Combined diffusion-driven and convective flow in a tilted square container

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Asymptotic analytical solutions are derived for the configuration posed by Quon [Phys. Fluids, 26, 632 (1983)] for diffusion-driven flow in a tilted square container when the diffusive parameter $R$ is small. The key regions of the asymptotic structure are outlined and leading-order solutions are determined in most of those regions. The analysis follows that in Page & Johnson [J. Fluid Mech., 629, 299 (2009)] and Page [in press, Quart. J. Mech. Appl. Math., 2011] but includes an additional ‘$R^{1/4}$-layer’ region. Analytical solutions are compared with numerical results for small $R$ and display excellent agreement. It is also shown that the solutions and flow structure are applicable over a wide range of Prandtl numbers.

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I. INTRODUCTION

Wunsch¹ and Phillips² showed that slow upslope flow can be generated in an otherwise quiescent linearly stratified fluid when an insulating boundary surface is tilted at an angle from horizontal (where gravity is vertical). As explained by Phillips³, this flow arises due to local bending of isotherms near an insulating surface, with steady motion maintained when the resulting buoyancy forces balance the viscous drag from the wall. Quon⁴ provided a test-bed for the phenomenon by examining the flow in a tilted square container with a linear temperature profile specified on two opposite boundaries and insulating conditions on the other two sides. Near the two insulating boundaries motion resulted from the same process as in Wunsch¹ and Phillips² while near the other boundaries it was driven by convection, due to differences between the boundary and internal temperatures. Numerical results by Quon³ show that the resulting flow was quite complicated and subsequently Quon⁴ was unable to find analytical solutions apart from when the insulating boundaries were either almost vertical or almost horizontal. Ulloa & Ochoa⁵ also examined numerical solutions for the same geometry at moderate Rayleigh numbers but with insulating conditions on one, two or three of the sides.

More recently, so-called ‘diffusion-driven flows’ have become of renewed experimental interest, with a detailed analysis by Peacock et al.⁶ for a range of slopes of the insulating boundary. Novel experiments by Allshouse et al.⁷ also show that the same phenomenon can drive the horizontal motion of a neutrally-buoyant wedge-shaped object in a stratified fluid. On the theoretical side, for small values of Wunsch’s parameter $R$ (which describes the relative importance of diffusive to buoyancy effects) Page & Johnson⁸,⁹ outline asymptotic theories for linear and nonlinear ‘diffusion-driven flows’. Page¹⁰ provides more details of that theory and applies it to some additional cases.

In this paper, the approach of Page¹⁰ is used to determine asymptotic solutions at $R ≪ 1$ for the problem posed in Quon³,⁴ at almost all tilt angles $\beta$. This enables the interaction of convection with a simple ‘diffusion-driven flow’ to be examined, and elucidates the latter in a closed container. In the process this analysis also reveals some important differences from previous studies, such as Page¹⁰, which concentrated on the ‘diffusion-driven’ effects alone. In particular, the differences in the boundary conditions require that an additional flow region, referred to as an $R^{1/4}$ layer here, is needed to complete the analysis. The outcome is a complete asymptotic flow structure for $R ≪ 1$ and over a range of values of the Prandtl number $\sigma$, with appropriate matching of analytical solutions between all regions. Apart from in two small parameter ranges for $\beta$, this completes the unsolved analytical problem posed by Quon⁴. The solutions are consistent with numerical calculations for the streamlines and temperature profiles from solving the full nonlinear equations when $R = 10^{-4}$ and $\sigma = 1$. The flow structure and the analytical solutions assist in understanding the processes that drive the motion, and explain some of the complicated flow features apparent in the numerical results of Quon⁴.

II. GOVERNING EQUATIONS AND GEOMETRY

Consider the steady two-dimensional flow of an incompressible viscous stratified fluid in a square container that has sides of length $L^*$ and is inclined at an angle $\beta > 0$. A Cartesian coordinate system $(x^*, z^*)$ is defined with the gravitational acceleration $g^*$ in the negative $z^*$ direction. The corresponding velocity components are denoted as $(u^*, w^*)$. A steady variation of the temperature $T^*(x^*, z^*)$ is maintained by specifying boundary conditions on the normal temperature gradient around the container walls, and the resulting temperature variations within the container drive a broad-scale motion. The Boussinesq approximation is used, based upon a constant background density $\rho_{00}$. The coefficient of thermal

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expansion $\alpha^*$, thermal diffusivity $\kappa^*$ and kinematic viscosity $\nu^*$ are all taken to be constant.

The non-dimensional governing equations are obtained by scaling lengths with $L^*$, so that $(x, z) = (x^*, z^*)/L^*$. A suitable temperature scale $\Delta T^*$, for example based upon $L^*$ and the imposed temperature gradient on the boundaries, is used to define a non-dimensional temperature variation $T(x, z)$ such that

$$T^*(x^*, z^*) = T_{00} + (\Delta T^*) \, T(x, z),$$

where $T_{00}$ is some ‘background’ temperature. Velocity components $(u, w)$ for the flow are non-dimensionalized using $U^* = N^* L^*$, based upon the buoyancy frequency $N^* = (g^* \alpha^* \Delta T^*/L^*)^{1/2}$. Variations of pressure $p$ from a hydrostatic background are scaled using $\rho_0^* (N^* L^*)^2$.

In terms of the parameter $R = \sqrt{\nu^* \kappa^*/N^* L^4}$ used by Wunsch and the Prandtl number $\sigma = \nu^*/\kappa^*$, the governing equations for steady flow are

$$\begin{align*}
u u_x + w u_z &= -p_x + R \sqrt{\sigma} \nabla^2 u, \\
u w_x + w w_z &= -p_z + R \sqrt{\sigma} \nabla^2 w, \\
u T_x + w T_z &= (R/\sqrt{\sigma}) \nabla^2 T,
\end{align*}$$

where dependent variables in a subscript relate to derivatives and $\nabla^2$ is the usual two-dimensional Laplacian. The continuity equation $u_x + w_z = 0$ allow a streamfunction $\psi$ to be defined with

$$u = \psi_z \quad \text{and} \quad w = -\psi_x.$$  

In this paper, differences in $\psi$ values are used to describe the nondimensional ‘mass flux’ between points of the two-dimensional flow field.

The nonlinear equations (2) are identical to those in Page & Johnson and equivalent to (1-3) in Wunsch when his $\epsilon = 1$. It is assumed that $R \ll 1$ here, with $\sigma$ taken to be $O(1)$ (although it will be seen in Sec. V that large values of $\sigma$ can also be handled). Quon presented numerical results based on equivalent unsteady equations, written in terms of his parameter $A = 1/R^2$.

Quon assumed the large $A$ and large $\sigma$ limit and neglected the left-hand sides of (2a-2b), describing the flow regions in terms of his $\epsilon = R^2$ and $\mu = R$.

As in Quon, a steady density variation is imposed by requiring that $T$ is linear in $z$ on the upper left and lower right boundaries of the tilted square container, as shown in Fig. 1 when $\beta = \pi/4$, and the diffusion-driven flow is generated by applying $T_n = \partial T/\partial n = 0$ on the other two edges, where $n$ is the outward normal. Without loss of generality, the average value of $T$ around the boundary is taken to be zero with $T = (z - z_0)$ on the two fixed-temperature edges, where $z_0$ is the vertical midpoint of the container. Nonslip boundary conditions are applied for $(u, w)$ on all boundaries, with $\psi = \partial \psi/\partial n = 0$.

These boundary conditions differ from those in Page, where $T_n$ was specified on every edge, but much of the flow structure remains similar – apart from in one new region. However, the fixed-temperature conditions lead to some more complicated flow features.

