ATMOSPHERIC SCIENCE AND GEOPHYSICAL FLUID DYNAMICS RESEARCH GROUP

The bathtub vortex

Supervisor: Michael Page

Level: Honours

Background: It is a common myth that the direction of rotation of bath water as it drains down a plughole is determined by the rotation of the Earth – in particular the hemisphere in which it is performed. Careful experiments have been performed to confirm that the rotation of the Earth has only a miniscule effect on the motion (Shapiro, 1962; Trefethen *et al*, 1965) and it can also be demonstrated theoretically using simple scaling arguments. More recently, Tyvand & Haugen (2005) published a computational study of the problem and also demonstrated that the influence of the rotation of the Earth is negligible. But why does the water often flow down the plughole in a consistent direction, and why is that direction opposite in the Australia to what is commonly observed in the United Kingdom, for example?

Objectives: The project will review previous analytical, experimental and computational work on this problem for simplified geometries. Among other things, this will help identify the principal determinants of the direction of motion of the fluid at the outlet and assist in identifying the key physical principles based on a mathematical analysis of the equations of motion.

Expectations:

- To undertake a thorough review of existing primary literature sources on this problem.
- To identify the dominant forces that affect the fluid motion, derive the corresponding governing equations and assess the relative size of the relevant nondimensional parameters.
- To simplify and solve the equations in appropriate cases, consider whether simplified models can adequately represent the phenomenon and perhaps undertake simple computational experiments.

Assumed knowledge:

- MTH3011 Partial differential equations; and
- MTH2051/3051 Introduction to computational mathematics; and
- MTH3360 Fluid dynamics.

Reading:

- Andersen A. *et al*, Anatomy of a bathtub vortex, *Phys. Review Letters* **91**, 104501, 2003.
- Shapiro, A.H., Bath-tub vortex, *Nature* **196**, pp1080-1081, 1962.
- Thefethen, L.M. *et al*, The bath-tub vortex in the southern hemisphere, *Nature* **207**, pp1084-1085, 1965.
- Tyvand, P.A. and Haugen, K.B., An impulsive bathtub vortex, *Phys. Fluids* **17**, 062105, 2005.
- Yukimoto, S. *et al*, Structure of a bathtub vortex: importance of the bottom boundary layer, *Theor. Comput. Fluid Dyn.* **24**, pp323-327, 2010.

Last reviewed: 24 September 2010

Diffusively-driven stratified flows

Supervisor: Michael Page

Level: Honours

Background: In 1970, two independent studies (by Wunsch and Phillips) of the behaviour of a linear density-stratified fluid in a closed container showed that motion can be generated simply due to the container having a sloping boundary surface, and furthermore that the fluid flows uphill! This remarkable phenomenon is a result of the curvature of the lines of constant density near any sloping surface, in order that a zero normal-flux condition on the density to be satisfied along that boundary.

Since that time a number of studies have since considered the consequences of this type of 'diffusively-driven' flow, including in the deep ocean and with turbulent effects included. More recently, Peacock *et al* (2004) undertook an experimental study of the phenomenon in a closed container and Page & Johnson (2008, 2009) extended the work to consider the broader-scale mass recirculation that is generated.

Objectives: The project will review and compare previous analytical, experimental and computational work on this problem for various geometries. The analytical approach introduced in Page & Johnson (2008) will be used to predict the form of the steady linear flow at the initial stages of the experiments by Peacock *et al* (2004) for various bottom slopes and compare that with computational results from a numerical model of the full governing equations.

Expectations:

- The previous work on diffusively-driven flows will be reviewed in relation to the geometries studied and the key non-dimensional parameters.
- An analytical solution will be found for the steady 'outer flow' in a similar configuration to that considered in experiments by Peacock *et al* (2004), based on the theory in Page & Johnson (2008).
- A computational model will be developed using MATLAB for a steady linear driven flow for the same type of container and those results analysed for various angles of inclination of the lower wall.

Assumed knowledge:

- MTH3011 Partial differential equations (required); and
- MTH2051/3051 Introduction to computational mathematics (preferred); and
- MTH3360 Fluid dynamics (preferred).

Reading:

- Page, M.A.. & Johnson, E.R., On steady linear diffusion-driven flow, *J. Fluid Mech.* **606**, pp433-443, 2008.
- Page, M.A.. & Johnson, E.R., Steady nonlinear diffusion-driven flow, *J. Fluid Mech.* **629**, pp299-309, 2009.
- Peacock, T., Stocker, R. & Aristoff, J. M., An experimental investigation of the angular dependence of diffusion-driven flow, *Phys. Fluids* **16**, pp3503-3505, 2004.
- Phillips, O.M., On flows induced by diffusion in a stably stratified fluid, *Deep-Sea Research* **17**, pp. 435-443, 1970.
- Wunsch, C., On oceanic boundary mixing, *Deep-Sea Research* **17**, pp293-301, 1970.

