Q1. Consider a gas composed of 80% H$_2$ and 20% He by mass. After deciding on the level of ionization, calculate the mean molecular weight $\mu$.

Q2. For a perfect monatomic gas undergoing adiabatic changes show that

$$\frac{dP}{P} + \gamma \frac{dV}{V} = 0 \quad \text{and} \quad \frac{dP}{P} + \frac{\gamma}{1-\gamma} \frac{dT}{T} = 0.$$

Q3. Consider a perfect monatomic gas with radiation pressure, so that the total pressure is

$$P = P_g + P_r = \frac{\rho RT}{\mu} + \frac{1}{3} aT^4$$

and the internal energy per gram is

$$U = U_g + U_r = c_v T + \frac{aT^4}{\rho} = \frac{1}{\gamma - 1} \frac{RT}{\mu} + \frac{aT^4}{\rho}.$$ 

Hence show that:

a) $$\frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0$$

b) $$\frac{dP}{P} + \frac{\Gamma_2}{1 - \Gamma_2} \frac{dT}{T} = 0$$

c) $$\frac{dT}{T} + (\Gamma_3 - 1) \frac{dV}{V} = 0$$

where: $\Gamma_1 = \frac{32 - 24\beta - 3\beta^2}{24 - 21\beta}$, $\Gamma_2 = \frac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}$, $\Gamma_3 = \frac{32 - 27\beta}{24 - 21\beta}$

Q4. Using the results of Q3 show that $\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1}$.

Q5. Plot the variation of the generalised gammas (defined in Q1) as a function of $\beta$. Note how they all reduce to 5/3 when $P_r \to 0$, but reduce to 4/3 when $P_g \to 0$.

Q6. A very important quantity is the “adiabatic temperature gradient”, $\nabla_{ad}$, defined by

$$\nabla_{ad} = \left( \frac{d\ln T}{d\ln P} \right)_{entropy}$$

a) For a perfect monatomic gas show that: $\nabla_{ad} = \frac{\gamma - 1}{\gamma} = \frac{2}{5}$.

b) By adding radiation pressure show that: $\nabla_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2}$. 
Q7. Show that, for a perfect monatomic gas with radiation pressure:

\[ P = \left[ \left( \frac{R}{\mu} \right)^{4.3} \frac{1 - \beta}{a \beta^4} \right]^{1/3} \rho^{4/3}. \]

Q8. a) From the first law of thermodynamics show that

\[ \left( \frac{\partial S}{\partial V} \right)_T = \frac{1}{T} \left[ \left( \frac{\partial U}{\partial V} \right)_T + P \right], \]

\[ \left( \frac{\partial S}{\partial T} \right)_V = \frac{1}{T} \left( \frac{\partial U}{\partial T} \right)_V. \]

b) Hence show that:

\[ \left( \frac{\partial U}{\partial V} \right)_T = T \left( \frac{\partial P}{\partial T} \right)_V - P. \]

Q9. Recall the definition of the specific heat for some thermodynamic quantity \( A \) held constant:

\[ c_A = \left( \frac{dQ}{dT} \right)_A. \]

Hence show that

\[ c_P - c_V = T \left( \frac{\partial V}{\partial T} \right)_P \left( \frac{\partial P}{\partial T} \right)_V. \]

(You will need the result of the previous question.)

Q10. Define \( \alpha \) and \( \delta \) by

\[ \alpha = \left( \frac{\partial \ln \rho}{\partial \ln P} \right)_T \quad \text{and} \quad \delta = - \left( \frac{\partial \ln \rho}{\partial \ln T} \right)_P. \]

Show that

a) \( \alpha = 1/\beta \)

b) \( \delta = (4 - 3\beta)/\beta \)

c) \( c_P - c_V = \frac{P\delta^2}{\rho T^\alpha}. \)

Q11. Hence, using the results of this sheet, derive the useful expression \( \nabla_{ad} = \frac{P\delta}{T\rho c_P} \) for the adiabatic gradient.

Q12. Show that another expression for the adiabatic gradient is

\[ \nabla_{ad} = \frac{R\delta}{\beta \mu c_P} = \left( \frac{\gamma - 1}{\gamma} \right) \left( \frac{4 - 3\beta}{\beta^2} \right). \]

JCL: 10-September-2011