Riemann Normal Coordinates

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Abstract

This is just a collection of my notes on Riemann normal coordinates.

1. Introduction

The basic idea behind Riemann normal coordinates is to use the geodesics through a given point to define the coordinates for nearby points. Let the given point be O and consider some nearby point P. If P is close enough to O then there exists a unique geodesic joining O to P. Let a^μ be the components of the unit tangent vector to this geodesic at O and let s be the geodesic arc length measured from O to P. Then the Riemann normal coordinates of P are defined to be $x^\mu = sa^\mu$. These coordinates are well defined provided the geodesics do not cross (which we can always ensure by choosing the neighbourhood of O to be sufficiently small).

One trivial consequence of this definition is that all geodesics through O are of the form $x^\mu(s) = sa^\mu$ and that the a^μ are constant along each geodesic. This implies, by direct substitution into the geodesic equation, that $\Gamma^\alpha_{\mu\nu} = 0$ at O which in turn implies that $g_{\mu\nu,\alpha} = 0$ at O. Suppose now that we were to expand the metric as a Taylor series in x^μ about O. In that series there would only be the zero, second and higher derivatives of the $g_{\mu\nu}$. Thus the leading terms of the metric could be expressed as a sum of a constant part plus a curvature part. If the curvature is weak this can be interpreted as an expansion of the metric in powers (and derivatives) of the curvature. Likewise one can imagine similar expansions of other geometrical quantities (eg. geodesics, arc length) in terms of a flat space part plus a curvature contribution.

The purpose of these notes is develop such expansions and to apply them in the calculation of such things as the geodesic distance between a pair of points and the angle subtended at a vertex of a geodesic triangle.

We will start by using Taylor series expansions of the metric and geodesic equations to obtain various useful formula between the metric, connection and Riemann tensors at O. There is nothing new in any of these formula. What is new (well I can't find them in the literature –

though I haven't looked too far) are the equations for the angles, equations for the geodesics and the derivation of the equation for the geodesic length.

2. Conformal coordinates

There is a potential fly in the ointment in that our proposed series expansion in powers of the curvature (and its derivatives) may not formally appear to converge. This technical difficulty can be overcome by introducing a conformal transformation of the original metric.

Let the typical length scale of the patch containing O be ϵ. Let the coordinates of the patch be x^μ and let the coordinates of O be $x^\mu_\star$. Now define a new set of coordinates y^μ by

$$x^\mu = x^\mu_\star + \epsilon y^\mu$$

Then

$$\begin{aligned} ds^2 &= g_{\mu\nu}(x) dx^\mu dx^\nu \\ &= \epsilon^2 g_{\mu\nu}(x_\star + \epsilon y) dy^\mu dy^\nu \end{aligned}$$

Define the conformal metric $d\tilde{s}$ by

$$\begin{aligned} d\tilde{s}^2 &= g_{\mu\nu}(x_\star + \epsilon y) dy^\mu dy^\nu \\ &= \tilde{g}_{\mu\nu}(y, \epsilon) dy^\mu dy^\nu \end{aligned}$$

In both coordinate systems the geometry of the patch is described by the metric and the boundary of the patch. In the original x^μ coordinates only the boundary depends on ϵ. Whereas in the conformal coordinates y^μ the boundary is fixed but the metric depends on ϵ.

From the above it is easy to see that, at O,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} \ , \qquad \tilde{g}_{\mu\nu,\alpha} = \epsilon g_{\mu\nu,\alpha} \ , \qquad \tilde{g}_{\mu\nu,\alpha\beta} = \epsilon^2 g_{\mu\nu,\alpha\beta} \tag{2.1}$$

where the partial derivatives on the left are with respect to y and those on the right are with respect to x. For each higher derivative an extra power of ϵ will appear.

Since $\Gamma^\alpha_{\mu\nu} = 0$ at O the $R_{\mu\nu\alpha\beta}$ are composed of just the $g_{\mu\nu}$ and its second derivatives. Thus from the above we immediately obtain

$$\tilde{R}_{\mu\nu\alpha\beta} = \epsilon^2 R_{\mu\nu\alpha\beta}$$

and as $R_{\mu\nu\alpha\beta}$ is independent of ϵ, we see that

$$\tilde{R}_{\mu\nu\alpha\beta} = \mathcal{O}(\epsilon^2)$$

for $\epsilon << 1$.

Clearly, then, as $\epsilon \to 0$ the conformal metric is flat.

There are now two ways to look at the patch. We can view it as patch of length scale ϵ with a curvature independent of ϵ. Or we can view it as patch of fixed size but with a curvature that varies as ϵ^2. This later view is useful since in using it we can be sure that the series expansions around flat space are convergent (for a sufficiently small ϵ).

I will use these conformal coordinates for the majority of these notes and I will drop the tilde and revert to x^μ as the generic coordinates (even while I will am working in the conformal frame.)

