

# ASP3012: Stars and Galaxies

## Stars Exercise Sheet 4

**Q1.** For the full PP chains, show that the equilibrium abundance of  $\text{He}^3$  is given by:

$$(\text{He}^3)_e = \frac{1}{2\lambda_{33}} \left[ -\lambda_{34}\text{He}^4 + [\lambda_{34}(\text{He}^4)^2 + 2\lambda_{pp}\lambda_{33}\text{H}^2]^{1/2} \right].$$

**Q2.** Consider the PP reactions, with the usual notation.

- a) Show that the rate of  $\text{He}^4$  production by the PPI chain, compared to the combined PPII and PPIII chains, is:

$$\frac{\text{PPI}}{\text{PPII} + \text{PPIII}} = \frac{r_{33}}{r_{34}},$$

where the  $\text{He}^3(\text{He}^3, 2\text{p})\text{He}^4$  rate is  $r_{33}$  and the  $\text{He}^3(\alpha, \gamma)\text{Be}^7$  rate is  $r_{34}$ .

- b) Defining

$$\alpha = \frac{\lambda_{34}^2}{\lambda_{33}\lambda_{pp}} \left( \frac{\text{He}^4}{\text{H}} \right)^2$$

show that

$$\frac{\text{PPI}}{\text{PPII} + \text{PPIII}} = \frac{(1 + \frac{2}{\alpha})^{1/2} - 1}{4},$$

once the  $\text{He}^3$  has come to equilibrium.

**Q3.** Lets look further at the full PP reactions.

- a) Write down the differential equations for H,  $\text{D}^2$ ,  $\text{He}^3$ ,  $\text{He}^4$ ,  $\text{Be}^7$ ,  $\text{Li}^7$ , for the full PP chains.
- b) By assuming that the sum ( $\text{Li}^7 + \text{Be}^7$ ) is in equilibrium, show that

$$\frac{d\text{He}^4}{dt} = \lambda_{33} \frac{(\text{He}^3)^2}{2} + \lambda_{34}\text{He}^3\text{He}^4$$

- c) It is convenient to write the  $\text{He}^4$  production rate for the PP chains as the value for the PPI chain multiplied by a correction factor which allows for the operation of the PPII and PPIII chains. Show that when both  $\text{D}^2$  and  $\text{He}^3$  are in equilibrium, as is the sum of ( $\text{Li}^7 + \text{Be}^7$ ), then we find:

$$\frac{d\text{He}^4}{dt} = \frac{1}{2}\lambda_{pp} \frac{\text{H}^2}{2} \left[ 1 + \frac{2\lambda_{34}(\text{He}^3)_e\text{He}^4}{\lambda_{pp}\text{H}^2} \right]$$

- d) Hence show that

$$\frac{d\text{He}^4}{dt} = \frac{1}{2}\lambda_{pp} \frac{\text{H}^2}{2} \Phi(\alpha)$$

where

$$\Phi(\alpha) = 1 - \alpha + \alpha \left( 1 + \frac{2}{\alpha} \right)^{1/2}.$$

**Q4.** Let  $F_{PPI}$  be the fraction of  $\text{He}^4$  produced by the PPI reactions. Likewise for  $F_{PPII}$  and  $F_{PPIII}$ . The from Q1 we have:

$$\frac{F_{PPI}}{F_{PPII} + F_{PPIII}} = \frac{\left(1 + \frac{2}{\alpha}\right)^{1/2} - 1}{4}$$

Hence show that

$$F_{PPI} = \left[ \left(1 + \frac{2}{\alpha}\right)^{1/2} - 1 \right] \left[ \left(1 + \frac{2}{\alpha}\right)^{1/2} + 3 \right]^{-1}$$

**Q5.** We saw in lectures that the CN cycle could be written in matrix form as

$$\frac{d\tilde{U}}{dt} = \Lambda \tilde{U}$$

where

$$\tilde{U} = \begin{bmatrix} C^{12} \\ C^{13} \\ N^{14} \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} -\frac{1}{\tau_{12}} & 0 & \frac{1}{\tau_{14}} \\ \frac{1}{\tau_{12}} & -\frac{1}{\tau_{13}} & 0 \\ 0 & \frac{1}{\tau_{13}} & -\frac{1}{\tau_{14}} \end{bmatrix}.$$

The solution is

$$\tilde{U}(t) = Ae^{\lambda_1 t} \tilde{U}_1 + Be^{\lambda_2 t} \tilde{U}_2 + Ce^{\lambda_3 t} \tilde{U}_3,$$

where the  $\lambda_i$  are the eigenvalues and the  $\tilde{U}_i$  the eigenvectors of  $\Lambda$ .

a) Show that  $\lambda_1 = 0$ ,  $\lambda_2 = \frac{-\Sigma + \Delta}{2}$  and  $\lambda_3 = \frac{-\Sigma - \Delta}{2}$  where  $\Delta$  and  $\Sigma$  are defined in the lecture notes.

b) Show that  $\tilde{U}_1$  is proportional to  $\begin{bmatrix} \tau_{12} \\ \tau_{13} \\ \tau_{14} \end{bmatrix}$ .

**Q6.** In discussing the CN cycle we derived the differential equation:

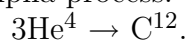
$$\frac{dN^{13}}{dt} = \frac{C^{12}}{\tau_{12}} - \frac{N^{13}}{\tau_{\beta}(13)}$$

for the number abundance of  $N^{13}$  as a function of time when  $T < 10^8 \text{K}$ . If the timescale of interest is short enough for both  $C^{12}$  and  $\tau_{12}$  to be considered constant, show that

$$N^{13}(t) = \frac{C^{12}}{\tau_{12}} \tau_{\beta}(13) \left[ 1 - e^{-t/\tau_{\beta}(13)} \right],$$

where we have assumed that  $N^{13}(0)=0$  (because it is an unstable isotope).

**Q7.** Helium burning occurs via the triple-alpha process:



Show that this releases about 1/10 as much energy per unit mass (of fuel destroyed) as H burning.