![Figure 1](image.jpg)

**FIG. 1.** The configuration in Sec. III, where $\beta = \pi/4$, showing the flow structure for $R \ll 1$. The insulating boundaries on which $T_n = 0$ are shaded and the internal arrows show the flow direction.

### III. THE OVERALL FLOW STRUCTURE FOR $\beta = \pi/4$

To illustrate the typical structure of these flows when $R \ll 1$, the $\beta = \pi/4$ case of the problem is considered initially. Other values of $\beta$ are considered in Sec. IV.

As in Page & Johnson, the origin of a Cartesian coordinate system $(x, z)$ is at the lower corner of the container, see Fig. 1. A non-trivial steady flow is induced by imposing boundary conditions $T_n = 0$ on the lower left-hand boundary $x = x_-(z)$ over $0 < z < z_0$, where $z_0 = 1/\sqrt{2}$, with $T = (z - z_0)$ on the lower right-hand boundary $x = x_+(z)$ over $0 < z < z_0$. For all cases considered in this paper the boundary conditions have a rotational symmetry about the midpoint $(0, z_0)$, with $T(x, z) = -T(-x, 2z_0 - z)$ and $\psi(x, z) = \psi(-x, 2z_0 - z)$, and therefore solutions are listed only for $z \leq z_0$.

Page & Johnson outline the three main regions that occur for diffusion-driven flow when $R \ll 1$ in the situation where $T_n$ is specified on every boundary. These are: broad-scale ‘outer flow’ regions in which $T$ and $w$ are functions of $z$ to leading order; thin ‘buoyancy layers’ of thickness $O(R^{1/2})$ on the sloping boundaries; and ‘$R^{1/3}$ layers’ of thickness $O(R^{1/3})$ located near lines of constant $z$ at which $T_n$ is discontinuous. Page analyzed those regions in more detail and also described a fourth $O(R^{1/2} \times R^{1/2})$ ‘corner region’, which is relatively passive to leading order but can affect the higher-order flow.

In Sec. III(D) it is shown that a fifth type of region, referred to as an ‘$R^{1/4}$ layer’ here, is required for the type of problem considered by Quon, where $T$ is specified on some boundaries. Like the ‘$R^{1/3}$ layers’ these occur near lines of constant $z$ but they are wider, with thickness of $O(R^{1/4})$. Quon had also noted the presence of these for the $\beta \approx \pi/2$ case in his Sec. IV(B).
A. The outer-flow region

The recirculating flow for the problem here is effectively driven by the mass flux along the two buoyancy layers for which \( T_n = 0 \), shown as shaded on the lower left and the upper right-hand side of the container in Fig. 1. This has similarities to the cases examined in Page\(^6\), where an \( O(R) \) mass flux was generated through imposed \( O(1) \) variations of the normal temperature gradient \( T_n \) all around the boundary. As for the case in Page & Johnson\(^6\), a broader ‘outer flow’ motion occupies the main body of the container, where \( (x, z) \) is \( O(1) \), the velocities \( (u, w) \) are \( O(R) \) and temperature variations \( T \) are \( O(1) \). As in Page\(^10\), the solution in that region for \( R \ll 1 \) is expanded as

\[
\begin{align*}
p &= p_0(z) + R^{1/2} p_1(x, z) + \ldots, \quad (4a) \\
T &= T_0(z) + R^{1/2} T_1(x, z) + \ldots, \quad (4b) \\
\psi &= R \psi_2(x, z) + R^{1/2} \psi_3(x, z) + \ldots, \quad (4c)
\end{align*}
\]

where \( \psi \) is given by (3) and the subscripts are used here to indicate the relevant power of the expansion parameter; in this case \( R^{1/2} \). From substitution into (2),

\[
\begin{align*}
0 &= -p_0 x, \quad 0 = -p_0 z + T_0, \quad (5a) \\
w_2 T_0 x + w_2 T_0 z &= (1/\sqrt{\sigma}) \nabla^2 T_0, \quad (5b)
\end{align*}
\]

where double subscripts are used to indicate partial derivatives of terms in (4). It follows from (5) that \( p_0, T_0 \) and \( w_2 \) are all independent of \( x \). The fixed-temperature boundary conditions on the two sides imply that

\[
T_0 = f_0(z) = (z - z_0)
\]

and hence \( p_0 = \frac{1}{2} (z - z_0)^2 \) and \( w_2 = 0 \). This produces no leading-order motion itself but the \( T_n = 0 \) condition on the other two sides will lead to an \( O(R) \) mass flux along those edges, with \( \psi_2(x, z) = g_0(z)/\sqrt{\sigma} \) found to be a nonzero constant.

As in Page\(^10\), \( g_0(z) \) is determined from the boundary values \( \psi_{E \pm}(z) \) of the (unscaled) streamfunction at the outer edges of the buoyancy layers near \( x = x_\pm \). (As shown in Fig. 1, the minus sign subscript refers to the left-hand boundary and the plus to the right-hand boundary.) Since \( w = -\psi_z \) is independent of \( x \) to \( O(R) \), it follows that

\[
R w_2 = -\psi_{E+}(z) - \psi_{E-}(z) = 0
\]

so \( \psi_{E+}(z) = \psi_{E-}(z) \) to that order, and hence

\[
R g_0(z) = \sqrt{\sigma} \psi_{E-}(z) = \sqrt{\sigma} \psi_{E+}(z).
\]

The resulting leading-order flow-field is similar to that for Case 1b in Page\(^10\) (but flipped horizontally), with no motion in the outer region and an \( O(R) \) mass flux circulating clockwise around the boundary.

More generally, outer-flow motion can occur at higher order due to temperature variations along the buoyancy layers. At the next order \( T_1 = f_1(z) \) again depends on \( z \) only, with \( w_3 = f_1''(z)/\sqrt{\sigma} \) and

\[
\psi_{E+}(z, x) = \frac{(z - x_0) f_1''(z)}{\sqrt{\sigma}} + g_1(z)/\sqrt{\sigma},
\]

where \( x_0(z) = (x_+ + x_-)/2 \) is the vertical centerline of the domain. This solution applies for any value of \( \beta \) but for \( \beta = \pi/4 \) it has \( x_0 = 0 \). It is also equivalent to (3.11) of Page & Johnson\(^6\), based on their linearized theory when their \( \epsilon \) is \( O(R^{1/2}) \). As in Page\(^10\), \( f_1 \) and \( g_1 \) in (9) are determined by higher-order terms in \( \psi_{E \pm}(z) \).

It now remains to determine the actual values of \( \psi_{E \pm}(z) \) at the outer edges of the buoyancy layers, depending on the boundary conditions imposed on either \( T \) or \( T_n \).

B. The buoyancy layers

The ‘buoyancy layers’ are the most important regions of these flows as they provide the driving force for the circulation, via the specified normal temperature gradient of \( T_n = 0 \) on the sloping boundaries. As shown by both Wunsch\(^1\) and Phillips\(^2\), these regions have a scale thickness of \( O(R^{1/2}) \) when \( R \ll 1 \). In general, motion arises in them because the imposed values of \( T_n \) do not match the leading-order outer-flow temperature \( T_0(z) \), which leads to localized temperature perturbations of \( O(R^{1/2}) \). Those variations induce local velocities of \( O(R^{1/2}) \) through buoyancy forces, which then generate a mass flux of \( O(R) \) around the container. A similar effect occurs near edges where \( T \) is specified, but in that case motion is driven by the convection which results from the \( O(R^{1/2}) \) temperature differences between the boundary and the outer flow.