3. Riemann normal coordinates

In these coordinates the geodesics through O must all be of the form

$$x^\mu(s) = a^\mu_1 s$$

for some set of numbers a^μ_1. By direct substitution into the geodesic equation

$$0 = \frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}(x)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{3.1}$$

and its derivatives, one obtains, at the origin O,

$$0 = \Gamma^\mu_{\alpha\beta} \tag{3.2}$$

$$0 = \Gamma^\mu_{\alpha\beta,\nu} + \Gamma^\mu_{\beta\nu,\alpha} + \Gamma^\mu_{\nu\alpha,\beta} \tag{3.3}$$

It is easy to see, by continuing in this way, that all symmetric derivatives of the connection vanish at the origin in Riemann normal coordinates. Some authors take this to be the definition of Riemann normal coordinates. I prefer the definition given at the beginning of these notes as it has a nice geometric feel to it.

There are two series expansions that we will use frequently. The first is an expansion of various objects, such as the metric, in powers of x^μ. The second expansion arises by constraining this first expansion to points on a typical geodesic. Each such expansion will give us some useful relations between the metric, the connection and the Riemann tensor.

3.1. Metric

Consider a Taylor series expansion of the metric around the origin O, namely,

$$g_{\mu\nu}(x) = g_{\mu\nu} + g_{\mu\nu,\alpha\beta}\frac{x^\alpha x^\beta}{2} + \mathcal{O}(\epsilon^3)$$

There is no linear term because $g_{\mu\nu,\alpha} = 0$ at the origin. It is a simple algebraic exercise to show, given (3.2) and (3.3), that

$$\Gamma^\mu_{\alpha\beta,\nu} = -\frac{1}{3}\left(R^\mu{}_{\alpha\beta\nu} + R^\mu{}_{\beta\alpha\nu}\right) \tag{3.1.1}$$

from which it follows that

$$g_{\mu\nu,\alpha\beta} = -\frac{1}{3}\left(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha}\right) \tag{3.1.2}$$

and finally

$$R_{\mu\nu\alpha\beta} = g_{\alpha\nu,\mu\beta} - g_{\alpha\mu,\nu\beta}$$

Substituting these into the above we obtain

$$g_{\mu\nu}(x) = g_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}\, x^\alpha x^\beta + \mathcal{O}(\epsilon^3) \tag{3.1.3}$$

From this we can also easily verify that

$$g^{\mu\nu}(x) = g^{\mu\nu} + \frac{1}{3}R^\mu{}_\alpha{}^\nu{}_\beta\, x^\alpha x^\beta + \mathcal{O}(\epsilon^3)$$

We will adopt the convention of raising and lowering indices with the flat space part of the metric $g_{\mu\nu}$. This incurs no error when applied at the origin. However, for other points, there is an associated truncation error and this must be accounted for – thus one must tread carefully.

3.2. Connection

We can also propose a Taylor series expansion for the connection about the origin O, namely,

$$\Gamma^\mu_{\alpha\beta}(x) = \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\alpha\beta,\rho}x^\rho + \Gamma^\mu_{\alpha\beta,\rho\tau}\frac{x^\rho x^\tau}{2} + \cdots \tag{3.2.1}$$

Our first observations are that

$$\begin{aligned}\Gamma^\mu_{\alpha\beta} &= 0\\ \Gamma^\mu_{\alpha\beta,\rho} &= \mathcal{O}(\epsilon^2)\\ \Gamma^\mu_{\alpha\beta,\rho\tau} &= \mathcal{O}(\epsilon^3)\end{aligned}$$

and in general the n-th derivative of Γ will be $\mathcal{O}(\epsilon^{n+1})$ at O. This follows by simple inspection of the standard formula for computing the metric connection and the previously stated asymptotic behaviour of the conformal metric (2.1).

We are only interested in the leading term in the above expansion, and so after using (3.1.1) we obtain

$$\Gamma^\mu_{\alpha\beta}(x) = -\frac{1}{3}\left(R^\mu{}_{\alpha\beta\nu} + R^\mu{}_{\beta\alpha\nu}\right)x^\nu + \mathcal{O}(\epsilon^3)$$

3.3. Geodesics

For the geodesics, we employ a series expansion in s, the distance measured along the geodesic,

$$x^\mu(s) = a^\mu_0 + a^\mu_1 s + a^\mu_2 \frac{s^2}{2} + a^\mu_3 \frac{s^3}{6} + \cdots \tag{3.3.1}$$

We will defer for the moment stating the nature of the truncation error. Our primary aim here is to determine as many of the a^μ_i as we can in terms of just $g_{\mu\nu}$ and $R_{\mu\nu\alpha\beta}$. This can be done by demanding that the above expansion for $x^\mu(s)$ is a solution of the geodesic equation (3.1).

