The Math of Traffic

*Kaleidoscopic overview of Research in Traffic Flow Modeling and Control in Delft*

Mathematics of Transport Networks - Melbourne, 19 juni 2013
Societal urgency: accessibility

Accessibility and Traffic Congestion

• History of traffic queues: from ‘unique sightseeing event’ to major and very common nuisance!

• Costs of traffic congestion in The Netherlands 4.6 billion Euros (2012), for Australia around 8.3 billion dollars (2005)
Societal urgency: accessibility

Reliability of Transport and Network Robustness

- In particular in peak-hours, travel times are hard to predict beforehand
- Trip planners have to take this uncertainty into consideration, resulting in extra cost (VOR = VOT!)
- Moreover, critically loaded networks are often not very robust (relatively small perturbations have very severe effects)
- Examples of robustness issues:
  - Extreme impact of weather (snow)
  - Impacts of incident on critical links
Societal urgency: Safety & Security
Emergencies and Evacuations

- Increasing risks of flooding of highly urbanized Randstad area
- Focus traditionally on prevention, but times are changing!

- Simple simulation
- Normal evacuation plans are inadequate and yield too long evacuation times (> 48 hours)
- How can we improve these plans or otherwise mitigate impacts of an emergency?
Example EVAQ application
Assessing and improving evacuation plans

- Flood strikes from West to East in six hours in which 120,000 residents / 48,000 cars need to be evacuated
- Capacity of outlinks = 8000 veh/h
- Spatio-temporal dynamics of hazard are known
- Evacuation instructions entail departure time, safe destination, and route to destination for specific groups of evacuees (e.g. per area code)
- Use shortest route to closest destination not overloading route
Evacuation of Walcheren

Assessing standard evacuation plan...

Number of evacuated people around 41000 (~34%)
Optimization objectives

Objective applied in this research

• Maximizing function of the number of arrived evacuees in each time period:

\[ J(u) = \int_{0}^{T} q_u(t) \, dt \]

- \( q_u(t) \): number of arrived evacuees in time period \( t \)
- \( u \): evacuation scheme

• Evacuate as many people as possible
• Use of evacuation simulation model EVAQ to compute \( J(u) \) as function of \( u \)
• NP hard problem: Ant Colony optimization
Example results

Strategy comparison

- Optimization of evacuation plan yields very significant improvement compared to other scenarios

![Chart showing comparison of evacuation strategies]

- Computation times are large, even for small network (10 hrs)
Optimal pedestrian evacuation

Similar problem, different approaches

• Optimal departure time & routing:

$$-\frac{\partial W}{\partial t} = L(t,x,v^*) + v^* \nabla W + \frac{\sigma^2}{2} \Delta W$$

where $v^* = -c_0 \nabla W$

• Network loading:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho \cdot v) = 0$$

• Fixed point problem…
Math and traffic / transportation

Examples of using mathematical techniques

- Evacuation case is example of (off-line) model-based optimization (in this case: evacuation instructions; but also: design, planning)
- Example applications of mathematical techniques:
  - Model-based analysis of traffic and transportation phenomena, e.g. to understand key mechanisms or to determine key decision variables by fitting models
  - Mathematical modeling and simulation for off-line applications (scenario assessment, (network) designs, new ITS measures, etc.)
  - Improving data quality using data fusion by Kalman filtering
  - On-line traffic prediction and analysis of scenarios
  - On-line model-based optimization in for control purposes
- Let’s take a look at some other examples…
Traffic instabilities

- Field data analysis (bottom figure) and physical experiments (top movie) show that in certain density regimes, traffic is unstable.

- Small disturbances amplify as they travel from one vehicle to the next.

- Eventually, disturbance grows into so-called wide moving jam, moving upstream in opposite direction of traffic at speed of 18 km/h.

- Outflow of wide-moving jam is about 30% less than free flow capacity.
Understanding Traffic Instability
Using relatively simple models...

• CHM car-following model describes acceleration of vehicle in response to distance to predecessor, and speed:

\[
\frac{d}{dt} v_i(t + T_r) = \kappa \cdot \Delta v_i(t)
\]

• Parameters are reaction time \( T_r \) and sensitivity \( \kappa \)
Understanding Traffic Instability

Using relatively simple models...