On each sloping boundary at an angle \( \alpha \) from horizontal, a rotated coordinate system \((\hat{x}, \hat{z})\) is defined, with \( \hat{x} = x \cos \alpha + z \sin \alpha \) and \( \hat{z} = -x \sin \alpha + z \cos \alpha \). From Fig. 1, \( \alpha_+ = -\pi/4 \) on the lower left-hand boundary and \( \alpha_+ = \pi/4 \) on the lower right-hand side. The corresponding velocity components are \( (\hat{u}, \hat{w}) \), with \( \psi = 0 \) and \( \hat{u} = \hat{w} = 0 \) on \( z = 0 \). For \( R \ll 1 \) and \( \alpha \neq 0 \) an expansion of the buoyancy-layer solution is sought with

\[
\begin{align*}
T &= \hat{T}_0(\hat{x}, \zeta) + R^{1/2} \hat{T}_1(\hat{x}, \zeta) + \ldots, \quad (10a) \\
\hat{u} &= R^{1/2} \hat{u}_1(\hat{x}, \zeta) + R^{3/2} \hat{u}_2(\hat{x}, \zeta) + \ldots, \quad (10b) \\
\hat{w} &= R \hat{w}_2(\hat{x}, \zeta) + R^{3/2} \hat{w}_3(\hat{x}, \zeta) + \ldots, \quad (10c) \\
\psi &= R \hat{\psi}_2(\hat{x}, \zeta) + R^{3/2} \hat{\psi}_3(\hat{x}, \zeta) + \ldots, \quad (10d)
\end{align*}
\]

in terms of a boundary-layer coordinate \( \zeta = \hat{z}/R^{1/2} \), with \( \hat{u}_1 = \hat{w}_2 = 0 \) for example.

Details of the solution are in Page\(^10\) but \( \hat{T}_0 \) is a function of \( \hat{z} \) only with \( \hat{T}_0(\hat{z}) = \sin \alpha \hat{T}_0(z) \) from matching to the outer flow at the same height \( z \). At the next order,

\[
\begin{align*}
\hat{u}_1(\hat{x}, \zeta) &= \hat{U}_1(x) \exp(-\zeta/\delta) \sin(\zeta/\delta) \quad \text{and} \quad (11a) \\
\hat{T}_1(\hat{x}, \zeta) &= 2\sqrt{\zeta} \hat{U}_1(x) \exp(-\zeta/\delta) \cos(\zeta/\delta) / \delta^2 \sin \alpha + \zeta \cos \alpha + \hat{T}_1(x), \quad (11b)
\end{align*}
\]
in terms of $\delta = \sqrt{2/|\sin \alpha|}$, where $\bar{U}_1(\bar{x})$ is determined by the boundary conditions on $\zeta = 0$ and $\bar{T}_1(\bar{x})$ matches to the outer flow. For $T_1(\bar{x}, 0) = -T_n = 0$ the (unscaled) streamfunction $\psi_E = R \bar{\psi}_3(\bar{x}, \infty) = R \bar{\delta} \bar{U}_1(\bar{x})/2$ at the outer edge of the layer is therefore

$$\psi_E(z) = \frac{R}{\sqrt{\sigma}} \cot \alpha + O(R^{3/2}), \quad (12)$$

and on boundaries where $T_1(\bar{x}, 0) = 0$ it is given by

$$\psi_E(z) = -\frac{R^{1/2} \sgn(\sin \alpha)}{\sqrt{2\sigma |\sin \alpha|}} [T - T_0(z)] + O(R^{3/2}). \quad (13)$$

The latter is $O(R)$ when $T_1(z)$ is nonzero. Together, (12) and (13) allow the outer flow to be found without the need for a detailed consideration of the buoyancy layer.

Page$^{10}$ also examined the next-order terms for a case where $T_2(\bar{z})$ was constant, as it is here, and noted that no additional $O(R^{3/2})$ mass flux leaves the edge of the buoyancy layer in those circumstances. As a result, the errors in (12) and (13) are actually $O(R^2)$ here.

C. The outer-flow solution

Using (12) and (13) in (8) gives that $f_1(z) = \sqrt{2}$ and $g_0(z) = -1$ when $\beta = \pi/4$, and hence

$$T_1(z) = \sqrt{2} \quad \text{and} \quad \psi_2(\bar{x}, z) = -1/\sqrt{\sigma}. \quad (14)$$

Since there are also no $O(R^{3/2})$ terms in (12) and (13) in this case, it also follows that $\psi_3(x, z) = g_1(z)/\sqrt{\sigma} = 0$.

These positive temperature perturbations $T_1(z)$ for $z < z_0$ arise as a result of the diffusion-driven flow up the lower left-hand boundary, as indicated by the arrows in Fig. 1, but they also mean that for any fixed $z$ value the temperature is larger in the outer flow region than at the right-hand boundary. The resultant cooling of the nearby fluid at the right-hand boundary leads to convection down the slope on that side of the container, compensating for the same mass flux up the slope on the left-hand side. It is in that sense that the flow in this paper represents a combined diffusion-driven and convective motion.

It was noted earlier that $T$ must change sign across the horizontal centerline $z = z_0$, due to the symmetry of the problem, and therefore $T_1$ is discontinuous. Page & Johnson$^{8,9}$ considered flows with a discontinuity in $T_{22}$ across such lines and noted that these can be resolved via 'R$^{1/3}$ layers'. Page$^{10}$ examined these layers in more detail, and also showed they can also resolve a discontinuity of $T_2$ in the higher-order outer flow in some circumstances, using a similar analysis to Moore & Saffman$^{11}$ for 'E$^{1/3}$ layers' in rapidly-rotating flows. However, an $O(R^{1/2})$ discontinuity in $T$ here would require unreasonably-large temperature perturbations of $O(1)$ in the 'R$^{1/3}$ layers', due to the resulting $(m = -1)$ point singularity, and violate Moore & Saffman's hypothesis of 'minimum singularity'; like the 'Kutta condition' in airfoil theory, they anticipate that the pressure gradient arising from any discontinuity should be no larger than that in the rest of the flow.

Instead, the discontinuity in $T_1$ must be resolved in another way. For the equivalent rotating-fluid problem, Moore & Saffman$^{11}$ note that Stewartson$^{12}$ showed 'E$^{1/4}$ layers' can also be present in some circumstances, and by analogy (see Veronis$^{13}$, for example) that possibility should also be considered here. This can happen when $O(R^{3/2})$ terms on the right-hand side of (2c)

$$R^{3/2} f_1''(\bar{z}) = -\frac{\psi_E(\bar{z}) - \psi_E(\infty)}{x_+(\bar{z}) - x_-(\bar{z})} \quad (15)$$

balance the $O(R)$ vertical velocities on the left-hand side over a region of thickness $O(R^{1/4})$ near $z = z_0$. This also addresses an important difference between the outer-flow solutions here and those in Page & Johnson$^{8,9}$, as the latter were always solutions of second-order differential equations and included some arbitrary constants that could be used to ensure $T$ and $T_2$ were continuous.

D. The R$^{1/4}$ layers

As noted above, for the mixed boundary conditions in Quon's problem a layer of thickness $O(R^{1/4})$ is needed near $z = z_0$, called an 'R$^{1/4}$ layer' here. These layers were not present for the problems in Page & Johnson$^{8,9}$ and Page$^{10}$, where $T_n$ was specified all around the container, but they can exist whenever $\psi_E$ is given by (13) on at least one boundary.