The basic steps are to substitute (3.3.1) into (3.2.1) and to then substitute all of these quantities into the geodesic equation (3.1). The result is a polynomial in s and as this must be identically zero for all s, we equate the separate coefficients of powers of s to zero. For the first two terms s^0 and s^1 we obtain, respectively,

$$\begin{aligned}
0 &= a^\mu_2 + \Gamma^\mu_{\alpha\beta,\rho} a^\rho_0 a^\alpha_1 a^\beta_1 + \frac{1}{2}\Gamma^\mu_{\alpha\beta,\rho\tau} a^\rho_0 a^\tau_0 a^\alpha_1 a^\beta_1 \\
0 &= a^\mu_3 + \Gamma^\mu_{\alpha\beta,\rho}\left(a^\rho_1 a^\alpha_1 a^\beta_1 + 2a^\rho_0 a^\alpha_1 a^\beta_2\right) + \Gamma^\mu_{\alpha\beta,\rho\tau}\left(2a^\rho_1 a^\tau_0 a^\alpha_1 a^\beta_1 + 2a^\rho_0 a^\tau_0 a^\alpha_2 a^\beta_1\right)
\end{aligned}$$

The term $\Gamma^\mu_{\alpha\beta,\rho} a^\rho_1 a^\alpha_1 a^\beta_1$ is zero in view of (3.3). Thus, to order ϵ^3, we obtain

$$\begin{aligned}
a^\mu_2 &= -\Gamma^\mu_{\alpha\beta,\rho} a^\rho_0 a^\alpha_1 a^\beta_1 + \mathcal{O}(\epsilon^3) \\
a^\mu_3 &= \mathcal{O}(\epsilon^3)
\end{aligned}$$

Clearly this process can be developed in full to obtain recurrence relations amongst all of the remaining a^μ_i. We will not need these but what we do require is their behaviour in ϵ. It is not hard to see that in the generic equation for a^μ_n the leading terms will be

$$0 = a^\mu_n + \Gamma^\mu_{\alpha\beta,\rho} a^\rho_{n-2} a^\alpha_1 a^\beta_1 + \cdots$$

Since we have already established that a^μ_2 is $\mathcal{O}(\epsilon^2)$ it follows that a^μ_n will be $\mathcal{O}(\epsilon^n)$ for $n \geq 2$.

Assembling the above results leads finally to

$$x^\mu(s) = a^\mu_0 + a^\mu_1 s + \frac{1}{3} R^\mu{}_{\alpha\beta\rho}\, a^\rho_0 a^\alpha_1 a^\beta_1 s^2 + \mathcal{O}(\epsilon^3) \tag{3.3.2}$$

Notice that a^μ_0 and a^μ_1 remain undetermined – they can only be computed from appropriate boundary or initial conditions.

3.3.1. Geodesic boundary value problem

In this case we are looking for the geodesic which passes through two given points. Let the coordinates of initial point be x_i^μ and those for the final point be x_j^μ. Suppose the geodesic distance between the two points is L_{ij}. The L_{ij} can not be freely specified as they must be derivable from the metric and the coordinates. A equation for L_{ij} will be given in a later section (3.4).

Our aim is to solve for a_0^μ and a_1^μ such that

$$
\begin{aligned}
x^\mu(s=0) &= x_i^\mu = a_0^\mu + \mathcal{O}(\epsilon^3) \\
x^\mu(s=L_{ij}) &= x_j^\mu = a_0^\mu + a_1^\mu L_{01} + \frac{1}{3} R^\mu{}_{\alpha\beta\rho}\, a_0^\rho a_1^\alpha a_1^\beta L_{01}^2 + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

The first equation is easy to solve, namely, $a_0^\mu = x_i^\mu + \mathcal{O}(\epsilon^3)$. However, the second equation does appear to pose a bit of a problem – it looks like a nasty quadratic equation for each of the a_1^μ. Fortunately this equation can be solved by an iterative method to within $\mathcal{O}(\epsilon^3)$. The starting point is to first substitute for a_0^μ to obtain

$$
x_j^\mu = x_i^\mu + a_1^\mu L_{01} + \frac{1}{3} R^\mu{}_{\alpha\beta\rho}\, x_i^\rho a_1^\alpha a_1^\beta L_{01}^2 + \mathcal{O}(\epsilon^3)
$$

Since $R^\mu{}_{\alpha\beta\rho} = \mathcal{O}(\epsilon^2)$ we obtain the first approximation

$$
a_1^\mu = \frac{1}{L_{01}}(x_j^\mu - x_i^\mu) + \mathcal{O}(\epsilon^2)
$$

This can now be substituted back into the previous equation leading to the second approximation

$$
a_1^\mu = \frac{1}{L_{ij}} \left(\Delta x_{ij}^\mu - \frac{1}{3} R^\mu{}_{\alpha\beta\rho}\, x_i^\rho \Delta x_{ij}^\alpha \Delta x_{ij}^\beta \right) + \mathcal{O}(\epsilon^3) \tag{3.3.3}
$$

where $\Delta x_{ij}^\mu = x_j^\mu - x_i^\mu$. Now this approximation, being $\mathcal{O}(\epsilon^3)$ accurate, is sufficient for our purposes so there is no need to proceed to higher order approximations. Combining these results for a_0^μ and a_1^μ and substituting into (3.3.2) leads to the following equation for the geodesic passing through the two points x_i^μ and x_j^μ

$$
x^\mu(s) = x_i^\mu + \lambda \Delta x_{ij}^\mu - \frac{\lambda(1-\lambda)}{3} R^\mu{}_{\alpha\beta\rho}\, x_i^\rho \Delta x_{ij}^\alpha \Delta x_{ij}^\beta + \mathcal{O}(\epsilon^3) \tag{3.3.4}
$$

where $\lambda = s/L_{ij}$.

