ASP3012
2011

ASP3012: Stars and Galaxies
Stars Exercise Sheet 3

Q1. By making the substitution $\theta = \phi/\xi$ in the Lane-Emden equation, show that that the solution for $n = 1$ is

$$\theta(\xi) = \frac{\sin(\xi)}{\xi}.$$

Verify that $\left(\frac{d\theta}{d\xi}\right) = 0$ at the origin.

Q2. By integrating the Lane-Emden equation from $\xi = 0$ to $\xi = \xi_1 \ll 1$, and taking the limit as $\xi_1 \to 0$ show that $\left(\frac{d\theta}{d\xi}\right)_{\xi=0} = 0$.

Q3. Derive the series solution to the Lane-Emden equation, up to order ξ^6, by following the steps below. Begin with a power series of the form $\theta(\xi) = \sum_{i=0}^{\infty} c_i \xi^i$.

a) Show that if $\theta(\xi)$ is a solution of the Lane-Emden equation then so is $\theta(-\xi)$. Thus there can only be even powers of ξ in the series. i.e. $\theta(\xi) = \sum_{i=0,2,4...}^{\infty} c_i \xi^i$.

b) Apply the initial conditions to show that $c_0 = 1$.

c) Substitute the resulting series into the Lane-Enden equation (using the binomial expansion to evaluate θ^n) and show that

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \frac{n(8n-5)}{42\times360}\xi^6 + \ldots$$

Q4. Obtain the Maclaurin series expansions for the three polytropic solutions ($n = 0, 1$ and 5):

$$\theta_0(\xi) = 1 - \frac{1}{6}\xi^2$$
$$\theta_1(\xi) = \frac{\sin(\xi)}{\xi}$$
$$\theta_5(\xi) = \left(1 + \frac{1}{3}\xi^2\right)^{-1/2}$$

Q5. Verify the above analytic solutions by substituting directly into the Lane-Emden equation.

Q6. Show that, for $n = 5$, $\lim_{\xi\to\infty}\left(-\xi^2\frac{d\theta}{d\xi}\right) = \sqrt{3}$.

Q7. Consider the Eddington Standard Model. Show that the mass is given by $18.0\frac{\sqrt{1-\beta}}{\mu^2\beta^2}M_\odot$.

Q8. Define variables U and V by:

$$U = \frac{d \ln M(r)}{d \ln r}$$
$$V = -\frac{d \ln P(r)}{d \ln r}$$

a) Show that in the centre of a star $U = 3$ and $V = 0$

b) For a polytrope of index n show that

$$U = -\frac{\xi \theta^n}{\theta'}$$
$$V = -\frac{(n+1)\xi\theta'}{\theta}$$

where the prime denotes differentiation with respect to ξ.

Q9. We have not discussed the important "isothermal equation of state" where the temperature remains constant. Such a situation arises at very low density. If material is heated usually cools by emitting a photon. For example, molecular H_2 can be heated and the extra energy may appear as vibration or rotation of the molecule. This cools by emitting a photon which would normally be reabsorbed by another particle in the gas. But if the density is sufficiently low it is likely to escape rather than interact with any matter. The material is said to be "optically thin", with the result that the heat is converted to photons which escape: the temperature remains constant. This equation of state is particularly relevant to regions where star formation is in its early stages.

Clearly we have $P = K\rho^\gamma = \frac{\rho \Re T}{\mu} \propto \rho$ which means that $\gamma = 1$ for such a gas. Since $\gamma = 1 + 1/n$ we must have $n \to \infty$. Hence the analysis leading to the Lane-Emden equation is invalid. But another analogous equation can be derived.

a) Using the reduced equations of stellar structure, derive the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{\Re T}{\mu}\frac{d\rho}{dr}\right) = -4\pi G\rho.$$

b) Use the substitutions $\rho = \rho_c e^{-\psi}$ and $r = \alpha\xi$ to determine the "isothermal sphere equation":

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi}, \quad \text{where } \alpha = \left[\frac{\Re T}{4\pi G\rho_c\mu}\right]^{1/2}.$$

c) Show that $\psi(0) = 0$ and $\left(\frac{d\psi}{d\xi}\right)_{\xi=0} = 0.$

d) No analytic solutions are available, so numerical solutions must be obtained. As in the Lane-Emden equation, the differential equation is singular so we require a series solution near the origin. Show that

$$\psi(\xi) = \frac{1}{6}\xi^2 - \frac{1}{120}\xi^4 + \frac{1}{1890}\xi^6 + \ldots$$