ORDINARY DIFFERENTIAL EQUATIONS

The general form of a first order differential equations is

$$\frac{dy}{dx} = f(x, y)$$

with initial condition $y(a) = y_a$. We seek the solution $y = y(x)$ for $x > a$. This is shown in Figure 1, and is known as an "initial value problem". (Boundary value problems are more complicated, and will be discussed briefly later.)
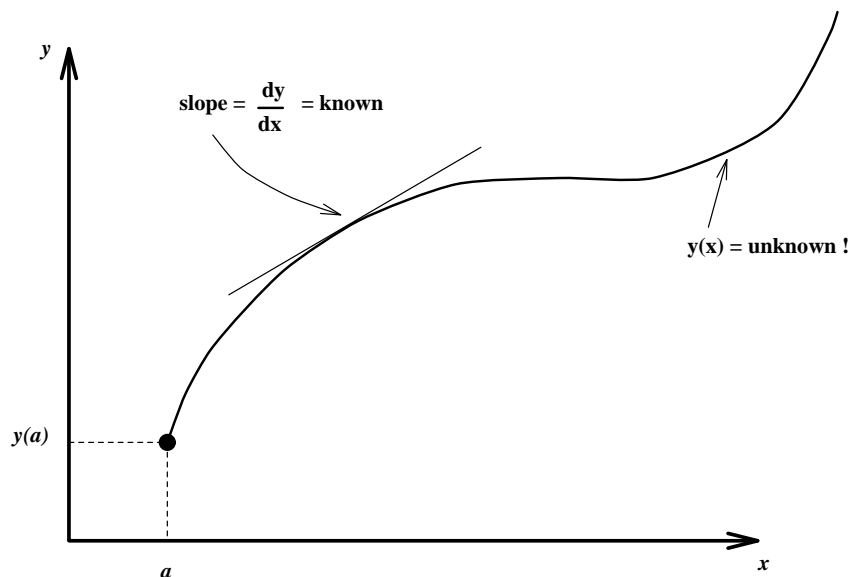

Figure 1: Geometrical interpretation of the problem.

Although some of these equations may be solved analytically (e.g. when f is linear in y, the equation is separable) most are not solvable by analytical techniques. Indeed, even when there is an analytical solution it is often of little practical use to us. For example, the d.e.:

$$\frac{dy}{dx} + 2xy = 1$$

with $y(0) = 0$ has solution

$$y(x) = e^{-x^2} \int_0^x e^{t^2} dt.$$

This integral must be evaluated numerically, and is no easier than doing the initial problem numerically from the start.

Note that numerical integration is simply a special case of solving a d.e. since

$$I = \int_a^b f(x) dx$$

is equivalent to solving for $I = y(b)$ with

$$\frac{dy}{dx} = f(x)$$

and $y(a) = 0$. Note that there is no y in the right-hand-side of the differential equation.

1. The Euler Method

This is the simplest (and least accurate) method, but it illustrates the general principles underlying the better schemes. We shall divide the area of integration, $[a, b]$, into n equal segments of width $h = (b - a)/n$, and define

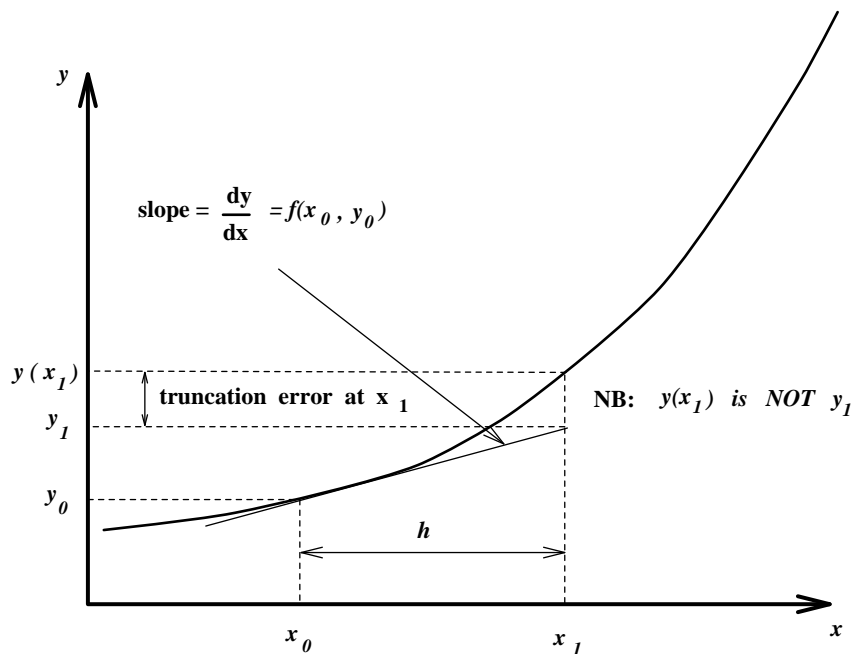$$x_i = a + ih \qquad\qquad i = 0, 1, \ldots, n.$$

This is shown in Figure 2.

Figure 2: The Euler Method.

Clearly, from the diagram,

$$y_1 = y_0 + \text{slope} \times h$$
$$= y_0 + hf(x_0, y_0)$$

and

$$y_2 = y_1 + hf(x_1, y_1)$$

which yields the general formula

$$y_{i+1} = y_i + hf(x_i, y_i).$$

An alternative derivation of this Euler formula is possible from the Taylor Series:

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(x_i) + \ldots$$

If we neglect terms of order h^2 (i.e. we "truncate" the series here, so that the "(local) truncation error" is $O(h^2)$):

$$y_{i+1} = y_i + hy'(x_i)$$
$$= y_i + hf(x_i, y_i).$$

135

— at *each step* we introduce an error due to truncation of the series, called the "local truncation error", which is $O(h^2)$ in this method.