3.3.2. Geodesic initial value problem

In this instance we are looking for a geodesic for which the initial position and direction of the geodesic are given. Let the initial point have coordinates x_i^μ and let the initial tangent vector be m^μ. We can assume that m^μ is a unit vector.

The two equations from which we must solve for a_0^μ and a_1^μ are

$$\begin{aligned} x^\mu(s=0) &= x_i^\mu = a_0^\mu + \mathcal{O}(\epsilon^3) \\ \frac{dx^\mu}{ds}(s=0) &= m^\mu = a_1^\mu + \mathcal{O}(\epsilon^3) \end{aligned}$$

These equations are rather easy to solve, leading to the following equation for the geodesic

$$x^\mu(s) = x_i^\mu + s m^\mu + \frac{s^2}{3} R^\mu{}_{\alpha\beta\rho}\, x_i^\rho m^\alpha m^\beta + \mathcal{O}(\epsilon^3) \tag{3.3.5}$$

Incidently, from (3.3.2) and (3.3.3), we see that to obtain a geodesic which passes through the two points x_i^μ and x_j^μ one must choose

$$m^\mu = \frac{1}{L_{ij}} \left(\Delta x_{ij}^\mu - \frac{1}{3} R^\mu{}_{\alpha\beta\rho}\, x_i^\rho \Delta x_{ij}^\alpha \Delta x_{ij}^\beta \right) + \mathcal{O}(\epsilon^3) \tag{3.3.6}$$

3.4. Geodesic distance

Consider two points with coordinates x_i^μ and x_j^μ. Since there exists, by assumption, a unique geodesic joining this pair of points, the distance between them should also be uniquely defined in terms of their coordinates and the metric.

Our aim is to evaluate, along the geodesic,

$$L_{ij} = \int_0^1 \left(g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda$$

The equation for $x^\mu(\lambda)$ is simply (3.3.4), namely

$$x^\mu(\lambda) = x_i^\mu + \lambda \Delta x_{ij}^\mu - \frac{\lambda(1-\lambda)}{3} R^\mu{}_{\alpha\beta\rho}\, x_i^\rho \Delta x_{ij}^\alpha \Delta x_{ij}^\beta + \mathcal{O}(\epsilon^3)$$

for $0 < \lambda < 1$. This can be substituted into the expansion (3.1.3) for $g_{\mu\nu}(x)$ with the result

$$g_{\mu\nu}(x(\lambda)) = g_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta} \left(x_i^\alpha + \lambda \Delta x_{ij}^\alpha \right) \left(x_i^\beta + \lambda \Delta x_{ij}^\beta \right) + \mathcal{O}(\epsilon^3)$$

It is a simple matter to substitute these into the integrand, leading to

$$\left(\frac{dL}{d\lambda}\right)^2 = g_{\mu\nu}\Delta x^\mu_{ij}\Delta x^\nu_{ij} - \frac{1}{3}R_{\mu\alpha\nu\beta}\, x^\alpha_i x^\beta_i \Delta x^\mu_{ij}\Delta x^\nu_{ij} + \mathcal{O}(\epsilon^3)$$

The important point to note is that this result does not depend on λ. Thus the integrand is constant and so the integration is trivial. The result follows immediately,

$$L^2_{ij} = g_{\mu\nu}\Delta x^\mu_{ij}\Delta x^\nu_{ij} - \frac{1}{3}R_{\mu\alpha\nu\beta}\, x^\alpha_i x^\beta_i \Delta x^\mu_{ij}\Delta x^\nu_{ij} + \mathcal{O}(\epsilon^3) \tag{3.4.1}$$

From this result it is easy to establish, using the symmetries of $R_{\mu\nu\alpha\beta}$, the following equivalent equations for L^2_{ij}

$$\begin{aligned} L^2_{ij} &= g_{\mu\nu}\Delta x^\mu_{ij}\Delta x^\nu_{ij} - \frac{1}{3}R_{\mu\alpha\nu\beta}\, x^\alpha_i x^\beta_i x^\mu_j x^\nu_j + \mathcal{O}(\epsilon^3) && (3.4.2)\\ &= (g_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}\, \overline{x}^\alpha_{ij}\, \overline{x}^\alpha_{ij})\Delta x^\mu_{ij}\Delta x^\nu_{ij} + \mathcal{O}(\epsilon^3) && (3.4.3)\\ &= g_{\mu\nu}\Delta x^\mu_{ij}\Delta x^\nu_{ij} - \frac{1}{3}R_{\mu\alpha\nu\beta}\, \Delta x^\alpha_{0i}\Delta x^\beta_{0i}\Delta x^\mu_{0j}\Delta x^\nu_{0j} + \mathcal{O}(\epsilon^3) && (3.4.4) \end{aligned}$$

where $\overline{x}^\mu_{ij} = (x^\mu_i + x^\mu_j)/2$ and where x^μ_0 are the coordinates of the origin (which in our case are zero). We will make use of this last form when deriving the (generalised) cosine law.