Since $T$ variations are of size $O(R^{1/2})$ over an $O(R^{1/4})$ length scale in $z$ in this case, it follows that both $w$ and $\psi$ are $O(R)$ with $u$ of $O(R^{3/4})$. Defining local coordinates $\bar{x} = (x + z_0)$ and $\bar{z} = (z - z_0)/R^{1/4}$, and noting $T_0(z_0) = 0$, a solution can be expanded as

$$p = R^{1/2} \hat{z}^2/2 + R^{3/4} \hat{\rho}_3(\bar{x}, \bar{z}) + \ldots, \quad (16a)$$

$$T = R^{1/4} \hat{z} + R^{1/2} \bar{T}_2(\bar{x}, \bar{z}) + \ldots, \quad (16b)$$

$$\psi = R \bar{\psi}_4(\bar{x}, \bar{z}) + R^{3/4} \bar{\psi}_5(\bar{x}, \bar{z}) + \ldots, \quad (16c)$$

where $\bar{u}_3 = \bar{\psi}_4$ and $\bar{u}_4 = -\bar{\psi}_5$. From (2), all of $\hat{\rho}_3$, $\bar{T}_2$ and $\bar{w}_4 = \bar{T}_{222}/\sqrt{\sigma}$ are independent of $\bar{x}$ leading order. Using a similar approach to (15), $\bar{T}_2 = \bar{f}_2(\bar{z})$ satisfies the second-order differential equation

$$\bar{f}_2''(\bar{z}) = \bar{f}_2(\bar{z})/\sqrt{\bar{\sigma}} - 1/\sqrt{2} \quad (17)$$

when $\bar{z} < 0$. Since $\bar{T}_2$ is zero at $\bar{z} = 0$ from symmetry and tends to $T_1(0^-) = \infty$ then

$$\bar{T}_2(\bar{z}) = \sqrt{2} [1 - \exp(\bar{z}/\delta)], \quad \text{where} \quad \delta = \sqrt{\bar{\sigma}}. \quad (18)$$

Using that $\bar{\psi}_4(0, \bar{z}) = -1/\sqrt{\sigma}$, based on (12), then

$$\bar{\psi}_4(\bar{x}, \bar{z}) = -[1 - (\bar{x}/\sqrt{2}) \exp(\bar{z}/\delta)]/\sqrt{\sigma}. \quad (19)$$
The solutions for $\tilde{z} > 0$ are obtained using the symmetry properties given earlier, and in particular note that both $T_2$ and $T_2'$ are continuous at $\tilde{z} = 0$.

From (19), $\tilde{w}_4 = \text{sgn}(\tilde{z}) \exp(-|\tilde{z}|/\delta) \sqrt{2/\sigma}$ and there is a vertical mass flux into both $R^{1/4}$ layers as $\tilde{z} \to 0$. Also, since $\tilde{w}_4 \to 0$ as $\tilde{z} \to -\infty$ and $\psi_4$ is positive in the lower layer, this $O(R)$ mass flux is carried towards the right of that region. At the edge of the right-hand buoyancy layer, where $\tilde{x} = \sqrt{2}$, (19) gives that

$$\tilde{\psi}_4(\sqrt{2}, \tilde{z}) = -[1 - \exp(\tilde{z}/\delta)]/\sqrt{\sigma}, \quad (20)$$

and this approaches zero as $\tilde{z} \to 0$. The right-hand buoyancy layer therefore starts empty near the right-hand corner and entrains fluid from the adjacent $R^{1/4}$ layer as $\tilde{z}$ decreases, with $\psi$ approaching its outer-flow value of $-R/\sqrt{\sigma}$ for large negative $\tilde{z}$. The fluid that is circulating around the boundary therefore enters the lower $R^{1/4}$ layer from above and feeds into the lower right-hand buoyancy layer, with the opposite happening in the upper $R^{1/4}$ layer and feeding into upper left-hand buoyancy layer.

For this to be possible there must be a thinner layer near $\tilde{z} = 0$ which feeds fluid into both the lower and upper $R^{1/4}$ layers. Also, since $\psi_4(0,0-) = -1/\sqrt{\sigma}$ and $\psi_4(0,0+) = 0$, there must be a point mass source near the left-hand corner $(x, z) = (-z_0, z_0)$, along with an equal source near the right-hand corner at $(z_0, z_0)$. This is a similar situation to that in Page & Johnson\textsuperscript{8,9}, where an $R^{1/3}$ layer near $z = z_0$ redistributes mass across the horizontal centerline of the container; although in their case it fed into the outer flow directly whereas here it feeds into the inner edges of the $R^{1/4}$ layers at $\tilde{z} = 0$.

In Fig. 2(a), the streamlines for the $R^{1/4}$-layer solution $\tilde{\psi}_4$ from (19) are shown for $R = 10^{-4}$. These illustrate the flow features described above, with the recirculating fluid diverted into an $R^{1/3}$ layer near $z = z_0$ via mass sources at the left and right-hand corners, and then fed back into the upper left and lower right-hand buoyancy layers via the two $R^{1/4}$ layers. There is also evidence of this same effect for Case 3 in Fig. 8(c) of Quon\textsuperscript{5} (at $R = 2.7 \times 10^{-4}$ and $\sigma = 7.14$) and perhaps more clearly in Fig. 2(b) here, where numerical solutions for the streamlines are shown for $R = 10^{-4}$ and $\sigma = 1$ (based on solving the full equations (2) using the same method as in Page\textsuperscript{10}). In Fig. 2(c) the solution $T_2(z)$ from (18) is plotted and it flattens out towards the top and bottom corners where the outer solution (14) applies. The corresponding temperature perturbations $[T - T_0]$ from the numerical solutions (truncated to a $25 \times 25$ grid) are shown in Fig. 2(d) and clearly the gradients in these are largest around the edges but then $[T - T_0]$ is a function of $z$ only over most of the domain. A comparison between the $R^{1/4}$-layer solution (18) and more-detailed numerical results is provided in Sec. IV(E) below.

**FIG. 2.** (a) Streamlines for the $R^{1/4}$-layer solution $\tilde{\psi}_4$ when $\beta = \pi/4$ at $R = 10^{-4}$ and $\sigma = 1$; (b) numerical solution for $\sqrt{\sigma} \psi/R$ from the full equations (2) at the same parameters; (c) the corresponding perturbation temperature $T_2(z)$; (d) numerical solution for the temperature difference $[T - T_0]$.

### E. The $R^{1/3}$ layers

As noted by Page & Johnson\textsuperscript{8,9}, thin horizontal $R^{1/3}$ layers can occur in the flow under some circumstances, for example when there are sources (or sinks) of mass at particular points on the boundary. Page\textsuperscript{10} also observed these layers in circumstances when there were no sources or sinks but for the problem here it is apparent from the $R^{1/4}$ layer flow near $z = z_0$ that they are needed to redistribute fluid across the container from sources in the left and right-hand corners.

Page & Johnson\textsuperscript{8} proposed that for an $R^{1/3}$-layer flow that arises from sources or sinks, an analysis similar to Moore & Saffman\textsuperscript{11} requires that $T$ and $T_2$ must be continuous across the width of the $R^{1/3}$ layers. In the current context that implies that both $f_2$ and $f_2'$ must be continuous between abutting $R^{1/4}$ layers at $\tilde{z} = 0$ (but that $\sigma \propto f_2''$ may be discontinuous). Page\textsuperscript{10} justified this proposition in more detail and also concluded that it remains valid for the nonlinear case as well.

