ASP3012: Stars and Galaxies
Stars Exercise Sheet 5

Q1. Consider stars on the Upper Main Sequence, where the mass-radius relation tells us that

$$R \sim M^{0.6}$$

and the mass-luminosity relation tells us that

$$L \sim M^{3.5}.$$

Assume the Upper Main Sequence starts at $1M_\odot$. Estimate the luminosity and radius of a $2M_\odot$ and a $5M_\odot$ star.

Q2. A good approximation to the pressure gradient in a star is given by

$$\frac{dP}{dr} = -\frac{4\pi}{3} G\rho_c^2 r e^{-x^2}$$

for some scale-length a and central density ρ_c, and where $x = r/a$.

a) Show that $P(r) = P_c e^{-x^2}$ where $P_c = \frac{2\pi}{3} G\rho_c^2 a^2$ (you may assume that the stellar radius $R \gg a$).

b) Show that $m(r) = \frac{4\pi\rho_c a^3}{3}\Phi(x)$ where $\Phi^2(x) = 6 - 3(x^4 + 2x^2 + 2)e^{-x^2}$.

c) Show that $\rho(r) = \rho_c x^3 e^{-x^2}/\Phi(x)$

d) Hence show that, near the centre, $\Phi(x) \simeq \left[x^6 - \frac{3}{4}x^8 + \ldots\right]^{1/2}$

e) Hence show that, near the centre, $\rho(r) \simeq \rho_c(1 - \frac{5}{8}x^2 + \ldots)$

f) And again near the centre, $T(r) \simeq T_c(1 - \frac{3}{8}x^2 + \ldots)$

g) Taking an energy generation formula of the form $\epsilon = \epsilon_0 \rho X^2 T^n$ show that

$$L(r) = 4\pi a^3 \epsilon_0 \rho_c^2 T_c^n X^2 \left[\frac{x^3}{3} - \frac{(5+3n)x^5}{40} + \ldots\right]$$

h) Use this formula to estimate the luminosity of the Sun, assuming $n \simeq 3.5$, $a \simeq R_\odot/5$, $T_c \simeq 1.5\times10^7$K, $\rho_c \simeq 150$g cm^{-3}, and $\epsilon_0 \simeq 4\times10^{-26}$. For X you should use the average of the surface value $X = 0.7$ and the central value $X = 0.4$, and integrate from $r = 0$ to $r = 0.1R_\odot$.

i) Now use the formula to estimate the luminosity of a $10M_\odot$ star. Here the energy generation is via the CN cycle, so we will use $n \simeq 18$, $a \simeq R_\odot/5$, $T_c \simeq 3\times10^7$K, $\rho_c \simeq 100$g cm^{-3}, and $\epsilon_0 \simeq 4\times10^{-130}$. For X you should use the value in the central convective core, say $X = 0.5$. Energy generation in these stars is very centrally condensed, so integrate from $r = 0$ to $r = 0.05R_\odot$.

Q3. Consider a red giant with a core showing a linear density distribution:

$$\rho(r) = \rho_c(1 - r/R).$$

a) Show that the core mass is given by

$$M_c \simeq \frac{4\pi r_s^3}{3}\rho_c$$

where r_s is the position of the hydrogen shell (i.e. the edge of the core).

b) Show that the pressure in the core is

$$P_c = P_s + \frac{2\pi}{3}G\rho_c^2 r_s^2.$$

c) Hence show that if $P_c \gg P_s$ then the temperature of the isothermal core is

$$T_c = \frac{2\pi G\mu}{3\Re}r_s^2\rho_c.$$

d) Hence show that

$$\frac{r_s}{R_\odot} \simeq 0.11\left(\frac{M_c}{M_\odot}\right)^{1/3}\left(\frac{\rho_c}{1000}\right)^{-1/3}$$

where ρ_c is measured in g/cm^3.

e) And finally, show that

$$T_c \simeq 9.8\times10^7\mu\left(\frac{M_c}{M_\odot}\right)^{2/3}\left(\frac{\rho_c}{1000}\right)^{1/3}.$$