3.5. Generalised Cosine law

Consider a geodesic triangle with vertices i, j and k. We would like to be able to compute the angles subtended at each vertex in terms of the usual quantities, the metric, the coordinates etc. We will develop the appropriate equations in two stages. First, we will consider the simple case of computing the angle at a vertex coincident with the origin. We shall then generalise this result to the case were all three vertices are distinct from the origin.

To start the ball rolling consider a geodesic triangle with vertices i, j and O, the origin. We seek an equation for the angle between the geodesic segments joining O to i and O to j. The unit tangent vectors to these geodesic segments are, from equation (3.3.1),

$$\begin{aligned} v^\mu_{iO} &= \Delta x^\mu_{iO}/L_{iO}\\ v^\mu_{jO} &= \Delta x^\mu_{jO}/L_{jO} \end{aligned}$$

Now let θ_{ij} be the angle subtended at O. Then

$$\cos\theta_{ij} = g_{\mu\nu}v^\mu_{iO}v^\nu_{jO} = g_{\mu\nu}\Delta x^\mu_{iO}\Delta x^\nu_{jO}/(L_{iO}L_{jO})$$

We can obtain two useful variants of this equation by writing, first, $\Delta x^\mu_{iO} = \Delta x^\mu_{jO} + \Delta x^\mu_{ij}$ and second, $\Delta x^\mu_{jO} = \Delta x^\mu_{iO} - \Delta x^\mu_{ij}$. This gives

$$\begin{aligned}
L_{iO} L_{jO} \cos\theta_{ij} &= g_{\mu\nu} \left(\Delta x^\mu_{jO} + \Delta x^\mu_{ij}\right) \Delta x^\nu_{jO} \\
&= g_{\mu\nu} \Delta x^\mu_{iO} \left(\Delta x^\nu_{iO} - \Delta x^\nu_{ij}\right)
\end{aligned}$$

Adding these two equations leads to

$$2L_{iO} L_{jO} \cos\theta_{ij} = L^2_{iO} + L^2_{jO} - g_{\mu\nu} \Delta x^\mu_{ij} \Delta x^\nu_{ij} \tag{3.5.1}$$

However, from equation (3.4.4) we see that

$$g_{\mu\nu} \Delta x^\mu_{ij} \Delta x^\nu_{ij} = L^2_{ij} + \frac{1}{3} R_{\mu\alpha\nu\beta}\, \Delta x^\mu_{iO} \Delta x^\nu_{iO} \Delta x^\alpha_{jO} \Delta x^\beta_{jO} + \mathcal{O}(\epsilon^3)$$

Thus we have

$$2L_{iO} L_{jO} \cos\theta_{ij} = L^2_{iO} + L^2_{jO} - L^2_{ij} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, \Delta x^\mu_{iO} \Delta x^\nu_{iO} \Delta x^\alpha_{jO} \Delta x^\beta_{jO} + \mathcal{O}(\epsilon^3) \tag{3.5.2}$$

With this equation we have achieved our first aim : to obtain an equation when the vertex resides at the origin. To obtain an equation applicable to the general case, where the vertex is not at the origin, we can imagine transforming to a second set of Riemann normal coordinates with an origin at some other point, say O'. We can do this simply by shifting the coordinates, eg. $x^\mu \to x^\mu + c^\mu$. The coordinates, metric and Riemann components at the respective origins will therefore be related by

$$x'^\mu = x^\mu + c^\mu$$

$$g'_{\mu\nu}(O') = g_{\mu\nu}(O) - \frac{1}{3} R_{\mu\alpha\nu\beta}(O)\, c^\alpha c^\beta + \mathcal{O}(\epsilon^3)$$

$$R'_{\mu\nu\alpha\beta}(O') = R_{\mu\alpha\nu\beta}(O) + \mathcal{O}(\epsilon^1)$$

Now the important observation is that the above equation (3.5.2) is covariant with respect to this transformation (whereas (3.5.1) is not). That is, it applies to any three vertices of a geodesic triangle. Let us now relabel the vertices as i, j and k. Then the angle subtended at vertex k can be computed from

$$2L_{ik} L_{jk} \cos\theta_{ij} = L^2_{ik} + L^2_{jk} - L^2_{ij} - \frac{1}{3} R_{\mu\alpha\nu\beta}\, \Delta x^\mu_{ik} \Delta x^\nu_{ik} \Delta x^\alpha_{jk} \Delta x^\beta_{jk} + \mathcal{O}(\epsilon^3) \tag{3.5.3}$$

Note added in Oct 2007 : I did these calculations sometime in 1996, but recently, (yes, it took a while and you might wonder what was I doing) I found that J.L.Synge has reported the same result using the method of world functions. See references [5,6]. Synge does the calculations upto and including $\mathcal{O}(L^6)$.

3.6. Parallel transport

Suppose a vector v^μ is to be parallel transported along a curve described by $x^\mu(s)$. Can the values of v^μ along this curve be expressed in terms of primary quantities such as $g_{\mu\nu}$ and $R_{\mu\nu\alpha\beta}$? Clearly the answer is yes, and what follows is a simple derivation employing, once again, series expansions.