As it happens, the solution (18) already has both $f_2$ and $f_2'$ continuous across $\tilde{z} = 0$ with $f_2''$ discontinuous such that $\Delta f_2'' = \sqrt{2}$. This leads to the observed difference in vertical velocity of size $\Delta w = R\sqrt{2/\sigma}$ between the outer edges of the $R^{1/3}$ layer, with a mass flux of $R/\sqrt{\sigma}$ flowing out above and below. The resulting streamlines for an $R^{1/3}$-layer flow are plotted in Fig. 3(a) using the sum of a pair of single-source solutions of the type given in Fig. 2(b) of Page\textsuperscript{10}.

When $\beta < \pi/4$ it will be seen below that a one-sided
FIG. 3. Streamlines for $R^{1/3}$-layer flows based on the solutions in §3.3 of Page\textsuperscript{10} for (a) unit mass sources at both ends with equal expulsion from both sides, as in Sec. III(E); (b) a single unit mass source with expulsion on one side only, as in Sec. IV(A3).

single-source solution is needed for each of the two $R^{1/3}$ layers that occur in that case, each of which have streamlines of the form shown in Fig. 3(b). That situation will be described in more detail in Sec. IV A(3) below.

F. $R^{1/2} \times R^{1/2}$ corner regions

As proposed by Quon\textsuperscript{4}, an additional region of radius $O(R^{1/2})$ can occur in the corners, where adjacent buoyancy layers meet. These type of ‘corner regions’ have also been described by Page\textsuperscript{10} for cases where $T_n$ is specified on all boundaries. For the problem posed here $\psi$ values are $O(R)$ in this region, with both velocity components $u$ and $w$ of $O(R^{1/2})$ and $T$ variations of $O(R^{1/2})$. Apart from the leading-order (hydrostatic) temperature and pressure contributions to (2b), it follows that all terms in (2) have the same relative magnitude and so the flow in the corner regions is governed by three coupled nonlinear equations.

For the $\beta = \pi/4$ case, Quon\textsuperscript{4} indicated that the overall flow could not be described without solving in these corner regions. The results here suggest that was overstating their importance, as the leading-order solutions above were able to be determined without any detailed consideration of them. For the small $\beta$ case Quon argued that ‘their influence was negligibly small’, and the same seems to be true for $\beta = \pi/4$ as well. That view is further supported by the analytical and numerical results in Page\textsuperscript{10} which suggest that the leading-order flow in each corner region is, in most cases, simply responsive to the leading-order flow in any surrounding buoyancy and $R^{1/3}$ layers. For example, as the flow in a buoyancy layer approaches a side corner region it has a typical buoyancy-layer velocity profile, and it then leaves via an $R^{1/3}$ layer that spreads out in accordance with the form of the similarity solution (3.22) in Page\textsuperscript{10}. The top/bottom corner regions have quite different flow features, as they mostly just turn the fluid by 90° between adjacent buoyancy layers, but they also appear to be passive at leading order.

Despite that, Page\textsuperscript{10} demonstrated the corner regions can sometimes be important at higher order and in particular that integrals over them can assist in determining unknown constants in the higher-order outer solution. In the current context a similar analysis suggests there would be an $O(R^{1/2})$ discontinuity in temperature gradient between the $R^{1/4}$ layers across $z = z_0$, with corresponding $O(R^{3/4})$ perturbations in $T$. As for Cases 1b and 2 in Page\textsuperscript{10}, that is associated with $\psi$ perturbations of $O(R^{7/6})$ which are driven by the left and right-hand corner regions – referred to as ‘corner-induced circulations’ in that paper. It is difficult to prove definitively that similar effects occur for the problem in this paper, as the corner integrals cannot be evaluated analytically for the boundary conditions here (as well as being a third-order term in $T$), but evidence presented in Sec. V indicates that ‘corner-induced circulations’ do exist at higher order for this problem.

With that caveat, it will be assumed in this paper that the corner regions are essentially passive in most situations, and that the leading-order solutions over most of the container can be determined without the need for any detailed analysis of the corners.

IV. QUON’S PROBLEM AT OTHER $\beta$ VALUES

The flow configuration described in Sec. III is now considered at other values of $\beta$. There are three outer-flow regions in this case but using symmetry it is only necessary to consider the lower region $0 < z < z_1$, where $z_1 = \min\{\sin \beta, \cos \beta\}$, and the middle region up to the horizontal centerline, for $z_1 < z \leq z_0$. Much of the analysis in Secs. III(A-C) remains valid for the lower region when $\beta \neq \pi/4$ but the cases $\beta < \pi/4$ and $\beta > \pi/4$ are examined separately below because the expressions for $\psi_E(z)$ in the middle region will differ in those cases, and hence the nature of the solution changes.

A. Case 1: $0 < \beta < \pi/4$

The flow structure for this case is shown in Fig. 4(a) using the same color-coding as in Fig. 1. Details that differ from those in Sec. III are outlined below.
1. Outer flow

In the lower region of the outer flow the slope of the left-hand boundary is \( \alpha_- = \beta - \pi/2 \), and (12) gives that \( \psi_{E-}(z) = -R \tan \beta/\sqrt{\sigma} \). Similarly, (13) for \( \alpha_+ = \beta \) implies that \( \psi_{E+}(z) = -R T_1(z)/\sqrt{2\sigma \sin \beta} \) on the right-hand boundary. As in Sec. III(C),

\[
T_1(z) = \sqrt{2 \sin \beta \tan \beta} = T_{1\infty} \quad \text{(21a)}
\]

\[
\psi_2(x, z) = -\tan \beta/\sqrt{\sigma} = \Psi_{2\infty} \quad \text{(21b)}
\]

for \( 0 < z < z_1 \). Since the boundary conditions in this lower region are the same for all values of \( \beta < \pi/2 \), this part of the solution applies for \( \beta \geq \pi/4 \) as well.

In the middle region of the outer flow, for \( z_1 < z \leq z_0 \), the boundary conditions at both ends of the \( x \) range are based on \( T_n = 0 \), and hence \( \psi_{E\pm}(z) = -R \tan \beta/\sqrt{\sigma} \) with an error of at most \( O(R^2) \). From (15), it follows that the outer-flow streamfunction is again constant in this case, including at \( O(R^{3/2}) \), and that \( T_1''(z) \) must be zero. The solution which has \( T_1 \) continuous at \( z_1 \) is

\[
T_1(z) = T_{1\infty}(z - z_0)/(z_1 - z_0) \quad \text{(22a)}
\]

\[
\psi_2(x, z) = \Psi_{2\infty} \quad \text{(22b)}
\]

over \( z_1 < z \leq z_0 \). At the next order \( T_2(z) \) is also linear in \( z \) and \( \psi_3(x, z) \) is zero, but they are not needed here.