- Stability analysis of shows for which parameters we get asymptotic instability that is, disturbances grow as they traverse from one vehicle to the next
- It turns out that string stability is determined by:
Understanding Transit disturbances

Propagation of delays through transit networks

- Description of scheduled rail network as a Discrete Event System:

\[ x_i(k) = \max \left( \max_j (a_{ij} + x_j(k - \mu_{ij})), d_i(k) \right) \]

- Max-plus algebra allows us to rewrite system as a linear system:

\[ x_i(k) = \bigoplus_{j=1,...,n} (a_{ij} \otimes x_j(k - \mu_{ij})) \oplus d_i(k) \]

\[ x(k) = A \otimes x(k) \oplus d \]
Understanding Transit disturbances
Propagation of delays through transit networks

**Stable**: 5 min initial delay Hilversum

**Unstable**: 5 min initial delay Coevorden
Understanding Transit disturbances

Propagation of delays through transit networks

- Stability of delay propagation can be analyzed by looking at eigenvalues of $A$

$$A \otimes v = \lambda \otimes v$$

*minimum period length for network*

*periodic minimal timetable for all trains*
State estimation

Making sense of real-time traffic data...
State estimation and data fusion

Estimate traffic state from different data sources

- Problems using Kalman filter approach using LWR model because of problematic linearization
- Use of Lagrangian formulation (change of coordinate system)

\[
\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0 \quad \text{Godunov}
\]

\[
\frac{\partial s}{\partial t} + \frac{\partial v(s)}{\partial n} = 0 \quad \text{Upwind}
\]

- Advantages of Lagrangian formulation:
  - Easy numerical discretization (upwind) with almost no num diffusion
  - A natural set of observation equations to deal with Lagrangian sensing data (probe vehicle, trajectory-based data)
  - Advantageous properties of application EKF (compared to Godunov)
Modeling

Not an exact science!
Traffic and Transport Models

- Traffic operations result from human decision making and complex multi-actor interactions at different behavioral levels.

- Human behavior is ‘not easy to capture and predict’

- System is highly complex, non-linear, has chaotic features, etc.

- Challenge is to develop theories and models that represent and predict operations sufficiently accurate for application at hand.

- But how is this achieved? Induction vs deduction...

...for Distinction Sake, a Deceiving by Words, is commonly called a Lye, and a Deceiving by Actions, Gestures, or Behavior, is called Simulation...

Robbert South (1643–1716)
Deduction

Modeling approaches

- Starts with an axiom, an assumed truth, a theory (which come from an observations, logic, other theories)
- Typical in (theoretical) physics, mathematics
- Example: special theory of relativity (Einstein postulated that the speed of light is the same for all observers, regardless of their motion relative to the light source – observations proved him right)
Induction

Modeling approaches

- Starts with observations (phenomena, patterns, etc.)
- Typical in social sciences and biology
- Example: Darwin’s theory of evolution by natural selection (Darwin observed populations finks diverging in different habitats and postulated natural selection as the motor – modern genetics, biology and many, many other scientific disciplines proved him right)

Observations

Phenomena, patterns

Tentative hypotheses

About underlying relations / theories

Testing / operationalizing

Qualitatively / quantitatively

New theory

Until falsified...
Traffic and Transportation Theory?
Inductive or deductive?

• Traffic flow theory is **largely** based on **induction** (with a bit of deduction): theory building is for a large part based on empirical or experimental observations.

• Our theories and models are as good as the quality of their predictions (and should be assessed with that in mind!)
  • Do they predict the **key phenomena and traffic flow features** we observe in the real world?
  • Do they incorporate a **(mathematical) structure** that provide insight into how these phenomena emerge?

• Let us consider some of these phenomena, starting with the father of traffic flow theory...
Bruce Greenshields...

The discovery of the Fundamental Diagram

• First traffic data collection using cameras and may hours of manual labour...