Last reviewed: 24 September 2010

APPLIED AND COMPUTATIONAL MATHEMATICS

Sloshing in a cylindrical cup

Supervisor: Michael Page

Level: Honours

Background: If you carry around a full cup of tea or coffee it is quite difficult to avoid spilling it, and it is even more difficult when you have a hot drink on a plane that is encountering turbulence. In this project we look at the basic mechanisms for the surface motions of a liquid in a cylindrical container, starting with the (easier) small-amplitude case and examining the response at different frequencies. This involves solving partial differential equations in polar coordinates using analytical methods.

Objectives: The project will involve determining the linearised equations for small-amplitude wave motion in a container of constant depth, for example using the standard treatments in Acheson (1990) and/or Lighthill (1978). The problem will be considered for a cylindrical container and solved using separation of variables (involving Bessel functions). The consequences of that solution will be analysed and the results plotted for various initial conditions and types of forcing that illustrate the key properties of the solution, including its susceptibility to forcing at certain frequencies.

Expectations:

- To understand the linear theory for surface-wave motion in a fluid of constant depth and to adapt the governing equations to cylindrical coordinates.
- To solve those equations using analytical methods, determine the resonant frequencies and plot the form of the unsteady solution for a range of appropriate initial conditions and/or types of forcing.

Assumed knowledge:

- MTH3011 Partial differential equations (required); and
- MTH3360 Fluid dynamics (required); and
- MTH2051/3051 Introduction to computational mathematics (preferred).

Reading:

- Acheson, D.J., Elementary fluid dynamics, Oxford, 1990.
- Lighthill, M.J. Waves in fluids, Cambridge, 1978.

Last reviewed: 24 September 2010

Resolving corner singularities in viscous flow

Supervisor: Michael Page

Level: Honours

Background: The solution of PDEs in domains with sharp corners can often involve singularities at those corners, where either the solution or some of its higher derivatives may become infinite as the corner is approached. A simple example of this is the solution for very viscous flow past a thin flat plate, for which Carrier and Lin (1948) demonstrated the velocity is proportional to $r^{1/2}$ as $r \to 0$ near the end of the plate. Similar types of behaviour can also occur near angular corners (Moffatt, 1966).

Numerical methods for solving PDEs typically assume that the solution is 'well-behaved' with all derivatives finite at every point in the domain and on the boundary, so they need to be modified when singularities are known to be present. A recent paper by Shi et al (2004) describes one way of doing this in the case of one particular viscous-flow problem, where the solution satisfies the 'bi-harmonic equation' $\nabla^4 \psi = 0$ and the domain involves an infinitely-long flat plate. The aim of this project is to compare their method with some other possible approaches that also take into account our knowledge about the nature of the singularity at the leading edge of the plate.

Objectives: The project will commence with a review of some of the existing primary literature sources on both the nature of singularities of viscous flows at corners and the treatment of no-slip boundary conditions in numerical models of those problems. The approach used by Shi et al (2004) will be studied in detail for the infinite plate problems and their results reproduced for some of the cases which they have considered. Their results will also be compared with some other simpler approaches. If time permits, the results will be extended to flow past a flat plate of finite length, including when the oncoming flow is an angle.

Expectations:

- A review will be undertaken of some of the existing primary literature sources on viscous-flow singularities at corners in order to understand the properties which are required to be resolved in the numerical solutions.
- The method used by Shi et al (2004) will be examined in detail and implemented for some of the test problems undertaken in their study. This method will also be compared with some other possible approaches that also take into account the nature of the singularity.
- The outcome of the project will be a working computational model for the solution of viscous flow past a flat plate with the singularities at the leading edge resolved accurately. If possible, the method will be extended to a finite-length plate which is inclined to the oncoming flow.

Assumed knowledge:

- MTH3011 Partial differential equations; and
- MTH3020 Complex analysis and integral transforms; and
- MTH2051/3051 Introduction to computational mathematics.

Reading:

- Carrier, G.F. and Lin, C.C., On the nature of the boundary layer near the leading edge of a flat plate, *Quart. Appl. Math.* **6**, pp63-38, 1948.
- Moffatt, H.K., Viscous and resistive eddies near a sharp corner, *J. Fluid Mech.* **18**, pp1-18, 1966.
- Shi, J.-M *et al*, A combined analytical-numerical method for treating corner singularities in viscous flow predictions, *Int. J. Numerical Meth. Fluids* **45**, pp659-688, 2004.

