Mathellaneous

Norman Do

Scissors congruence and Hilbert's third problem

What is area?

Despite being such a fundamental notion of geometry, the concept of area is very difficult to define. As human beings, we have an intuitive grasp of the idea which suffices for our everyday lives; however, as mathematicians, we have only relatively recently been able to identify what we actually mean by the word. Efforts to define and generalize the notions of length, area and volume have led to the development of the branch of mathematics known today as measure theory.

Part of the difficulty in defining area lies in the fact that subsets of the plane can be quite wild in comparison to the objects we encounter in the physical world. In particular, there exist certain sets for which we cannot ascribe an area without disturbing one or more of the fundamental premises that govern measure theory. These so-called non-measurable sets make any efforts to characterize area rather complicated. To avoid such intricacies, let us propose the more modest task defining area for polygons in the plane.

A seemingly elementary approach to the problem is to use the method taught to children in primary school. This involves drawing the shape on graph paper and then counting the number of squares which lie within the figure. This, of course, gives a lower bound for the area, while counting the squares which contain some part of the shape in question will yield an upper bound. The idea now is to consider finer and finer graph paper, thus giving more and more accurate lower and upper bounds on the area in the hope that they will converge to the same number. It is this number which is defined to be the area. It turns out that this definition of area, although unsatisfactory for more complicated subsets of the plane, is well-defined for every polygon in the plane. In this way, we have constructed an area function which, given a polygon P, returns a real number $a(P)$.

It is quite unfortunate that this definition of area, even for polygons, should involve infinite processes and continuity arguments. Could we perhaps find an alternative definition, one which requires purely elementary techniques? A common trend in contemporary mathematics is to define an object by the properties which it satisfies. For example, we may like to extract enough properties of the area function defined above that there can be only one such function which satisfies them all. In fact, the four properties listed below perform such a task and we present them as the area axioms.

Area axioms

- (Non-negative) If P is a polygon, then $a(P) \geq 0$.
- (Additive) If P_1 and P_2 are polygons with no common interior points, then $a(P_1 \cup P_2) = a(P_1) + a(P_2)$.
- (Invariant) If P_1 is a polygon and P_2 is the polygon obtained by subjecting P_1 to a rigid motion, then $a(P_1) = a(P_2)$.
- (Normalized) If S is the unit square in the plane, then $a(S) = 1$.

We say that two polygons are *scissors congruent* if one can be cut into finitely many polygons which can be rearranged to give the other. In other words, scissors congruent polygons are common solutions to a jigsaw puzzle with the same set of pieces. Let us denote the fact that polygons P and Q are scissors congruent by $P \sim Q$. Also, note that we can extend the definition to finite unions of polygons with mutually disjoint interiors. In this case, we denote the union by $P_1 + P_2 + \cdots + P_n$, where $P_1, P_2, \ldots, P_n$ denote the individual polygons. The notion of scissors congruence, first for polygons, and then for polyhedra, will be the main topic of investigation in this article.

Scissors congruence in the plane

Our exploration into the areas of polygons has naturally led us to the notion of scissors congruence. It is a simple consequence of the area axioms that if two polygons are scissors congruent, then they have the same area. Now it is only natural to ask the converse.

> If two polygons have the same area, then are they necessarily scissors congruent?

This question was answered in the affirmative by F. Bolyai in 1832 and independently by P. Gerwien one year later. Before embarking on the proof, let us begin by presenting a few important lemmas on scissors congruence.

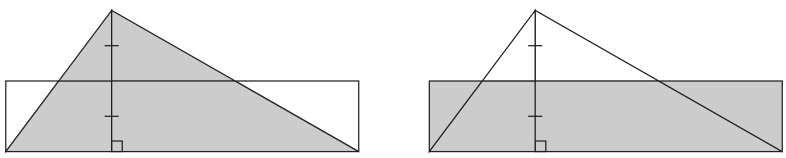
Lemma: Scissors congruence is an equivalence relation. In other words:

(1) for all polygons P, $P \sim P$;
(2) if $P \sim Q$, then $Q \sim P$; and
(3) if $P \sim Q$ and $Q \sim R$, then $P \sim R$.