• Studied relation between average vehicle speeds and vehicle density (= average distance\(^{-1}\)) and found an important relation
Bruce Greenshields
The discovery of the Fundamental Diagram

- Decreasing relation between speed and density
- When speed decreases, drivers drive closer

- Although the assumption of a linear relation turned out to be flawed, FD formed basis for contemporary traffic flow theory!
- With \( q = kv = Q(k) \) and conservation of vehicle equation we get a complete model of traffic flow!
First-order theory
Application of the FD

- Predicting queue dynamics using first order theory
- Predicts dynamics of congestion using FD
- Flow in queue = \( C - q_{\text{on-ramp}} \)
- Shock speed determined by:

\[
\omega_{12} = \frac{Q(k_2) - Q(k_1)}{k_2 - k_1}
\]
With improved data collection to better theory!

- Data collection system for collecting high-frequency images from the air (helicopter, drones)
- Algorithms for stabilization of images and geo-referencing
- Vehicle detection and tracking, resulting in high-resolution data on revealed driving behavior (long + lat)
- 15-30 min of data, 500 m roadway, 15 Hz, 40 cm resolution, all vehicles!
- Multiple data sets for variety of circumstances (congestion, merges, incidents, etc.)
Vehicle trajectory information

Example of findings

- New data has provided avalanche of new insights for regular and non-recurrent conditions:
  - Driver heterogeneity and adaptation effects (e.g. in case of incidents)
  - Benchmarking of car-following models
  - Discontinuous car-following behavior (action points)
  - Detailed analysis of lane changing and merging behavior
- Example analysis merging behavior:
  - **Accepted models for merging turn out to be flawed** since drivers actively select gap actively rather than passively accept it
  - Paradigm shift and new mathematical models yield increased predictive validity of microscopic flow models
  - Practically: distribution of merging points far less concentrated
Example of findings

- Although microscopic simulation models can be tuned such that most important macroscopic features can be represented, the microscopic processes often are not correctly described!
- Impacts of this observation, e.g., with respect to the predictive validity
- Consider how models are used!
More (big?) data, new insights

- Availability of large datasets from urban and motorway arterials leads to new insights into network dynamics
- Data from GPS (Yokohama) empirically underpins existence of Network Fundamental diagram
- Fundamental property of traffic network: production deteriorates at high loads!
More (big?) data, new insights

- Recent studies (TU Delft, ICL) show that network dynamics are a “bit more involved”
- Next to average density, spatial variation of density plays a crucial role in representing network traffic production and level of service...
- Congestion nucleation causes spatial variation to self-sustain & increase

![Graph showing network dynamics with color-coded density](image)
Network Dynamics

Features and phenomena that you need to capture!
Efficient and inefficient self-organization and network degradation

- For low network loads, interactions between traffic participants is very efficient.
- For high loads, inefficient phenomena self-organize / occur reducing performance.

There are severe limits to the self-organization capacities of the traffic system.
Characteristic features of traffic flow

Efficient self-organization in dilute flow conditions

- Dynamically formed walking lanes
- High efficiency in terms of capacity and observed walking speeds
- Experiments by Hermes group show similar results
- Phenomena is characteristic of a pedestrian flow, and needs to be included in model
Main behavioral assumptions (loosely based on psychology):

- Pedestrian can be described as optimal, predictive controllers who make short-term predictions of the prevailing conditions, including the anticipated behavior of the other pedestrians
- Pedestrians **minimize walking effort** caused by distance between peds, deviations from desired speed / direction, and acceleration
- Costs are discounted over time, yielding:

\[
J = \int_1^{\infty} e^{-\eta t} \left[ \frac{1}{2} \mathbf{a}^T \mathbf{a} + c_1 \frac{1}{2} (\mathbf{v}^0 - \mathbf{v})^T (\mathbf{v}^0 - \mathbf{v}) + c_2 \sum_q e^{-\frac{\|r_q - r\|}{R_0}} \right]
\]

- Use of differential game theory to determine the pedestrian acceleration behavior (i.e. the acceleration \( \mathbf{a} \))
Game-theory applications

To modeling interactions of traffic participants

- Next to walker behavior, other applications of differential game theory have been put forward
  - Car-following and merging behavior modeling
  - Cooperative driving control strategies for vehicle platoons
- Recent work involves interactions of large vessels, where game theory is used to describe the behavior of the bridge team under different scenarios (cooperative and single-sided interaction, demon-ship interaction)

- Note that the resulting optimization problem can be solved using Pontryagin’s minimum principle + dedicated numerical solver
- Computationally quite demanding!
Adding fraction terms

The simplest of models...

