ASP3012
2011

ASP3012: Stars and Galaxies
Stars Exercise Sheet 2

Q1. Consider a gas composed of 80% H_2 and 20% He by mass. After deciding on the level of ionization, calculate the mean molecular weight μ.

Q2. For a perfect monatomic gas undergoing adiabatic changes show that

$$\frac{dP}{P} + \gamma\frac{dV}{V} = 0 \qquad \text{and} \qquad \frac{dP}{P} + \frac{\gamma}{1-\gamma}\frac{dT}{T} = 0.$$

Q3. Consider a perfect monatomic gas with radiation pressure, so that the total pressure is

$$P = P_g + P_r = \frac{\rho \Re T}{\mu} + \frac{1}{3}aT^4$$

and the internal energy per gram is

$$U = U_g + U_r = c_V T + \frac{aT^4}{\rho} = \frac{1}{\gamma - 1}\frac{\Re T}{\mu} + \frac{aT^4}{\rho}.$$

Hence show that:

a) $\dfrac{dP}{P} + \Gamma_1 \dfrac{dV}{V} = 0$

b) $\dfrac{dP}{P} + \dfrac{\Gamma_2}{1-\Gamma_2}\dfrac{dT}{T} = 0$

c) $\dfrac{dT}{T} + (\Gamma_3 - 1)\dfrac{dV}{V} = 0$

where: $\Gamma_1 = \dfrac{32 - 24\beta - 3\beta^2}{24 - 21\beta}$ $\qquad \Gamma_2 = \dfrac{32 - 24\beta - 3\beta^2}{24 - 18\beta - 3\beta^2}$ $\qquad \Gamma_3 = \dfrac{32 - 27\beta}{24 - 21\beta}$

Q4. Using the results of Q 3 show that $\dfrac{\Gamma_1}{\Gamma_3 - 1} = \dfrac{\Gamma_2}{\Gamma_2 - 1}$.

Q5. Plot the variation of the generalised gammas (defined in Q1) as a function of β. Note how they all reduce to 5/3 when $P_r \to 0$, but reduce to 4/3 when $P_g \to 0$.

Q6. A very important quantity is the "adiabatic temperature gradient", ∇_{ad}, defined by

$$\nabla_{ad} = \left(\frac{d\ln T}{d\ln P}\right)_{entropy}$$

a) For a perfect monatomic gas show that: $\quad \nabla_{ad} = \dfrac{\gamma - 1}{\gamma} = \dfrac{2}{5}.$

b) By adding radiation pressure show that: $\quad \nabla_{ad} = \dfrac{\Gamma_2 - 1}{\Gamma_2}.$

Q7. Show that, for a perfect monatomic gas with radiation pressure:

$$P = \left[\left(\frac{\Re}{\mu} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right]^{1/3} \rho^{4/3}.$$

Q8. a) From the first law of thermodynamics show that

$$\left(\frac{\partial S}{\partial V} \right)_T = \frac{1}{T} \left[\left(\frac{\partial U}{\partial V} \right)_T + P \right]$$
$$\left(\frac{\partial S}{\partial T} \right)_V = \frac{1}{T} \left(\frac{\partial U}{\partial T} \right)_V$$

b) Hence show that: $\left(\frac{\partial U}{\partial V} \right)_T = T \left(\frac{\partial P}{\partial T} \right)_V - P.$

Q9. Recall the definition of the specific heat for some thermodynamic quantity A held constant:

$$c_A = \left(\frac{dQ}{dT} \right)_A .$$

Hence show that

$$c_P - c_V = T \left(\frac{\partial V}{\partial T} \right)_P \left(\frac{\partial P}{\partial T} \right)_V .$$

(You will need the result of the previous question.)

Q10. Define α and δ by

$$\alpha = \left(\frac{\partial \ln \rho}{\partial \ln P} \right)_T \quad \text{and} \quad \delta = -\left(\frac{\partial \ln \rho}{\partial \ln T} \right)_P$$

Show that

a) $\alpha = 1/\beta$

b) $\delta = (4 - 3\beta)/\beta$

c) $c_P - c_V = \dfrac{P\delta^2}{\rho T \alpha}.$

Q11. Hence, using the results of this sheet, derive the useful expression $\nabla_{ad} = \dfrac{P\delta}{T\rho c_P}$ for the adiabatic gradient.

Q12. Show that another expression for the adiabatic gradient is

$$\nabla_{ad} = \frac{\Re\delta}{\beta\mu c_P} = \left(\frac{\gamma - 1}{\gamma} \right) \left(\frac{4 - 3\beta}{\beta^2} \right)$$