# ASP3012: Stars and Galaxies
# Stars Exercise Sheet 4

**Q1.** For the full PP chains, show that the equilibrium abundance of $He^3$ is given by:

$$(\mathrm{He}^3)_e = \frac{1}{2\lambda_{33}}\left[-\lambda_{34}\mathrm{He}^4 + [\lambda_{34}(\mathrm{He}^4)^2 + 2\lambda_{pp}\lambda_{33}\mathrm{H}^2]^{1/2}\right].$$

**Q2.** Consider the PP reactions, with the usual notation.

a) Show that the rate of $He^4$ production by the PPI chain, compared to the combined PPII and PPIII chains, is:

$$\frac{\mathrm{PPI}}{\mathrm{PPII}+\mathrm{PPIII}} = \frac{r_{33}}{r_{34}},$$

where the $\mathrm{He}^3(\mathrm{He}^3,2\mathrm{p})\mathrm{He}^4$ rate is $r_{33}$ and the $\mathrm{He}^3(\alpha,\gamma)\mathrm{Be}^7$ rate is $r_{34}$.

b) Defining

$$\alpha = \frac{\lambda_{34}^2}{\lambda_{33}\lambda_{pp}}\left(\frac{\mathrm{He}^4}{\mathrm{H}}\right)^2$$

show that

$$\frac{\mathrm{PPI}}{\mathrm{PPII}+\mathrm{PPIII}} = \frac{(1+\frac{2}{\alpha})^{1/2}-1}{4},$$

once the $He^3$ has come to equilibrium.

**Q3.** Lets look further at the full PP reactions.

a) Write down the differential equations for H, $D^2$, $He^3$, $He^4$, $Be^7$, $Li^7$, for the full PP chains.

b) By assuming that the sum ($Li^7$ + $Be^7$) is in equilibrium, show that

$$\frac{d\mathrm{He}^4}{dt} = \lambda_{33}\frac{(\mathrm{He}^3)^2}{2} + \lambda_{34}\mathrm{He}^3\mathrm{He}^4$$

c) It is convenient to write the $He^4$ production rate for the PP chains as the value for the PPI chain multiplied by a correction factor which allows for the operation of the PPII and PPIII chains. Show that when both $D^2$ and $He^3$ are in equilibrium, as is the sum of ($Li^7$ + $Be^7$), then we find:

$$\frac{d\mathrm{He}^4}{dt} = \frac{1}{2}\lambda_{pp}\frac{\mathrm{H}^2}{2}\left[1+\frac{2\lambda_{34}(\mathrm{He}^3)_e\mathrm{He}^4}{\lambda_{pp}\mathrm{H}^2}\right]$$

d) Hence show that

$$\frac{d\mathrm{He}^4}{dt} = \frac{1}{2}\lambda_{pp}\frac{\mathrm{H}^2}{2}\Phi(\alpha)$$

where

$$\Phi(\alpha) = 1-\alpha+\alpha\left(1+\frac{2}{\alpha}\right)^{1/2}.$$

**Q4.** Let $F_{PPI}$ be the fraction of $He^4$ produced by the PPI reactions. Likewise for $F_{PPII}$ and $F_{PPIII}$. The from Q1 we have:

$$\frac{F_{PPI}}{F_{PPII}+F_{PPIII}} = \frac{\left(1+\frac{2}{\alpha}\right)^{1/2}-1}{4}$$

Hence show that

$$F_{PPI} = \left[\left(1+\frac{2}{\alpha}\right)^{1/2}-1\right]\left[\left(1+\frac{2}{\alpha}\right)^{1/2}+3\right]^{-1}$$

**Q5.** We saw in lectures that the CN cycle could be written in matrix form as

$$\frac{d\underset{\sim}{U}}{dt} = \Lambda\underset{\sim}{U}$$

where

$$\underset{\sim}{U} = \begin{bmatrix} C^{12} \\ C^{13} \\ N^{14} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} -\frac{1}{\tau_{12}} & 0 & \frac{1}{\tau_{14}} \\ \frac{1}{\tau_{12}} & -\frac{1}{\tau_{13}} & 0 \\ 0 & \frac{1}{\tau_{13}} & -\frac{1}{\tau_{14}} \end{bmatrix}.$$

The solution is

$$\underset{\sim}{U}(t) = Ae^{\lambda_1 t}\underset{\sim}{U}_1 + Be^{\lambda_2 t}\underset{\sim}{U}_2 + Ce^{\lambda_3 t}\underset{\sim}{U}_3,$$

where the $\lambda_i$ are the eigenvalues and the $\underset{\sim}{U}_i$ the eigenvectors of $\Lambda$.

a) Show that $\lambda_1 = 0$, $\lambda_2 = \frac{-\Sigma+\Delta}{2}$ and $\lambda_3 = \frac{-\Sigma-\Delta}{2}$ where $\Delta$ and $\Sigma$ are defined in the lecture notes.

b) Show that $\underset{\sim}{U}_1$ is proportional to $\begin{bmatrix} \tau_{12} \\ \tau_{13} \\ \tau_{14} \end{bmatrix}$.

**Q6.** In discussing the CN cycle we derived the differential equation:

$$\frac{dN^{13}}{dt} = \frac{C^{12}}{\tau_{12}} - \frac{N^{13}}{\tau_\beta(13)}$$

for the number abundance of $N^{13}$ as a function of time when $T < 10^8$K. If the timescale of interest is short enough for both $C^{12}$ and $\tau_{12}$ to be considered constant, show that

$$N^{13}(t) = \frac{C^{12}}{\tau_{12}}\tau_\beta(13)\left[1-e^{-t/\tau_\beta(13)}\right],$$

where we have assumed that $N^{13}(0)=0$ (because it is an unstable isotope).

**Q7.** Helium burning occurs via the triple-alpha process:

$$3He^4 \to C^{12}.$$

Show that this releases about 1/10 as much energy per unit mass (of fuel destroyed) as H burning.