- Under the assumption that the opponent peds do not react to the considered ped, we find a closed form expression for acc vector:

\[ \mathbf{a}_p(t) = \frac{\mathbf{v}_p^0 - \mathbf{v}_p}{\tau_p} - A_p^0 \sum_{q \neq p} \mathbf{n}_{pq} e^{-||\mathbf{r}_p - \mathbf{r}_q||/R_p^0} \]

- Resulting expression is same as original Social Forces model of Helbing
- Physical interactions (physical contact, pushing) can be modeled by adding physical forces between pedestrians
Interaction modeling

Use of differential game theory

- Simple model reproduces lane formation processes adequately

*Example shows lane formation process for homogeneous groups...*

*Heterogeneity yields less efficient lane formation (freezing by heating)*
Pedestrian flow capacity drop

• Adding friction between pedestrians causes severe reduction in capacity

• Capacity drop is due to arc formation in front of exit

• Gets worse when pedestrians are more anxious to get out (Helbing et al, Nature 2000)

• In line with results from pedestrian experiments (TU Dresden, TU Delft)

• Capacity drop also occurs in car-traffic: when congestion sets in, capacity reduces with 10-15%
Impact of spillback on throughput

- Example of impacts of spillback on A10 motorway
- Average daily collective delay of 300 veh-h
- Societal cost about 1 million Euros per year!
Spill-back and grid-lock

Urban networks

• Spill-back easily leads to grid-lock effects, as we saw earlier...
• Similarly, grid-lock can occur in pedestrian networks when network load is too high
• In this case, self-organization fails and capacity drops
Stochasticity...

Random nature of traffic
Which is the representative day?
Stochasticity

Supply factors

- Clearly, **traffic demand** is stochastic but what about capacity?
- Capacity = maximum (hourly) flow that can be sustained for a considerable time period
- What determines capacity?
  - Infrastructure
  - Driving behavior
  - Vehicle characteristics
  - Occurrence of incidents
- It is not reasonable to assume that capacity is deterministic!
Example: IDM
Explaining stochasticity?

- Vehicle trajectories collected from airborne platform (helicopter)
- IDM model by Treiber and Helbing:

\[
a = f(s, v, \Delta v) = a \cdot \left[ 1 - \left( \frac{v}{v_*} \right)^4 - \left( \frac{s_*(v, \Delta v)}{s} \right)^2 \right]
\]

where \( s_* = s_0 + \tau v + \frac{v \Delta v}{2 \sqrt{ab}} \)

- Find estimates for parameters that maximize the likelihood \( L \) of finding the actually observed car-following behavior
Pictures show CDFs of estimated parameters showing large heterogeneity in driving behavior!
Modeling approaches

Fitting models...
Some considerations
When choosing / developing a model

- Trivial: model requirements depend on application, which in turn prescribes:
  - Which behavioral processes to include
  - Type of validity (qualitative, quantitative, reproduce or predict?)
  - Which phenomena or features need to be reproduced
  - Math / computational properties of approach

---

**Location choice**

**Trip choice**

**Destination choice**

**Mode choice**

**Route choice**

**Departure time choice**

**Driving behavior**
Modeling approaches
Reproducing vs predicting

- Two dimensions:
  - Representation of traffic
  - Behavioral rules

<table>
<thead>
<tr>
<th></th>
<th>Individual particles</th>
<th>Continuum</th>
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<tr>
<td>Individual behavior</td>
<td>Microscopic (simulation) models</td>
<td>Gas-kinetic models (Boltzmann equations)</td>
</tr>
<tr>
<td>Aggregate behavior</td>
<td>Particle discretization models (Dynasmart)</td>
<td>Queuing models Macroscopic flow models</td>
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</table>
Relation between micro and macro

Micro, meso and macro?

- Microscopic models (aim to) **explain and predict** driving behavior (car-following, lane changing, etc.)
- Macroscopic features (e.g. capacity, jam-density, etc.) are thus predicted output of these models

**Example:**

- (CHM model)

- Ensuring correct reproduction of macroscopic features is often a difficult (calibration) process (parameters not directly observable)

- Macroscopic models generally (often) take macroscopic features as input and correct **representation** is thus ‘trivial’
How good are these models anyway?