Proof. The first two statements are immediately evident from the definition of scissors congruence. Now suppose that we trace out the cuts required to decompose Q into pieces which rearrange to give P as well as the cuts required to decompose Q into pieces which rearrange to give R. Then cutting along all of these lines will yield a finite set of pieces which can be rearranged to yield either P or R. It follows that P and R are scissors congruent. $\square$

Lemma: Every triangle is scissors congruent with some rectangle.

Proof. Cut along the altitude to the longest side of the triangle as well as along the perpendicular bisector of this altitude. This will decompose the triangle into two triangles and two quadrilaterals which can be rearranged to give a rectangle with the same base and half the height of the triangle. $\square$

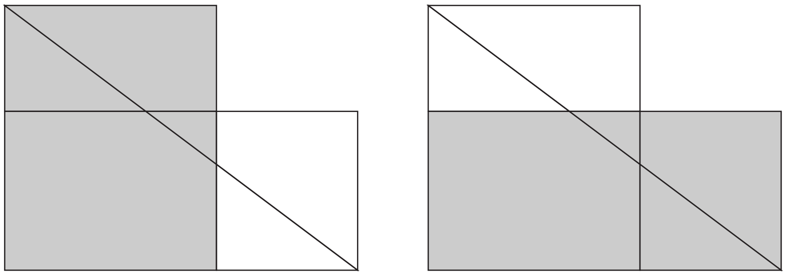


Lemma: Any two rectangles with the same area are scissors congruent.

Proof. Place the two rectangles in the plane so that they have one vertex in common, with two adjacent sides aligned as shown in the diagram. Cutting one of the rectangles along the lines shown produces a pentagon, a small triangle and a large triangle which can be rearranged to give the other rectangle.

The astute reader may have noticed that this particular decomposition may not always work if one of the rectangles is "too long". More explicitly, suppose that the rectangles have dimensions $\ell_1 \times h_1$ and $\ell_2 \times h_2$ where we may assume without loss of generality that

$h_1 < h_2 \le \ell_2 < \ell_1$. Then the construction works only when $\ell_1 \le 2\ell_2$. However, this can be arranged by repeatedly cutting the longer rectangle in half and stacking the two pieces on top of each other until the condition $\ell_1 \le 2\ell_2$ is satisfied. □



With these three lemmas under our metaphorical belts, we are now ready to prove the Bolyai-Gerwien Theorem.

Bolyai-Gerwien Theorem: Two polygons are scissors congruent if and only if they have the same area.

Proof. First, note that any polygon P can be decomposed into finitely many triangles. Furthermore, each of these triangles is scissors congruent to some rectangle and each of these rectangles is scissors congruent to a rectangle, one of whose sides has length 1. Therefore, we can write

$$P \sim R_1 + R_2 + \ldots + R_n,$$

where each R_i is a rectangle with one side of length 1.

Concatenating these rectangles together produces a single rectangle R which has one side of length 1. Of necessity, the other side of R must have length equal to the area of P. Therefore, if P and Q have the same area, they are both scissors congruent to the rectangle R, so they are scissors congruent to each other. □

What is volume?

It is simple enough to develop a theory of volume for polyhedra in an analogous manner to the theory of area for polygons. The area axioms transform naturally into volume axioms to give a rigorous definition of the volume of a polyhedron. However, it had been noted by Gauss that proofs for the volume of a tetrahedron had all used in some way or another continuity arguments, rather than entirely elementary methods. Such an elementary proof would require that polyhedra with the same volume be scissors congruent.

> If two polyhedra have the same volume, then are they necessarily scissors congruent?

Hilbert considered this question of such importance that he included it in his famous address to the International Congress of Mathematicians at Paris in 1900. As is well known, his address included a list of twenty-three unsolved problems in mathematics which he considered to be of great significance. This question of scissors congruence in three dimensions was the third on his list.