We start by writing $x^\mu(s)$ as a series in s

$$x^\mu(s) = a_0^\mu + a_1^\mu s + a_2^\mu \frac{s^2}{2} + \mathcal{O}(\epsilon^3)$$

We will also expand $\Gamma^\mu_{\alpha\beta}(x)$ around the origin $x^\mu = 0$

$$\begin{aligned}\Gamma^\mu_{\alpha\beta}(x) &= \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\alpha\beta,\rho} x^\rho + \mathcal{O}(\epsilon^3)\\ &= \Gamma^\mu_{\alpha\beta,\rho}\left(a_0^\rho + a_1^\rho s\right) + \mathcal{O}(\epsilon^3)\end{aligned}$$

Finally, we propose the following expansions for $v^\mu(s)$

$$v^\mu(s) = v_0^\mu + v_1^\mu s + v_2^\mu \frac{s^2}{2} + v_3^\mu \frac{s^3}{6} + \cdots$$

Each of these expansions can then be substituted into the parallel transport equation

$$0 = \frac{dv^\mu}{ds} + \Gamma^\mu_{\alpha\beta}(x) v^\alpha \frac{dx^\mu}{ds}$$

As is customary, we equate to zero the coefficients of successive powers of s. For s^0 and s^1 we obtain, respectively,

$$\begin{aligned}0 &= v_1^\mu + \Gamma^\mu_{\alpha\beta,\rho} v_0^\mu a_1^\beta a_0^\rho + \mathcal{O}(\epsilon^3)\\ 0 &= v_2^\mu + \Gamma^\mu_{\alpha\beta,\rho}\left(v_1^\alpha a_1^\beta a_0^\rho + v_0^\alpha a_1^\beta a_1^\rho + v_0^\alpha a_2^\beta a_0^\rho\right) + \mathcal{O}(\epsilon^3)\end{aligned}$$

Using (3.1.1) and the fact that $\Gamma^\mu_{\alpha\beta,\rho}$ is $\mathcal{O}(\epsilon^2)$ we obtain

$$\begin{aligned}v_1^\mu &= \frac{1}{3}\left({R^\mu}_{\alpha\beta\rho} + {R^\mu}_{\beta\alpha\rho}\right) v_0^\alpha a_1^\beta a_0^\rho + \mathcal{O}(\epsilon^3)\\ v_2^\mu &= \frac{1}{3}\left({R^\mu}_{\alpha\beta\rho} + {R^\mu}_{\beta\alpha\rho}\right) v_0^\alpha a_1^\beta a_1^\rho + \mathcal{O}(\epsilon^3)\end{aligned}$$

Substituting these back into the expansion for $v^\mu(s)$ leads to

$$v^\mu(s) = v_0^\mu + \frac{1}{3}\left(R^\mu{}_{\alpha\beta\rho} + R^\mu{}_{\beta\alpha\rho}\right)\left(v_0^\alpha a_1^\beta\right)\left(s a_0^\rho + \frac{s^2}{2} a_1^\rho\right) + \mathcal{O}(\epsilon^3)$$

In the case where the curve is a geodesic joining the points x_i^μ to x_j^μ, we can use the results of section (3.3), namely

$$\begin{aligned} a_0^\mu &= x_i^\mu + \mathcal{O}(\epsilon^3) \\ a_1^\mu &= \frac{1}{L_{ij}}\left(\Delta x_{ij}^\mu - \frac{1}{3} R^\mu{}_{\alpha\beta\rho}\, x_i^\rho \Delta x_{ij}^\alpha \Delta x_{ij}^\beta\right) + \mathcal{O}(\epsilon^3) \end{aligned}$$

to obtain

$$v^\mu(s) = v_0^\mu + \frac{1}{3}\left(R^\mu{}_{\alpha\beta\rho} + R^\mu{}_{\beta\alpha\rho}\right)\left(v_0^\alpha \Delta x_{ij}^\beta\right)\left(\lambda x_i^\rho + \frac{\lambda^2}{2}\Delta x_{ij}^\rho\right) + \mathcal{O}(\epsilon^3)$$

where $\lambda = s/L_{ij}$ and $\Delta x_{ij}^\mu = x_j^\mu - x_i^\mu$. In particular, when $s = L_{ij}$ we obtain

$$v_j^\mu = v_i^\mu + \frac{1}{3}\left(R^\mu{}_{\alpha\beta\rho} + R^\mu{}_{\beta\alpha\rho}\right) v_i^\alpha\, \Delta x_{ij}^\beta\, \overline{x}_{ij}^\rho + \mathcal{O}(\epsilon^3) \tag{3.6.1}$$

where $\overline{x}_{ij}^\mu = (x_i + x_j)/2$. Note that we have also changed the notation slightly so that v_i^μ and v_j^μ refer to the values of v^μ at the two end points of the geodesic. The v_i^μ and v_j^μ should not be confused with the terms in the original expansion for $v^\mu(x)$.

3.7. From generic to Riemann normal coordinates

It is very unlikely, except for highly specialised metrics, that the coordinates will be Riemann normal coordinates. How then can we transform a generic set of coordinates into Riemann normal coordinates? We shall now develop such a transformation.

The first step is to apply the conformal transformations of section (2) to the generic coordinates. It is easy to see that the conformal relations, (2.1), carry over to the conformally transformed coordinates z^μ.