Last reviewed: 24 September 2010

On the roll-up of a 'vortex sheet'

Supervisor: Michael Page

Level: Honours

Background: The position of a thin interface between two fluids that are moving at different speeds is known to be unstable in an inviscid fluid. This is known as Kelvin-Helmholtz instability, and typically it leads to counter-rotating vortices on the interface. Such vortices are often observed between atmospheric layers, and also in the wake of moving vehicles.

In this project the Kelvin-Helmholtz instability is considered for an idealised problem of a thin straight interface between two infinite fluids that have different constant velocities parallel to the interface. For a two-dimensional inviscid fluid this problem can be posed in the complex plane and determined computationally using complex-valued functions (see for example Krasny 1986 and Baker & Pham 2006). It can also be shown that the subsequent roll-up of the vortex develops as the singularity of one of the functions approaches the real axis, where the interface is initially located.

Objectives: The project will review and compare some of the existing primary literature sources on vortex roll-up and the desingularisation of the equations, including both the approach of Krasny (1986) and the introduction of additional factors, for example including viscosity. Krasny's results will be reproduced and then extended to other initial conditions. Key features in the roll-up will be identified beyond the critical time at which it 'overturns' and begins to form a spiral shape.

Expectations:

- A review will be undertaken of some of the existing primary literature sources on this problem in order to identify the current state of knowledge.
- The equations that determine the motion of the interface will be derived and the corresponding complex-plane formulation justified.
- MATLAB (for example) will be used to recalculate and extend the results of the periodic problem in Krasny (1986), where a small parameter δ was introduced in order to 'desingularise' the interface.
- Krasny's approach will be extended to the simpler problem of the roll-up due to an isolated disturbance, for example using recent work by Baker & Pham (2006).

Assumed knowledge:

- MTH3011 Partial differential equations (required); and
- MTH3020 Complex analysis and integral transforms (required); and
- MTH3360 Fluid dynamics (preferred).

Reading:

- Baker, G.R. and Pham L.D., A comparison of blob methods for vortex sheet roll-up, *J. Fluid Mech.* **547**, pp297-316, 2006.
- Krasny, R., Desingularisation of periodic vortex sheet roll-up, *J. Comp. Phys.* **65**, pp292-313, 1986.

Last reviewed: 24 September 2010

The Immersed Interface Method for solving elliptic PDEs

Supervisor: Michael Page

Level: Honours

Background: Most of the simple approaches to solving elliptic partial differential equations numerically using finite-difference techniques rely upon the solution being smooth and continuous everywhere in the domain. In addition, the domain must have a simple, regular geometry – such as a rectangle or a circle. Over the last decade or so, a couple of approaches have been developed which allow discontinuities of the solution and/or consider domains with an irregular shape. In the latter case, these methods can be a simpler alternative to using finite-element methods.

The two most common approaches used for these problems are the Immersed Boundary Method, which was first developed by Charles Peskin in the 1970s, and the Immersed Interface Method. Some aspects of these techniques are similar, but there are important differences. This project examines the basis of the Immersed Interface Method and uses it to examine the accuracy of the approach for some simple test problems with exact solutions.

Objectives: The project will review and compare some of the existing primary literature sources on both the Immersed Interface Method and the Immersed Boundary Method. Some trials of the Immersed Interface Method will be undertaken, initially based on the test problems in Li and Ito (2006) but then extended to a broader range of configurations, with the intention of identifying the advantages and limitations of the method.

Expectations:

- Primary literature sources on the Immersed Interface Method, including LeVeque and Li (1994), and the Immersed Boundary Method will be reviewed, and the two approaches compared in detail.
- The material covered in the first three chapters of Li and Ito (2006) will be examined in detail and all numerical results checked using MATLAB. Some additional simple one and two-dimensional test problems involving elliptic PDEs will also be solved using a similar approach in order to test the capabilities of the method.
- The approach will be applied to some simple 'embedded' boundary-value problems in two-dimensional domains with internal boundaries.

Assumed knowledge:

- MTH3011 Partial differential equations (required); and
- MTH2051/3051 Introduction to computational mathematics (preferred).

Reading:

- LeVeque, R.J. and Li, Z., The immersed interface method for elliptic equations with discontinuous coefficients and singular sources, *SIAM J. Numer. Anal.* **31**, pp1019-1044, 1994.
- Li, Z. and Ito, K., The Immersed Interface Method, *SIAM*, 2006.

Last reviewed: 24 September 2010