It is clear from Hilbert's exposition on the matter that he did not expect the Bolyai-Gerwien theorem to carry over from polygons in the plane to polyhedra in space — and he was exactly right.

Dehn invariants

Hilbert's third problem was answered by his own student Max Dehn in 1900, the very year in which Hilbert had given his address at the International Congress of Mathematicians. The crux of the proof lies in constructing ingenious invariants which since been named after Dehn. But before we can progress to the definition of Dehn invariants, we need to develop a little background from linear algebra.

Given a finite set of real numbers $X = \{x_1, x_2, \ldots, x_n\}$, let $V(X)$ denote the set of linear combinations of numbers in X with rational coefficients.

$$V(X) = \{q_1x_1 + q_2x_2 + \cdots + q_nx_n : q_i \in \mathbb{Q}\}$$

Note that $V(X)$ is a vector space over $\mathbb{Q}$, so it makes sense to speak of $\mathbb{Q}$-linear functions $f : V(X) \to \mathbb{Q}$. In other words, we are interested in functions $f : V(X) \to \mathbb{Q}$ which satisfy

- $f(v_1 + v_2) = f(v_1) + f(v_2)$ for all $v_1, v_2 \in V(X)$; and
- $f(qv) = qf(v)$ for all $q \in \mathbb{Q}$ and $v \in V$.

In later discussion of Dehn invariants, we will need the following simple lemma.

Lemma: Let $X \subseteq Y$ be finite sets of real numbers. If $f : V(X) \to \mathbb{Q}$ is a $\mathbb{Q}$-linear function, then f can be extended to a $\mathbb{Q}$-linear function $g : V(Y) \to \mathbb{Q}$ such that $f(v) = g(v)$ for all $v \in V(X)$.

Proof. A $\mathbb{Q}$-linear function $f : V(X) \to \mathbb{Q}$ is determined by its values on a $\mathbb{Q}$-basis of $V(X)$. However, we know from the theory of vector spaces that every basis of $V(X)$ can be extended to a basis of $V(Y)$. □

Given a polyhedron P, let X_P denote the set of dihedral angles between adjacent faces of P, along with the number π. Let X be a finite set of real numbers which contains X_P and consider any $\mathbb{Q}$-linear function $f : V(X) \to \mathbb{Q}$ which satisfies $f(\pi) = 0$. Then we define the Dehn invariant of P with respect to f to be the real number

$$D_f(P) = \sum_{e \in P} \ell(e) \cdot f(\theta(e)),$$

where the sum extends over all edges e of the polyhedron, $\ell(e)$ denotes the length of e, and $\theta(e)$ denotes the dihedral angle at e. The reason for this construction will become more transparent after considering the following crucial theorem.

Dehn-Hadwiger Theorem: Let P and Q be polyhedra and let $X_P \cup X_Q \subseteq X$. If $f : V(X) \to \mathbb{Q}$ is a $\mathbb{Q}$-linear function with $f(\pi) = 0$ such that

$$D_f(P) \neq D_f(Q),$$

then P and Q are not scissors congruent.

Proof. Suppose that the polyhedron P can be decomposed into finitely many polyhedra $P_1, P_2, \ldots, P_n$ and let $Y = X \cup X_{P_1} \cup X_{P_2} \cup \ldots \cup X_{P_n}$. Then by the lemma above, we can extend $f : V(X) \to \mathbb{Q}$ to a $\mathbb{Q}$-linear function $g : V(Y) \to \mathbb{Q}$. Now we will show that the Dehn invariants are additive in the sense that

$$D_g(P) = D_g(P_1) + D_g(P_2) + \cdots + D_g(P_n).$$

Given a polyhedron P_i in the decomposition of P, let us associate a mass to each edge of P_i. For the edge e, the mass is given by the real number

$$\ell(e) \cdot g(\theta(e)),$$

where $\ell(e)$ denotes the length of e and $\theta(e)$ the dihedral angle at e in the polyhedron P_i. Now we make the following observation about the masses associated to an edge e in the decomposition of P.