Let the generic coordinates be z^μ. As the Riemann coordinates are based upon the geodesics passing through the origin it is appropriate to examine the same geodesics in the z^μ coordinates. So, consider a typical geodesic passing through the point $z^\mu = 0$. We shall now repeat much of the calculations of section (3.3) with the one variation that now the $\Gamma^\mu_{\alpha\beta} \neq 0$. Thus we start by proposing a series expansion for $z^\mu(s)$,

$$z^\mu(s) = a_1^\mu s + a_2^\mu \frac{s^2}{2} + a_3^\mu \frac{s^3}{6} + \cdots$$

and a Taylor series expansion, about $z^\mu = 0$, of the connection

$$\begin{aligned}\Gamma^\mu_{\alpha\beta}(z) &= \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\alpha\beta,\rho} z^\rho + \mathcal{O}(\epsilon^3)\\ &= \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\alpha\beta,\rho}\left(a^\rho_1 s + a^\rho_2 \frac{s^2}{2} + \cdots\right) + \mathcal{O}(\epsilon^3)\end{aligned}$$

These expansions can be substituted into the geodesic equation, (3.1), and, according to the now familar theme of this paper, the successive powers of s are equated to zero. For the first two terms we obtain

$$\begin{aligned}0 &= a^\mu_2 + \Gamma^\mu_{\alpha\beta} a^\alpha_1 a^\beta_1\\ 0 &= a^\mu_3 + 2\Gamma^\mu_{\alpha\beta} a^\alpha_1 a^\beta_2 + \Gamma^\mu_{\alpha\beta,\rho} a^\alpha_1 a^\beta_1 a^\rho_1\end{aligned}$$

This leads to

$$z^\mu(s) = a^\mu_1 s - \Gamma^\mu_{\alpha\beta} a^\alpha_1 a^\beta_1 \frac{s^2}{2} + \left(2\Gamma^\mu_{\alpha\tau}\Gamma^\tau_{\beta\rho} - \Gamma^\mu_{\alpha\beta,\rho}\right) a^\alpha_1 a^\beta_1 a^\rho_1 \frac{s^3}{6} + \mathcal{O}(\epsilon^3)$$

The tangent vector to the geodesic has components a^μ_1 at the origin. However, at the origin, we can align the axes of the Riemann normal coordinates with those of z^μ. Thus in the Riemann normal coordinates the tangent vector would also have components a^μ_1. Furthermore, in these Riemann normal coordinates we know that the geodesics are described by

$$x^\mu(s) = a^\mu_1 s$$

This allows us to eliminate each a^μ_1 term in the above equation for z^μ. In the process we also eliminate all explicit dependance on the parameter s,

$$z^\mu = x^\mu - \frac{1}{2}\Gamma^\mu_{\alpha\beta} x^\alpha x^\beta + \frac{1}{6}\left(2\Gamma^\mu_{\alpha\tau}\Gamma^\tau_{\beta\rho} - \Gamma^\mu_{\alpha\beta,\rho}\right) x^\alpha x^\beta x^\rho + \mathcal{O}(\epsilon^3) \qquad (3.7.1)$$

Thus we have arrived at a transformation between the x^μ and z^μ coordinates. In this form the equation is, however, not particularily useful since our aim was to construct the x^μ coordinates from the given z^μ. We will invert this equation by a succession of approximations in much the same way as we did in section (3.3).

Since $\Gamma^\mu_{\alpha\beta} = \mathcal{O}(\epsilon^1)$ we have the first approximation

$$x^\mu = z^\mu + \mathcal{O}(\epsilon^1)$$

Substitute this into (3.7.1) to obtain the second approximation

$$x^\mu = z^\mu + \frac{1}{2}\Gamma^\mu_{\alpha\beta} z^\alpha z^\beta + \mathcal{O}(\epsilon^2)$$

which can then be used to obtain the third and final approximation

$$x^\mu = z^\mu + \frac{1}{2}\Gamma^\mu_{\alpha\beta} z^\alpha z^\beta + \frac{1}{6}\left(\Gamma^\mu_{\alpha\tau}\Gamma^\tau_{\beta\rho} + \Gamma^\mu_{\alpha\beta,\rho}\right) z^\alpha z^\beta z^\rho + \mathcal{O}(\epsilon^3)$$

Note that the Γ's in this equation are those for the original z^μ coordinates.

This transformation produces a set of Riemann normal coordinates aligned to the the original coordinate axes at their coincident origins. We would like a somewhat more general transformation, one which allows for non-coincident origins and non-aligned coordinates axes. This is easily acheived by simple linear transformations. It is easy to see that this leads to

$$x^\mu = \Lambda^\mu{}_\eta \left(\Delta z^\eta + \frac{1}{2}\Gamma^\eta_{\alpha\beta}\Delta z^\alpha \Delta z^\beta + \frac{1}{6}\left(\Gamma^\eta_{\alpha\tau}\Gamma^\tau_{\beta\rho} + \Gamma^\eta_{\alpha\beta,\rho}\right)\Delta z^\alpha \Delta z^\beta \Delta z^\rho\right) + \mathcal{O}(\epsilon^3) \tag{3.7.2}$$

where $\Delta z^\eta = z^\eta - z^\eta_o$ and z^η_o coresponds to the origin of the Riemann normal coordinates. The matrix $\Lambda^\mu{}_\nu$ is chosen to align the axes of the Riemann normal coordinates to a preferred set of directions. For example, we may choose an orthogonal set of axes (at the origin). The $\Lambda^\mu{}_\nu$ could then be computed by way of a Gramm-Schmidt orthogonalisation procedure.

