ATMOSPHERIC SCIENCE AND GEOPHYSICAL FLUID DYNAMICS RESEARCH GROUP

The bathtub vortex

Supervisor: Michael Page

Background: It is a common myth that the direction of rotation of bath water as it drains down a plughole is determined by the rotation of the Earth – in particular the hemisphere in which it is performed. Careful experiments have been performed to confirm that the rotation of the Earth has only a miniscule effect on the motion (Shapiro, 1962; Trefethen *et al*, 1965) and it can also be demonstrated theoretically using simple scaling arguments. More recently, Tyvand & Haugen (2005) published a computational study of the problem and also demonstrated that the influence of the rotation of the Earth is negligible. But why does the water often flow down the plughole in a consistent direction, and why is that direction opposite in the Australia to what is commonly observed in the United Kingdom, for example?

Objectives: The project will review previous analytical, experimental and computational work on this problem for simplified geometries. Among other things, this will help identify the principal determinants of the direction of motion of the fluid at the outlet and assist in identifying the key physical principles based on a mathematical analysis of the equations of motion. A new type of model will then be examined in the context of the previous work. This will require both analytical and computational expertise.

Reading:

- Andersen A. *et al*, Anatomy of a bathtub vortex, *Phys. Review Letters* **91**, 104501, 2003.
- Shapiro, A.H., Bath-tub vortex, *Nature* **196**, pp1080-1081, 1962.
- Thefethen, L.M. *et al*, The bath-tub vortex in the southern hemisphere, *Nature* **207**, pp1084-1085, 1965.
- Tyvand, P.A. and Haugen, K.B., An impulsive bathtub vortex, *Phys. Fluids* **17**, 062105, 2005.
- Yukimoto, S. *et al*, Structure of a bathtub vortex: importance of the bottom boundary layer, *Theor. Comput. Fluid Dyn.* **24**, pp323-327, 2010.

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Diffusively-driven stratified flows

Supervisor: Michael Page

Background: In 1970, two independent studies (by Wunsch and Phillips) of the behaviour of a linear density-stratified fluid in a closed container showed that motion can be generated simply due to the container having a sloping boundary surface, and furthermore that the fluid flows uphill! This remarkable phenomenon is a result of the curvature of the lines of constant density near any sloping surface, in order that a zero normal-flux condition on the density to be satisfied along that boundary.

Since that time a number of studies have since considered the consequences of this type of 'diffusively-driven' flow, including in the deep ocean and with turbulent effects included. More recently, Peacock *et al* (2004) undertook an experimental study of the phenomenon in a closed container and Page & Johnson (2008, 2009) extended the work to consider the broader-scale mass recirculation that is generated.

Objectives: The project will review and compare previous analytical, experimental and computational work on this problem for various geometries. The analytical approach introduced in Page & Johnson (2008) will be used to predict the form of the steady flow for a variety of geometric configurations and the results compared with computational data from numerical solutions of the full governing equations. This will require both analytical and computational expertise, in particular a sound understanding of matched asymptotic expansions.

Reading:

- Page, M.A.. & Johnson, E.R., On steady linear diffusion-driven flow, *J. Fluid Mech.* **606**, pp433-443, 2008.
- Page, M.A.. & Johnson, E.R., Steady nonlinear diffusion-driven flow, *J. Fluid Mech.* **629**, pp299-309, 2009.
- Peacock, T., Stocker, R. & Aristoff, J. M., An experimental investigation of the angular dependence of diffusion-driven flow, *Phys. Fluids* **16**, pp3503-3505, 2004.
- Phillips, O.M., On flows induced by diffusion in a stably stratified fluid, *Deep-Sea Research* **17**, pp. 435-443, 1970.
- Wunsch, C., On oceanic boundary mixing, *Deep-Sea Research* **17**, pp293-301, 1970.

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APPLIED AND COMPUTATIONAL MATHEMATICS

Sloshing in a cylindrical cup

Supervisor: Michael Page

Background: If you carry around a full cup of tea or coffee it is quite difficult to avoid spilling it, and it is even more difficult when you have a hot drink on a plane that is encountering turbulence. In this project we look at the basic mechanisms for the surface motions of a liquid in a cylindrical container, starting with the (easier) small-amplitude case and examining the response at different frequencies. This involves solving partial differential equations in polar coordinates using analytical methods.