2. \( R^{1/4} \)-layer flow

The outer-flow solution (21a) has \( T \) continuous at \( z_1 \), but \( T_2 \) has a discontinuity of \( O(R^{1/2}) \). From Sec. III(E), \( T_2 \) should be continuous across the \( R^{1/4} \) layer at \( z = z_1 \), and as in Sec. III(D) that requires consideration of nearby \( R^{1/4} \) layers. A suitable expansion in such a layer, using a local variable \( \tilde{z} = (z - z_1)/R^{1/4} \), has

\[
T = T_0(z_1) + R^{1/4} \tilde{z} + R^{1/2} T_1(z_1) + R^{3/4} T_3(\tilde{z}) + \ldots ,
\]

and as in Sec. III(D) it follows that \( T_3 = f_3(\tilde{z}) \) satisfies

\[
f_3''(\tilde{z}) = \frac{\cos \beta}{2 \sin \beta} f_\beta(\tilde{z}) \quad \text{(23)}
\]

when \( \tilde{z} < 0 \), using that \( (x_+ - x_-) = \sec \beta \). A similar approach for \( \tilde{z} > 0 \) implies that \( f_\beta'(\tilde{z}) \) is zero everywhere and hence that no \( R^{1/4} \) layer is present on the upper side of the \( R^{1/3} \) layer near \( z = z_1 \). From matching with (22a), it also follows that \( f_\beta'(\tilde{z}) = 2T_{1\infty}/(\cos \beta - \sin \beta) \) for all \( \tilde{z} > 0 \), and that the suitable solution of (23) is

\[
T_3(\tilde{z}) = -\frac{2\delta T_{1\infty}}{\cos \beta - \sin \beta} \exp(\tilde{z}/\delta) \quad \text{(24)}
\]

for \( \tilde{z} < 0 \), where \( \delta = \sqrt{2 \tan \beta \sec \beta} \). The corresponding \( O(R^{5/4}) \) perturbation \( \tilde{\psi}_3 \) to the streamfunction is

\[
\tilde{\psi}_3(x, \tilde{z}) = -\tilde{x} T_3'(\tilde{z})/\sqrt{\sigma} \quad \text{(25)}
\]

for \( \tilde{z} < 0 \). The corresponding streamfunction \( \tilde{\psi}_5 \) given by (25) is shown in Fig. 4(e) for \( \beta = \pi/6 \) and \( R = 10^{-4} \), and it remains accurate as \( \beta \to \pi/4 \) and yields (19) in

![FIG. 4. (a) The flow structure for \( 0 < \beta < \pi/4 \), using the same labeling as in Fig. 1; (b) streamlines for the numerical solution of the full equations when \( \beta = \pi/6 \) at \( R = 10^{-4} \); and \( \sigma = 1 \); (c) surface plot of the \( R^{1/4} \)-layer solution \( \sqrt{\sigma} \tilde{\psi}_5 \), at the same parameters; (d) corresponding plot of the numerical solution \( \sqrt{\sigma} \Delta \psi/R^{5/4} \), where \( \Delta \psi = (\psi - R\Psi_{2\infty}) \).](image-url)
that limit. In Fig. 4(d) the equivalent plot based on the numerical solutions of the full equations is shown (with the buoyancy-layer values omitted for clarity) and there is clear correspondence with the features of the $R^{1/4}$-layer solution in Fig. 4(e), albeit with a smoothing effect due to the finite value of $R$.

3. $R^{1/3}$-layer flow

As noted above, the mass flux associated with the $R^{1/4}$-layer circulation is $O(R^{5/4})$ when $\beta$ is more than $O(R^{1/2})$ below $\pi/4$. To close the mass flux, an $R^{1/3}$-layer flow with horizontal velocities of $O(R^{11/12})$ is needed to feed some fluid from the side corners into the $R^{1/4}$ layers, and from there back into the buoyancy layers. As in Fig. 3(b), this $R^{1/3}$-layer flow expels fluid from only one side, since the motion is stagnant in the middle region of the outer flow. Unlike in Sec. III(D), this diversion is smaller than the overall $O(R)$ mass flux that proceeds clockwise around the boundaries and therefore most of the fluid passes through both side corner regions without being transported into the $R^{1/3}$ and $R^{1/4}$ layers.

In the context of $R^{1/3}$ layers, it is worth observing some differences between this flow and that considered by Page\textsuperscript{10}, where $T_n$ was specified on every boundary. In that situation there was an $O(R^{7/6})$ ‘corner-induced circulation’ in the $R^{1/3}$ layers which resolved a discontinuity of $O(R^{1/2})$ across $z_1$, in the higher-order outer-flow temperature gradient $T_x$. Since $R^{1/4}$ layers cannot occur in that situation, it was not possible resolve the discontinuity in $T_x$ by any means other than the $R^{1/3}$-layer solution $u_n$ in his equation (3.29). In this paper, however, it is possible to resolve it via a $R^{1/4}$-layer flow and thereby reduce the strength of the singularity in the $R^{1/3}$-layer solution from $m = -2/3$ to $m = -1/3$, consistent with Moore & Saffman’s\textsuperscript{11} ‘minimum singularity’ hypothesis.

B. Case 2: $0 < \beta \leq O(R^{1/3})$

The small $\beta$ case is also of interest. In that situation, from (21b) and (22b), the overall circulation has a mass flux of $O(\beta R)$ around the buoyancy layers. Those layers have thickness of $O(R^{1/4})$ on the almost-vertical $T_n = 0$ boundaries and $O(R/\beta^{1/2})$ near the almost-horizontal boundaries on which $T$ is fixed. As $\beta$ decreases the $R^{1/4}$ layer in Sec. IV(A1) narrows to thickness $O(\beta R^{1/4})$ with temperature perturbations of $O(\beta^{1/2} R^{1/2})$ which match onto the higher-order outer-flow layer of the same size. The lower $R^{1/4}$ layer also circulates a smaller mass flux of $O(\beta R^{5/4})$ between the $R^{1/3}$ and the right-hand buoyancy layer.

From the analysis in §3.3 of Page\textsuperscript{10} the thickness of the $R^{1/3}$ layer near $z = z_1$, is independent of $\beta$, and it is the same size as both the lower buoyancy layer and the $R^{1/4}$ layer once $\beta$ is $O(R^{1/3})$. The circulating mass flux in that situation is $O(R^{3/3})$ and both temperature perturbations and horizontal velocities are $O(\beta R)$. Quon\textsuperscript{3} described this lower layer for $\beta \ll R^{1/3}$ in his Sec. IV(A), for which the lower boundary conditions can be linearized and applied on $z = 0$, and provided a series solution that matched both those boundary conditions and the flux condition $\psi_{E\pm} = 0$ at $x = 0, 1$. That solution assumed that the mass flux was $O(\beta R)$ and temperature perturbations were $O(\beta R^{2/3})$, both of which have the same size as above once $\beta$ is $O(R^{1/3})$. It was also consistent with the features of the numerical results in Fig. 9 of Quon\textsuperscript{3}.

More generally, when $\beta$ is $O(R^{1/3})$ the lower boundary conditions must be imposed on a sloping boundary at $z = \tan \beta \approx x \beta$, rather than linearized to $z = 0$. Analytical solutions are not available for that situation but numerical results indicate that the flow has broadly the same features described by Quon for $\beta \ll R^{1/3}$ except that the $T$ perturbations in the lower $R^{1/3}$ layer near $z = 0$ lose their antisymmetry about $x = x_0$ as $\beta$ increases and are mostly positive across that layer. The thickness and general form of the layer remains the same as for the larger $\beta$ case in Sec. IV(A2), however, including that a proportion of the fluid recirculates back along the outer edge towards the right-hand corner.

C. Case 3: $\pi/4 < \beta < \pi/2$

For $\beta > \pi/4$ there are two significant changes to the description earlier: first, the middle part of the outer flow has fixed temperature conditions at both ends $x_{\pm}$ and hence $T_0$ must be zero across that region; and second, there are exponentially-decaying $R^{1/4}$-layer solutions on both sides of the two $R^{1/3}$ layers. Despite those changes the broad flow structure illustrated in Fig. 5(a) is essentially the same as for $0 < \beta < \pi/4$ in Sec. IV(A).

