Q1. For the full PP chains, show that the equilibrium abundance of He$^3$ is given by:

$$(\text{He}^3)^e = \frac{1}{2\lambda_{33}} \left[ -\lambda_{34}\text{He}^4 + [\lambda_{34}(\text{He}^4)^2 + 2\lambda_{pp}\lambda_{33}\text{H}^2]^{1/2} \right].$$

Q2. Consider the PP reactions, with the usual notation.

a) Show that the rate of He$^4$ production by the PPI chain, compared to the combined PPII and PPIII chains, is:

$$\frac{\text{PPI}}{\text{PPII} + \text{PPIII}} = \frac{r_{33}}{r_{34}},$$

where the He$^3$(He$^3, 2p$)He$^4$ rate is $r_{33}$ and the He$^3(\alpha, \gamma)$Be$^7$ rate is $r_{34}$.

b) Defining

$$\alpha = \frac{\lambda_{34}^2}{\lambda_{33}\lambda_{pp}} \left( \frac{\text{He}^4}{\text{H}} \right)^2$$

show that

$$\frac{\text{PPI}}{\text{PPII} + \text{PPIII}} = \frac{(1 + \frac{2}{\alpha})^{1/2} - 1}{4},$$

once the He$^3$ has come to equilibrium.

Q3. Let’s look further at the full PP reactions.

a) Write down the differential equations for H, D$^2$, He$^3$, He$^4$, Be$^7$, Li$^7$, for the full PP chains.

b) By assuming that the sum (Li$^7$ + Be$^7$) is in equilibrium, show that

$$\frac{d\text{He}^4}{dt} = \lambda_{33} \frac{(\text{He}^3)^2}{2} + \lambda_{34}\text{He}^3\text{He}^4$$

c) It is convenient to write the He$^4$ production rate for the PP chains as the value for the PPI chain multiplied by a correction factor which allows for the operation of the PPII and PPIII chains. Show that when both D$^2$ and He$^3$ are in equilibrium, as is the sum of (Li$^7$ + Be$^7$), then we find:

$$\frac{d\text{He}^4}{dt} = \frac{1}{2}\lambda_{pp} \frac{\text{H}^2}{2} \left[ 1 + \frac{2\lambda_{34}(\text{He}^3)^e\text{He}^4}{\lambda_{pp}\text{H}^2} \right]$$

d) Hence show that

$$\frac{d\text{He}^4}{dt} = \frac{1}{2}\lambda_{pp} \frac{\text{H}^2}{2} \Phi(\alpha)$$

where

$$\Phi(\alpha) = 1 - \alpha + \alpha \left( 1 + \frac{2}{\alpha} \right)^{1/2}.$$
Q4. Let \( F_{PPI} \) be the fraction of \( \text{He}^4 \) produced by the PPI reactions. Likewise for \( F_{PPII} \) and \( F_{PPIII} \). The from Q1 we have:

\[
\frac{F_{PPI}}{F_{PPII} + F_{PPIII}} = \frac{(1 + \frac{2}{\alpha})^{1/2} - 1}{4}
\]

Hence show that

\[
F_{PPI} = \left[ \left( 1 + \frac{2}{\alpha} \right)^{1/2} - 1 \right] \left[ \left( 1 + \frac{2}{\alpha} \right)^{1/2} + 3 \right]^{-1}
\]

Q5. We saw in lectures that the CN cycle could be written in matrix form as

\[
\frac{dU}{dt} = \Lambda \bar{U}
\]

where

\[
\bar{U} = \begin{bmatrix} C_{12} \\ C_{13} \\ N_{14} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} -\frac{1}{\tau_{12}} & 0 & \frac{1}{\tau_{14}} \\ 0 & -\frac{1}{\tau_{13}} & 0 \\ \frac{1}{\tau_{12}} & \frac{1}{\tau_{13}} & -\frac{1}{\tau_{14}} \end{bmatrix}.
\]

The solution is

\[
\bar{U}(t) = A e^{\lambda_1 t} \bar{U}_1 + B e^{\lambda_2 t} \bar{U}_2 + C e^{\lambda_3 t} \bar{U}_3,
\]

where the \( \lambda_i \) are the eigenvalues and the \( \bar{U}_i \) the eigenvectors of \( \Lambda \).

a) Show that \( \lambda_1 = 0 \), \( \lambda_2 = -\frac{\Sigma + \Delta}{2} \) and \( \lambda_3 = -\frac{\Sigma - \Delta}{2} \) where \( \Delta \) and \( \Sigma \) are defined in the lecture notes.

b) Show that \( \bar{U}_1 \) is proportional to \( \begin{bmatrix} \tau_{12} \\ \tau_{13} \\ \tau_{14} \end{bmatrix} \).

Q6. In discussing the CN cycle we derived the differential equation:

\[
\frac{dN_{13}}{dt} = \frac{C_{12}}{\tau_{12}} - \frac{N_{13}}{\tau_{\beta}(13)}
\]

for the number abundance of \( N_{13} \) as a function of time when \( T < 10^8 \text{K} \). If the timescale of interest is short enough for both \( C_{12} \) and \( \tau_{12} \) to be considered constant, show that

\[
N_{13}(t) = \frac{C_{12}}{\tau_{12}} \tau_{\beta}(13) \left[ 1 - e^{-t/\tau_{\beta}(13)} \right],
\]

where we have assumed that \( N_{13}(0) = 0 \) (because it is an unstable isotope).

Q7. Helium burning occurs via the triple-alpha process:

\[
3\text{He}^4 \rightarrow \text{C}^{12}.
\]

Show that this releases about 1/10 as much energy per unit mass (of fuel destroyed) as H burning.

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