Objectives: The project will involve determining the linearised equations for small-amplitude wave motion in a container of constant depth, for example using the standard treatments in Acheson (1990) and/or Lighthill (1978). The problem will be considered for a cylindrical container and solved using separation of variables (involving Bessel functions). The consequences of that solution will be analysed and the results plotted for various initial conditions and types of forcing that illustrate the key properties of the solution, including its susceptibility to forcing at certain frequencies. Analytical and computational solutions will then be sought for the nonlinear problem.

Reading:

- Acheson, D.J., Elementary fluid dynamics, Oxford, 1990.
- Lighthill, M.J. Waves in fluids, Cambridge, 1978.

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Resolving corner singularities in viscous flow

Supervisor: Michael Page

Background: The solution of PDEs in domains with sharp corners can often involve singularities at those corners, where either the solution or some of its higher derivatives may become infinite as the corner is approached. A simple example of this is the solution for very viscous flow past a thin flat plate, for which Carrier and Lin (1948) demonstrated the velocity is proportional to $r^{1/2}$ as $r \to 0$ near the end of the plate. Similar types of behaviour can also occur near angular corners (Moffatt, 1966).

Numerical methods for solving PDEs typically assume that the solution is 'well-behaved' with all derivatives finite at every point in the domain and on the boundary, so they need to be modified when singularities are known to be present. A recent paper by Shi et al (2004) describes one way of doing this in the case of one particular viscous-flow problem, where the solution satisfies the 'bi-harmonic equation' $\nabla^4\psi = 0$ and the domain involves an infinitely-long flat plate. The aim of this project is to compare their method with some other possible approaches that also take into account our knowledge about the nature of the singularity at the leading edge of the plate.

Objectives: The project will commence with a review of some of the existing primary literature sources on both the nature of singularities of viscous flows at corners and the treatment of no-slip boundary conditions in numerical models of those problems. The approach used by Shi et al (2004) will be studied in detail for the infinite plate problems and their results reproduced for some of the cases which they have considered. Their results will also be compared with some other simpler approaches, and some new approaches trialled in comparison to 'best practice'

The aim of the project will be the development of a generalised numerical method for the treatment of corner singularities that is both mathematically sound and applicable to viscous flows past a variety of obstacles with complicated shapes – including both smooth curves and sharp corners – and also for a range of Reynolds numbers and external forces, including rotating and/stratified flows.

Reading:

- Carrier, G.F. and Lin, C.C., On the nature of the boundary layer near the leading edge of a flat plate, *Quart. Appl. Math.* **6**, pp63-38, 1948.
- Moffatt, H.K., Viscous and resistive eddies near a sharp corner, *J. Fluid Mech.* **18**, pp1-18, 1966.
- Shi, J.-M *et al*, A combined analytical-numerical method for treating corner singularities in viscous flow predictions, *Int. J. Numerical Meth. Fluids* **45**, pp659-688, 2004.

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The Immersed Interface Method for solving hyperbolic PDEs

Supervisor: Michael Page

Background: Most of the simple approaches to solving hyperbolic partial differential equations numerically using finite-difference techniques rely upon the solution being smooth and continuous everywhere in the domain. In addition, the domain must have a simple, regular geometry – such as a rectangle or a circle. Over the last decade or so, a couple of approaches have been developed which allow discontinuities of the solution and/or consider domains with an irregular shape. In the latter case, these methods can be a simpler alternative to using finite-element methods.

The two most common approaches used for these problems are the Immersed Boundary Method, which was first developed by Charles Peskin in the 1970s, and the Immersed Interface Method. Some aspects of these techniques are similar, but there are important differences. This project examines the basis of the Immersed Interface Method and uses it to examine the accuracy of the approach for some simple test problems with exact solutions.

Objectives: The project will review and compare some of the existing primary literature sources on both the Immersed Interface Method and the Immersed Boundary Method. Some trials of the Immersed Interface Method will be undertaken, initially based on the test problems in Li and Ito (2006) but then extended to a broader range of configurations and flow types, including some with moving discontinuities. The intention of the project will be to identify the advantages and limitations of the method, especially for wave propagation problems.

Reading:

- LeVeque, R.J. and Li, Z., The immersed interface method for elliptic equations with discontinuous coefficients and singular sources, *SIAM J. Numer. Anal.* **31**, pp1019-1044, 1994.
- Li, Z. and Ito, K., The Immersed Interface Method, *SIAM*, 2006.

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