3.8. Alternative Riemann normal coordinate frames

Riemann normal coordinates can be constructed in a small region of any given point. Suppose we choose two distinct (but close) points and that we constructed a Riemann normal coordinate frame for each point. What is the transformation that maps one frame into the other? We shall develop this transformation by adapting the result of the previous section.

Let the two origins be O and O' and let their respective Riemann normal coordinates be x^μ and x'^μ. In the previous section we obtained a transformatiom from any generic coordinates z^μ into a specific set of Reimann normal coordinates x^μ. Clearly we are free to choose the z^μ to be the x'^μ. Thus we have

$$x'^\mu = \Lambda^\mu{}_\eta \left(\Delta x^\eta + \frac{1}{2}\Gamma^\eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta + \frac{1}{6}\left(\Gamma^\eta_{\alpha\tau}\Gamma^\tau_{\beta\rho} + \Gamma^\eta_{\alpha\beta,\rho}\right)\Delta x^\alpha \Delta x^\beta \Delta x^\rho\right) + \mathcal{O}(\epsilon^3) \tag{3.8.1}$$

where we have swapped the roles of x^μ and x'^μ simply for notational convenience (and to accord with convention that $x \to x'$).

Note that each of the $\Gamma^\mu_{\alpha\beta}$ and $\Gamma^\mu_{\alpha\beta,\rho}$ are evaluated at $x^\mu = x^\mu_{o'}$ ie. at $x'^\mu = 0$. We can evaluate each of these by way of a Taylor series about $x^\mu = 0$. This leads to

$$\Gamma^\eta_{\alpha\tau}(x_{o'}) = \Gamma^\eta_{\alpha\beta,\rho} x^\rho_{o'} + \mathcal{O}(\epsilon^3)$$
$$\Gamma^\eta_{\alpha\beta,\rho}(x_{o'}) = \Gamma^\eta_{\alpha\beta,\rho} + \mathcal{O}(\epsilon^3)$$

Since $\Gamma^{\eta}_{\alpha\beta,\rho}$ is $\mathcal{O}(\epsilon^2)$ we see that the quadratic terms in (3.8.1) are $\mathcal{O}(\epsilon^4)$ and thus may be neglected. The remaining linear terms can be expressed in terms of the Riemann tensor via (3.1.1). The final result is

$$x'^{\mu} = \Lambda^{\mu}{}_{\eta}\left(\Delta x^{\eta} - \frac{1}{3}R^{\eta}{}_{\alpha\beta\rho}\,\Delta x^{\alpha}\Delta x^{\beta}x^{\rho}_{o'}\right) + \mathcal{O}(\epsilon^3) \tag{3.8.2}$$

4. Appendix

This is a little appendix to prove two key results (3.1.1) and (3.1.2). Starting with

$$\begin{aligned} 0 &= \Gamma^{\mu}_{\alpha\beta} \\ 0 &= \Gamma^{\mu}_{\alpha\beta,\nu} + \Gamma^{\mu}_{\beta\nu,\alpha} + \Gamma^{\mu}_{\nu\alpha,\beta} \end{aligned}$$

and using the definition of the Riemann tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\rho\mu}\Gamma^{\rho}_{\beta\nu} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\rho}_{\beta\mu}$$

we obtain

$$R^{\alpha}{}_{\beta\mu\nu} + R^{\alpha}{}_{\mu\beta\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta,\nu}$$

From which it follows that

$$\Gamma^{\alpha}_{\beta\mu,\nu} = -\frac{1}{3}\left(R^{\alpha}{}_{\beta\mu\nu} + R^{\alpha}{}_{\mu\beta\nu}\right) \tag{3.1.1}$$

Now for the metric, we start first with the statement that the metric has zero covariant derivative

$$0 = g_{\mu\nu,\alpha} - g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha} - g_{\rho\nu}\Gamma^{\rho}_{\mu\alpha}$$

which when differentiated is

$$0 = g_{\mu\nu,\alpha\beta} - g_{\mu\rho}\Gamma^{\rho}_{\nu\alpha,\beta} - g_{\rho\nu}\Gamma^{\rho}_{\mu\alpha,\beta}$$

(recall that $g_{\mu\nu,\alpha} = 0$ at the origin). Upon using the results just obtained for the connection, we see that

$$\begin{aligned} g_{\mu\nu,\alpha\beta} &= -\frac{1}{3}\left(R_{\mu\alpha\nu\beta} + R_{\nu\alpha\mu\beta}\right) \\ &= -\frac{1}{3}\left(R_{\mu\alpha\nu\beta} + R_{\mu\beta\nu\alpha}\right) \end{aligned} \tag{3.1.2}$$

Note that each of the above expressions is applicable only at the origin.

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