1. Outer flow

The solution in the lower part of the outer-flow region is the same as described in Sec. IV(A1) given by (21). In the middle part of the region, where $z_1 < z < z_{1+}$, the boundary conditions at both ends of the $x$ range are based on the fixed $T$ conditions, and hence (13) gives

$$\psi_{E\pm}(z) = \mp R^{1/4}[T - T_0(z)]/\sqrt{2\sigma} \sin \beta$$

to leading order. From (8), it follows that $T_1(z)$ and $\psi_2(x, z)$ are both zero throughout the middle region.

The key feature of the $\beta > \pi/4$ case is immediately apparent from this leading-order solution. When $\beta$ is less than $\pi/4$ the flow is dominated by the $O(R)$ mass flux which circulates all around the boundary – rather like that for Cases 1b and 2 in Page\textsuperscript{10}, albeit with a diversion into the $R^{1/3}$ and $R^{1/4}$-layer structures. For $\beta > \pi/4$, however, the outer flow splits into two triangular gyres in which $\psi = R \psi_{2\infty}$ with no leading-order motion in the middle part. This is also evident in the streamlines of the...
2. $R^{1/4}$-layer flow

The outer-flow solution above has $T$ discontinuous but $T_2$ continuous at $z = z_1$, and so, as in Sec. IV(A2), it is necessary to consider $R^{1/4}$ layers near that level. The discontinuity in $T$ is $O(R^{1/2})$ so a suitable expansion in terms of $\tilde{z} = (z - z_1)/R^{1/4}$ has

$$T = T_0(z_1) + R^{1/4}\tilde{z} + R^{1/2}\tilde{T}_2(\tilde{x}, \tilde{z}) + \ldots$$

(27)

As in Sec. III(D), $\tilde{T}_2 = \tilde{f}_2(\tilde{z})$ for $\tilde{z} < 0$ satisfies

$$\tilde{f}_2'(\tilde{z}) = \sqrt{\sin \beta/2} \tilde{f}_2(\tilde{z}) - \sin \beta \tan \beta,$$  

(28)

since the layers have length $\cosec \beta$ and $\delta_0 = \sqrt{2}/\sin \beta$. For $\tilde{z} > 0$, $\tilde{f}_2'(\tilde{z}) = \sqrt{2\sin \beta} \tilde{f}_2(\tilde{z})$ with $\delta_+ = \sqrt{1/2}\sin \beta$. The required solution is

$$\tilde{T}_2(\tilde{z}) = \begin{cases} T_{1\infty}\left[1 - \sqrt{\frac{2\exp(\frac{\delta_0}{\sqrt{2}})}{1 + \sqrt{2}}} \right], & \tilde{z} < 0 \\ T_{1\infty}\left[\frac{\exp(-\frac{\delta_0}{\sqrt{2}})}{1 + \sqrt{2}}\right], & \tilde{z} > 0 \end{cases}$$

(29)

with streamfunction

$$\tilde{\psi}_4(\tilde{x}, \tilde{z}) = \begin{cases} \Psi_{2\infty}\left[1 - \sqrt{\frac{\sqrt{2}\sin \beta \exp(\frac{\delta_0}{\sqrt{2}})}{1 + \sqrt{2}}} \right], & \tilde{z} < 0 \\ \Psi_{2\infty}\left[\frac{\exp(\frac{\delta_0}{\sqrt{2}})}{1 + \sqrt{2}}\right], & \tilde{z} > 0 \end{cases}$$

(30)

where $\tilde{x} = (x + \sin \beta)$. Notice that $\tilde{\psi}_4$ is discontinuous at $\tilde{x} = 0$ here, in the corner, but continuous at $\tilde{x} = \cos \beta$. This is also apparent in the plot of $\psi_4$ in Fig. 5(c).

In contrast to the $\beta < \pi/4$ case, the corresponding vertical velocity $\tilde{w}_4$ is nonzero for $\tilde{z} > 0$ and, unlike the $\beta = \pi/4$ case, it is not antisymmetric. Instead, there is a different rate of outflow into the $R^{1/4}$ layers on each side of $\tilde{z} = 0$, with a total downwards mass flux of $\sqrt{2}R\tan \beta/\sqrt{\beta}$ on the lower side and an upwards flux of $\sqrt{2}$ times that on the upper side.

As in Sec. IV(B), (29) and (30) are not valid if $\beta$ is within $O(R^{1/4})$ of $\pi/4$ but in a similar manner to (26) that can be addressed by using a more-broadly applicable solution with $\tilde{T} \propto \sinh((z - z_0)/R^{1/4}\delta_+)$ when $\tilde{z} > 0$.

3. $R^{1/3}$-layer flow

From (30) at $\tilde{x} = 0$, there is a point mass source of strength $\sqrt{2}R\tan \beta/\sqrt{\beta}$ at the inner edge of the $R^{1/4}$ layers near the left-hand corner at $(x, z_0) = (-\sin \beta, \cos \beta)$. This has a mass flux of $\sqrt{2}$ times that from the diffusion-driven flow on the lower-left boundary. As is also evident from the streamlines for $\beta = \pi/3$ in Fig. 5(b), the other contribution to the source arises from the fluid which circulates between the upper $R^{1/4}$ layer near $z = z_1$, and the lower end of the upper left-hand buoyancy layer. This leads to a small counter-clockwise gyre just above the left-hand corner.

As noted above, the vertical velocity $\tilde{w}_4 = -\tilde{\psi}_4_{4\bar{z}}$ in the $R^{1/4}$ layers is a factor of $-\sqrt{2}$ different on each side of the $R^{1/3}$ layer from which it draws fluid. The resulting streamlines for the $R^{1/3}$-layer flow would therefore an asymmetric version of those in Fig. 2(b) of Page, although that level of detail is not evident in the streamlines on Fig. 5(b) – probably because the $R^{1/3}$ and $R^{1/4}$ layers have almost the same thickness when $R = 10^{-4}$. The corresponding smoothing is apparent when comparing the surface plot of the $R^{1/4}$-layer solution $\sqrt{\beta}\tilde{\psi}_4$ at the same parameters; (d) corresponding plot of the numerical solution $\sqrt{\beta}\Delta \psi/R$, where $\Delta \psi = (\psi - R\Psi_{2\infty})$.

As $\beta$ increases towards $\pi/2$ the size of the counter-clockwise gyre near the left-hand corner increases until it appears to dominate that corner region, and hence also affects the $R^{1/3}$ layer which extends along $z = z_1$ from that region. This is evident for $\beta = 80^\circ$ for Case 4 in Fig. 8(c) of Quon, at which point the lower $R^{1/4}$ layer is no longer very evident and the lower outer-flow region has essentially disappeared except very close to the bottom corner.

D. Case 4: $(\pi/2 - \beta) \leq O(R^{1/3})$

As $\beta$ approaches $\pi/2$ the mass flux of the diffusion-driven flow up the lower left boundary at $z = -x\cot \beta$ in
creases. In terms of small $\Delta \beta = (\pi/2 - \beta)$, it is $O(R/\Delta \beta)$ over a buoyancy layer of thickness $O(R/\Delta \beta)^{3/2}$.

Once $\Delta \beta$ has decreased to $O(R^{1/4})$ the lower outer-flow region has the same thickness as the lower $R^{1/4}$ layer and has essentially disappeared, but the buoyancy layer remains relatively thin and the perturbation temperature gradients are $O(1)$. Despite that, the structure of the flow is essentially the same as for larger $\Delta \beta$ apart from that the buoyancy layer matches directly on to the lower $R^{1/4}$ layer.

