 10 \leq B_n - 10 \leq G_1 \leq G_m \leq B_1 + 10 \leq B_n + 10,$$

which imply that $|B_1 - G_1| \leq 10$ and $|B_n - G_m| \leq 10$. Therefore, we may swap the pairs so that the boy with height B_1 is paired with the girl with height G_1 and the boy with height B_m is paired with the girl with height G_n without disturbing the height condition.

Now remove the couple with heights B_1 and G_1 from consideration and repeat the process with the couple with heights B_2 and G_2. Continuing in this manner, we will eventually have paired the shortest boy with the shortest girl, the second shortest boy with the second shortest girl, and so on, up to the tallest boy with the tallest girl, without disturbing the height condition.

Prisoner perplexity

Solution by: Ivan Guo

Call the prisoners $P_1, P_2, \ldots, P_{100}$ and let prisoner P_k assume that the sum of all of the numbers assigned to the prisoners is congruent to k modulo 100. Using this assumption, when the prisoner sees all of the remaining prisoners' numbers, they

can then subtract modulo 100 to determine their own number. Of course, using this strategy, exactly one of the prisoners will be correct and all of the others will be wrong.

Weighing coins

Solution by: Joseph Kupka

(1) Split the 68 coins into 34 pairs and weigh each coin against its partner. We know that the heaviest coin is one of the 34 coins which was heavier than its partner. Split these 34 coins into 17 pairs and weigh each coin against its new partner. Then take the 17 heavier coins and split them into 8 pairs and weigh each coin against its partner, keeping the leftover coin A to one side. Continue this splitting and weighing process with 4 pairs, then 2 pairs, then 1 pair and let B be the heavier coin from this last weighing. Now the heaviest coin must be A or B and we can determine which by weighing them against each other. This takes $34 + 17 + 8 + 4 + 2 + 1 + 1 = 67$ weighings.
We also know that the lightest coin is one of the 34 coins which was originally lighter than its partner. We can apply the same splitting and weighing process and this will determine the lightest coin in an extra $17 + 8 + 4 + 2 + 1 + 1 = 33$ weighings. Therefore, we have determined the heaviest coin and the lightest coin using a balance scale 100 times.

(2) Weigh 30 coins against 30 other coins, and discard the leftover coin.
- Case 1: If the scale balances, then exactly 1 counterfeit coin lies in each pan. Take 30 from one pan and divide these into three groups of 10. Call these groups A, B, C and weigh A against B.
    - Case 1A: If the scale balances, then A and B consist of genuine coins so a counterfeit coin lies in C. So we can simply weigh A against C to determine whether the counterfeit coins are heavier or lighter.
    - Case 1B: If the scale doesn't balance, then C consists of genuine coins. Now weigh C against the heavier of A and B. If they balance, then the counterfeit coins are lighter and if they don't balance, then the counterfeit coins are heavier.
- Case 2: If the scale doesn't balance, then the 30 coins from the heavier group contain 0, 1 or 2 counterfeit coins. Divide these 30 coins into three groups of 10. Call these groups A, B, C and weigh A against B and then B against C. If both of these weighings balance, then there are no counterfeit coins in these 30 coins and the counterfeit coins must be lighter. Otherwise, the counterfeit coins must be heavier.

(3) Divide the coins into three groups of four. Call these groups A, B, C and weigh A against B.
Case 1: If A and B balance, then these eight coins are all genuine and the four coins from group C include the counterfeit one. So weigh three from group A against three from group C. If they balance, then the remaining coin from group C must be the counterfeit one and one extra weighing will determine whether it is heavier or lighter. If they don't balance, then suppose

without loss of generality that the three coins from group C are heavier than the three coins from group A. Weigh two of these three coins from group C against each other. If they balance, then the third coin is counterfeit and heavier. If they don't balance, then the heavier coin is counterfeit.

Case 2: If A and B don't balance, then assume without loss of generality that group A is heavier than group B. Weigh two coins from group A and one coin from group B against the two other coins from group A and one other coin from group B. If they balance, then one of the remaining coins from group B is lighter and we can determine which one with one extra weighing. If they don't balance, then take the two coins from group A which were in the heavier pan and weigh them against each other. If they balance, then the coin from group B in the other pan must be lighter. If they don't balance, then the heavier of these two coins is counterfeit.
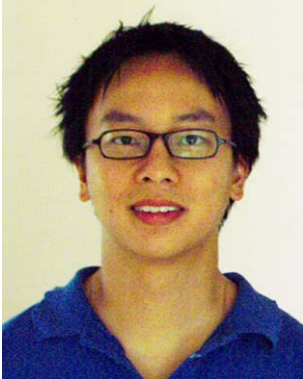
Norman Do is currently a CRM-ISM Postdoctoral Fellow at McGill University in Montreal. He is an avid solver, collector and distributor of mathematical puzzles. When not playing with puzzles, Norman performs research in geometry and topology, with a particular focus on moduli spaces of curves.