- If e is contained in an edge of P, then the dihedral angles of the pieces add up to the dihedral angle of P at e, and hence, the masses add up in the required way.
- If e is contained in the interior of a face of P, then the dihedral angles of the pieces add up to π, so the g-values of the angles in the pieces add up to $g(\pi) = 0$.
- If e is contained in the interior of P, then the dihedral angles of the pieces add up to 2π, so the g-values of the angles in the pieces add up to $g(2\pi) = 2g(\pi) = 0$.

Further to these observations, note that the Dehn invariants of two congruent polyhedra must be equal. Hence, if the polyhedra P and Q can be decomposed as $P = P_1 \cup P_2 \cup \ldots \cup P_n$ and $Q = Q_1 \cup Q_2 \cup \ldots \cup Q_n$ where P_i is congruent to Q_i for each i, then

$$\begin{aligned} D_g(P) &= D_g(P_1) + D_g(P_2) + \cdots + D_g(P_n) \\ &= D_g(Q_1) + D_g(Q_2) + \cdots + D_g(Q_n) \\ &= D_g(Q). \end{aligned}$$

However, since g was an extension of the function f, this yields $D_f(P) = D_f(Q)$, as desired. □

Hilbert's third problem solved

Example (Cube)

Let us consider a cube C of side length s. Since all of the dihedral angles of the cube are equal to $\frac{\pi}{2}$, we have $X_C = \{\frac{\pi}{2}, \pi\}$. Now to form the Dehn invariant for the cube, we need to consider a $\mathbb{Q}$-linear function $f : V(X) \to \mathbb{Q}$ satisfying $f(\pi) = 0$, where $X_C \subseteq X$. However, any such function f necessarily satisfies $f\left(\frac{\pi}{2}\right) = \frac{1}{2} f(\pi) = 0$, so the Dehn invariant is simply

$$D_f(C) = 12 s f\left(\frac{\pi}{2}\right) = 0.$$

Example (Regular tetrahedron)

Let us now consider a regular tetrahedron T of side length ℓ. Again, all of the dihedral angles of the regular tetrahedron are equal and a simple calculation demonstrates that this angle is $\phi = \arccos \frac{1}{3}$. However, there is no linear dependence between $\arccos \frac{1}{3}$ and π with rational coefficients. This follows from the fact that $\frac{1}{\pi} \arccos \frac{1}{3}$ is irrational, a fact whose proof we will postpone till later. Therefore, we can consider the function $f : V(X_T) \to \mathbb{Q}$ defined on the basis elements

$$f(\phi) = 1 \text{ and } f(\pi) = 0.$$

With this particular choice of f, we can calculate the Dehn invariant to be

$$D_f(T) = 6\ell f(\phi) = 6\ell \neq 0.$$

Solution to Hilbert's third problem

The function defined above when calculating the Dehn invariant for the regular tetrahedron is also a valid choice for calculating the Dehn invariant of the cube. We may fix the side lengths of the cube and regular tetrahedron so that they both have unit volume. We know that their Dehn invariants satisfy

$$\begin{aligned} D_f(C) &= 0 \\ D_f(T) &\neq 0. \end{aligned}$$

Therefore, by the Dehn-Hadwiger Theorem, the cube and the regular tetrahedron of unit volume are not scissors congruent.

Fact: The number
$$\frac{1}{\pi}\arccos\frac{1}{3}$$
is irrational.