The $R^{1/3}$ layers near $z = z_l \approx \Delta \beta$ maintain a constant thickness of $O(R^{1/3})$ in this limit and the lower buoyancy layer merges with it once $\Delta \beta$ has decreased to $O(R^{1/3})$. At that stage the $R^{1/3}$-layer equations in Page 10 must be solved, although with a sloping lower boundary at $z = -x\Delta \beta$. Analysis of an equivalent ‘small $\alpha$’ problem is in progress for a simpler geometry, albeit without mismatched boundary conditions at the left and right-hand corners.

For the current configuration, however, it is clear from the numerical results that as $\Delta \beta$ decreases the size of the gyre near the left-hand corner increases. Quon 4 provides a solution as $\Delta \beta \to 0$ in his Sec. IV(B) and the streamlines in his Fig. 5(a) show that there are two equal-sized gyres in that case. His analysis is claimed to be valid for $\Delta \beta \ll R^{2/3}$ but in fact it remains accurate for $\Delta \beta \ll R^{1/3}$, beyond which the linearization of the lower boundary is no longer appropriate.

E. Comparison with numerical solutions for $0 < \beta < \pi/2$

As noted above, the analytical solutions are valid for $O(R^{1/3}) < \beta < \pi/2 - O(R^{1/3})$ when $R \ll 1$. To examine their validity at finite $R$ they are compared here with numerical solutions of the full equations for $R = 10^{-4}$ and $\sigma = 1$ (calculated using the same method as in Page 10).

From the $R^{1/4}$-layer solutions, the maximum value of the scaled temperature perturbation $[T - T_0]$ should be $R^{1/2}T_{\infty}$ outside of the buoyancy-layer and corner regions. For each $\beta$ the maximum value of the ratio $[T - T_0]/R^{1/2}T_{\infty}$ along a cross-section from the bottom to the top corner is plotted on Fig. 6(a), shown as a dotted line. These support the analysis for a broad range of $\beta$, but also indicate that it requires modification on the approach to either 0 or $\pi/2$, consistent with comments in Secs. IV(B) and IV(D).

The mean value of $\psi/R\Psi_{2\infty}$ over the container is also plotted in Fig. 6(a). For $\beta < \pi/4$ the small differences from one are attributable to the decreased $|\psi|$ values in the $R^{1/3}$ and $R^{1/4}$ layers, which become more significant as $\beta$ increases. For $\beta > \pi/4$ the outer-flow solution $\psi_2$ is zero over the middle part of the domain and $\psi/R\Psi_{2\infty}$ approaches $\cot \beta$ (dashed line) for $R \ll 1$.

The largest value of the buoyancy-layer solution (11a) is $R^{1/2}\Psi_{2\infty}\exp(-\pi/4)/\sqrt{2}$. This is shown on Fig. 6(b) by a dashed line and it increases with $\beta$ and tends to infinity as $\beta \to \pi/2$. Also shown is the maximum value of $|\psi|/R\Psi_{2\infty}$ over the container (dotted line) and for $\beta < \pi/4$ this is close to the predicted maximum. For larger values of $\beta$ the overall maximum is achieved near the top and bottom corners, where no analytical solutions are available for comparison, and so the maximum value of $|\psi_{\max}|/R^{1/2}$ across the ‘mid-sectional’ maximum velocity max $|\psi_{\max}|/R^{1/2}$ with estimated values of the latter (dashed line);

(c) numerical solutions for $[T_z - 1]/R^{1/2}$ along a cross-section from the bottom to the top corner and the analytical solution $T_z(z)$ (dotted lines) for $\beta/\pi = 0.1, 0.15, 0.2, 0.25$; (d) the same quantities for $\beta/\pi = 0.25, 0.3, 0.35, 0.4$.

FIG. 6. (a) Plots as a function of $\beta$ of max $([T - T_0]/R^{1/2}T_{\infty})$ (dotted) and the mean value of $\psi/R\Psi_{2\infty}$ over the container, based on numerical solutions of the full equations for $R = 10^{-4}$ and $\sigma = 1$, along with $R \ll 1$ estimates of the latter (dashed line); (b) the scaled maximum velocity max $|\psi/R^{1/2}|$ and ‘mid-sectional’ maximum velocity max $|\psi_{\max}|/R^{1/2}$ with estimated values of the latter (dashed line); (c) numerical solutions for $[T_z - 1]/R^{1/2}$ along a cross-section from the bottom to the top corner and the analytical solution $T_z(z)$ (dotted lines) for $\beta/\pi = 0.1, 0.15, 0.2, 0.25$; (d) the same quantities for $\beta/\pi = 0.25, 0.3, 0.35, 0.4$.

The maximum values of $|\psi_{\max}|$ in Fig. 6(b) are similar to those at the equivalent angle $\pi/2 - \beta$ in Fig. 4(d) of Page 10, and also the experimental results in Fig. 4 of Peacock et al. 4. One difference between this flow and that for Case 2 in Page 10 is that the fluid does not become stagnant as $\beta \to \pi/2$ here, due to a mismatch between the $T = (z - z_0)$ boundary condition on the almost-vertical surfaces and $T_n = 0$ on the almost-horizontal surfaces. The resulting nonzero circulation is apparent from the nonzero values of $|\psi|$ and $|\psi_{\max}|$ in Fig. 6(b) at $\beta = \pi/2$.

The $z$ dependence of the temperature-gradient perturbations $[T_z - 1]/R^{1/2}$ is another critical test for the anal-
The analysis in this paper was derived for $\sigma$ of $O(1)$, but with a scaling of $\psi$ by $\sqrt{\sigma}$ throughout it is also appropriate for larger values of $\sigma$, including $\sigma \gg 1$. As a result, all of the leading-order solutions can be used for the $\sigma \to \infty$ cases considered in Quon$^4$ without modification as well as for the numerical cases in Quon$^3$ and Ulloa & Ochoa$^5$, where $\sigma = 7.14$. Indeed, the only apparent difference between the $\sigma = O(1)$ and $\sigma \gg 1$ analysis is that the nonlinear terms in (2a-b) vanish for the latter and, as indicated above, that should principally affect the motion in the corner regions only. Numerical results show that the size of $\sigma$ has some influence on the strength of the higher-order ‘corner-induced circulation’, as is apparent from Fig. 7(b) which shows the percentage difference between the scaled streamfunction values $\sqrt{\sigma} \psi/R$ for the $\sigma \gg 1$ and fully nonlinear forms of (2). As for Fig. 7(a), these differences are attributable to higher-order details of the solution in the corners, most likely because the area integral of (2c) has different $T_n$ values on the fixed-$T$ boundary as $\sigma$ varies.

VI. CONCLUDING REMARKS

New asymptotic solutions have been determined for the unsolved problem posed by Quon$^3,4$, for combined diffusion-driven and convective flow in a tilted square container in the low-diffusion limit $R \ll 1$. The analysis uses a similar approach as Page & Johnson$^8,9$ and Page$^{10}$ but the flow field is more complicated and the flow structure includes an additional flow region, referred to as an ‘$R^{1/4}$ layer’ here. Analytical solutions have been compared with numerical solutions of the full equations when $R = 10^{-4}$ and $\sigma = 1$ and they show similar qualitative features, and in particular support the accuracy of the calculated temperature profiles. Apart from in the corner regions, the same leading-order analytical solutions are applicable for large values of $\sigma$.

The same analysis and flow structure can be applied to a broad class of problems where $T$ or $\partial T/\partial n$ are specified on boundaries. For example, it can be used to determine analytical solutions at large Rayleigh number $Ra$ for the various cases considered by Ulloa & Ochoa$^5$ and should give a clearer indication of the observed flow properties than their numerical results alone.

The current analysis is not applicable when the container has an slope within $O(R^{1/4})$ of either horizontal or vertical but separate work is underway on that issue.

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