Some example approaches...

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<th>BPR functions</th>
<th>Queuing models</th>
<th>First-order theory</th>
<th>Micro-simulation</th>
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<tbody>
<tr>
<td>Capacity drop</td>
<td>N/A</td>
<td>EVAQ</td>
<td>Infinite wave speed</td>
<td>Yes, but often too small</td>
</tr>
<tr>
<td>Spill-back</td>
<td>N/A</td>
<td>Extended LTM</td>
<td>Yes</td>
<td>Only if model reproduces FD</td>
</tr>
<tr>
<td>Stochastic demand and supply</td>
<td>N/A</td>
<td>Quast</td>
<td>Only research models</td>
<td>Variation often too small</td>
</tr>
<tr>
<td>Congestion instability</td>
<td>N/A</td>
<td>N/A</td>
<td>Only research models</td>
<td>No absolute validity</td>
</tr>
</tbody>
</table>
Trade-offs!
It is not only accuracy that counts...

<table>
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<tr>
<th>Application</th>
<th>Key requirements</th>
<th>Examples</th>
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<td>Understanding phenomena</td>
<td>• Construct / face validity</td>
<td>Flow instability, train delay propagation analysis</td>
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<tr>
<td></td>
<td>• Analytical properties</td>
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<tr>
<td>Off-line assessment of (ITS) measures</td>
<td>• Predictive validity</td>
<td>Evacuation assessment and optimization</td>
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<td>State estimation (Kalman filters)</td>
<td>• Computational properties</td>
<td>Lagrangian multi-class modeling</td>
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<tr>
<td></td>
<td>• Content validity</td>
<td></td>
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<tr>
<td>On-line prediction and scenario assessment</td>
<td>• Predictive validity</td>
<td>Fastlane Multiclass Traffic macro model</td>
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<tr>
<td></td>
<td>• Computation speed</td>
<td></td>
</tr>
<tr>
<td>On-line optimization</td>
<td>• Computation speed / properties?</td>
<td>Reduced models, smart reformulations (Le et at, 2013)</td>
</tr>
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</table>
Reformulate and simplify

...or conservation of misery?

- Reformulation can lead to models with more favorable mathematical / computational properties
- Simplified models allowing favorable computational techniques:
  - Decomposition the NP-hard evacuation instruction optimization problem into three simple subproblems
  - Reformulating non-linear optimization problem for MPC control of urban networks as a LQ optimization problem (Le et al, 2013), or approximating it as a MILP problem (Bart De Schutter)
- Learning for the resulting optimal solutions:
  - Deriving heuristics for controlling motorway arterials (Specialist speed-limit controllers) or networks (Praktijkproef Amsterdam)
Instruction optimization

- Objective: get out as many inhabitants within \([0,T]\):

\[ J(u) = \int_{0}^{T} q(t) \, dt \]

- Bi-level problem: instructions yield response from evacuees and result in traffic operations
Simplifying the problem
Using decoupling of the problem...

Optimization of turning fractions

- Upper and lower bounds on turning fractions

Approximation of compliance behavior

- Instructed turning fractions
- Realized turning fractions

Optimization of route advice

- Intermediate optimized turning flows

Small reduction of effectiveness
- Very large impact on computation speed (upto 100 for simple Walcheren network)
- Application to other problems likely

The Math of Traffic
Final words...

*Stochastic nature of traffic*
Some final remarks...

Almost there!

- Importance of model choice in relation to application!
  - Ensure that your model captures the phenomena that are relevant for your application (e.g. optimization of ramp-meter signal requires a model to capture the capacity drop and spill-back!)
  - Think what type of validity you need (face, content, predictive) and which trade-off you need to make between accuracy / performance

- Still many challenges left to solve:
  - in modeling (predictive validity of microscopic models, modeling for safety assessment, modeling for ITS)
  - in estimation (making sense of all these data) and prediction
  - in optimization (network-wide control approaches anticipating on behavioral adaptation)
Innovations in data collection

- Development of a Virtual Traffic and Travel laboratory (VTT-Lab) for collecting data under a variety of experimental conditions