Proof. If the number was rational, then we could write
$$\frac{1}{\pi}\arccos\frac{1}{3} = \frac{p}{q}$$
where p and q are integers, with q positive. Then
$$q\arccos\frac{1}{3} = p\pi \Rightarrow \cos(q\arccos\frac{1}{3}) = \cos(p\pi) = \pm 1.$$
However, we will now proceed to show by induction that for $\phi = \arccos\frac{1}{3}$, $\cos n\phi = \frac{A_n}{3^n}$ where A_n is an integer not divisible by 3. It follows that $\cos q\phi \neq \pm 1$ and from this contradiction, we deduce the irrationality of $\frac{1}{\pi}\arccos\frac{1}{3}$.
We have $\cos\phi = \frac{1}{3}$, so the statement is certainly correct for $n = 1$. We also have
$$\cos 2\phi = 2\cos^2\phi - 1 = 2 \times \left(\frac{1}{3}\right)^2 - 1 = -\frac{7}{9},$$
so the statement is also true for $n = 2$. Now suppose that the statement is true for some pair of consecutive positive integers $k-1$ and k. Then by the sums to products formula, we can express $\cos(k+1)\phi$ as follows.
$$\begin{aligned}\cos(k+1)\phi &= 2\cos k\phi\cos\phi - \cos(k-1)\phi \\ &= \frac{2}{3}\frac{A_k}{3^k} - \frac{A_{k-1}}{3^{k-1}} \\ &= \frac{2A_k - 9A_{k-1}}{3^{k+1}} \\ &= \frac{A_{k+1}}{3^{k+1}}\end{aligned}$$
All that remains is to note that, by the induction hypothesis, $A_{k+1} = 2A_k - 9A_{k-1}$ is not divisible by 3. □

Further comments

- Hilbert's third problem was actually stated in two parts. More explicitly, he asks to specify

  > "two tetrahedra of equal bases and equal altitudes which can in no way be split up into congruent tetrahedra,"

  as well as to specify

  > "[two tetrahedra] which cannot be combined with congruent tetrahedra to form two polyhedra which themselves could be split up into congruent tetrahedra."

  The Dehn-Hadwiger Theorem can in fact be used to show that the tetrahedron spanned by three mutually orthogonal edges AB, AC, AD of unit length and the tetrahedron with three consecutive mutually orthogonal edges AB, BC, CD of unit length are not scissors congruent. This solves the first part of Hilbert's third problem.

Now let us call two polyhedra *glue congruent* if we can glue some finite set of congruent polyhedra onto both of them to give two polyhedra which are scissors congruent. A closer inspection of Dehn invariants shows that these two tetrahedra cannot be glue congruent either, thus solving the second part of Hilbert's third problem.

- The exposition given here on Dehn invariants considers only finite dimensional $\mathbb{Q}$-vector spaces which are subsets of $\mathbb{R}$. However, given the axiom of choice, we can treat all of $\mathbb{R}$ as a $\mathbb{Q}$-vector space with an uncountable basis. Hence, Dehn invariants can be formed by taking $\mathbb{Q}$-linear functions from $\mathbb{R}$ to $\mathbb{Q}$. However, by restricting ourselves to finite-dimensional vector spaces, the treatment here does not need to invoke the axiom of choice.
- In 1965, Sydler showed that the Dehn invariants form a complete set of invariants for the scissors congruence problem in three dimensions. In other words, two polyhedra are scissors congruent if and only if their Dehn invariants are the same for every suitable choice of the $\mathbb{Q}$-linear function f.

References

[1] V.G. Boltianskii, *Hilbert's Third Problem* (V.H. Winston, Halsted Press, Wiley Washington DC 1978).
[2] M. Dehn, *Ueber raumgleiche Polyeder*, Nachrichten von der Königl. Gesellschaft der Wissenschaften, Mathematisch-physikalische Klasse (1900), 345–354.
[3] D. Hilbert, *Mathematical problems*, Lecture delivered at the International Congress of Mathematicians at Paris in 1900, Bulletin Amer. Math. Soc. **8** (1902), 437–479.
[4] M. Aigner and G.M. Ziegler, *Proofs from the Book* (Springer-Verlag Heidelberg 2000).

Department of Mathematics and Statistics, The University of Melbourne, VIC 3010
E-mail: N.Do@ms.unimelb.edu.au