— the complete integration involves n steps, so that the "global" error is $\sim nh^2 \sim (nh)h$, and since $nh = (b-a) = $ constant, then the "global" error is $O(h)$.

e.g. Solve $\frac{dy}{dx} = y$ on $[0,1]$ with $y(0) = 1$ using Euler's method with $n = 10$.

$$x_i = 0.1i \qquad\qquad i = 0, 1, \ldots, 10$$
$$y_i = y_i + hf(x_i, y_i)$$
$$= y_i + 0.1y_i = 1.1y_i$$

The exact solution is easily found to be $y = e^x$, so we can calculate the Euler Method value and compare with the exact solution: Thus the solution is:

x_i	y_i	% error
0.0	1.00	0.0
0.1	1.10	0.5
0.2	1.21	0.9
0.3	1.33	1.4
0.4	1.46	1.9
0.5	1.61	2.3
0.6	1.77	2.8
0.7	1.95	3.2
0.8	2.14	3.8
0.9	2.36	4.0
1.0	2.59	4.6

Note that the local error in this case is $\sim \frac{h^2}{2} y'' \sim h^2/2 \sim 0.04$. This is verified in the table: the error increases by $\frac{1}{2}\%$ per step to a total of $\sim n \times \frac{1}{2}\% \simeq 5\%$.

Obviously the global error here $\propto h$, so we can get more accurate results by decreasing h. Figure 3 shows the solution found by the Euler method with $h = 0.1$ together with the analytic solution. Figure 4 shows the improvement we obtain when we decrease h to 0.05. Cearly we cannot decrease h indefinitely, to obtain a given accuracy, because of the growth of round-off errors when the number of calculations increases. We can do better than this !

2. Improved Euler Method ("Huen's Method")

The principal contribution to the error in Euler's method is the neglect of curvature in the intervals $[x_i, x_{i+1}]$. Rather than take a constant slope across the interval equal to the initial value, we can use the average across the interval. This is shown in Figure 5 below.

Hence we can write

$$y_{i+1} = y_i + h \left[\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} \right].$$

The problem with this equation is that it is *implicit*, with y_{i+1} appearing on both sides of the equation. To avoid this problem we will **predict** a value of y_{i+1} to use on the right-hand-side, and then use this to provide a **corrected** value. We predict by using the Euler method:

$$\bar{y}_{i+1} = y_i + hf(x_i, y_i)$$

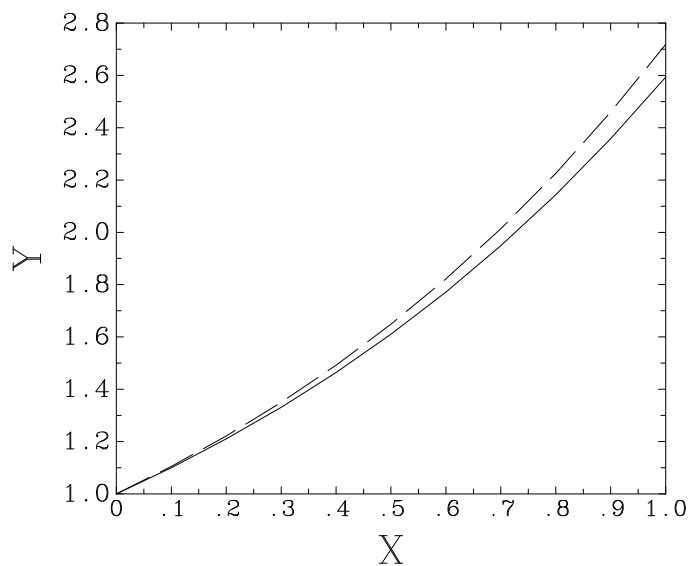Exact solution shown dashed,
Euler method with h=0.100 shown solid.

Figure 3: Results of Euler Method with $h = 0.1$.

Exact solution shown dashed,
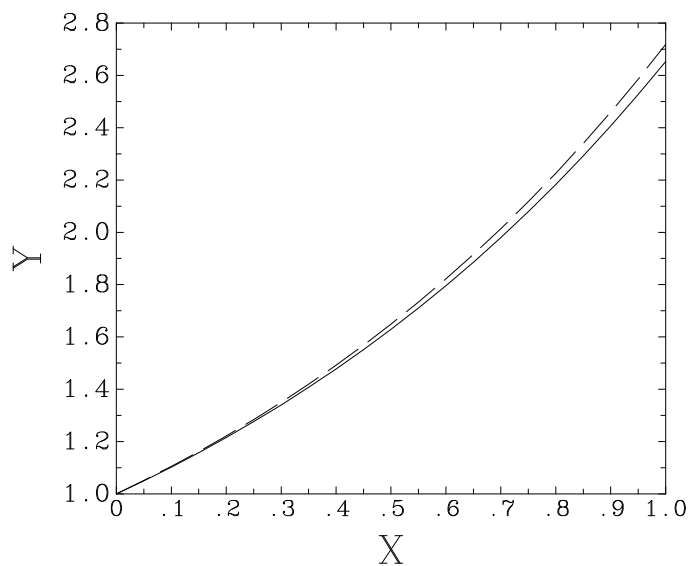Euler method with h=0.050 shown solid.

Figure 4: Results of Euler Method with $h = 0.05$.

and then use this in the above formula for y_{i+1}:

$$y_{i+1} = y_i + h \left[\frac{f(x_i, y_i) + f(x_{i+1}, \bar{y}_{i+1})}{2} \right].$$

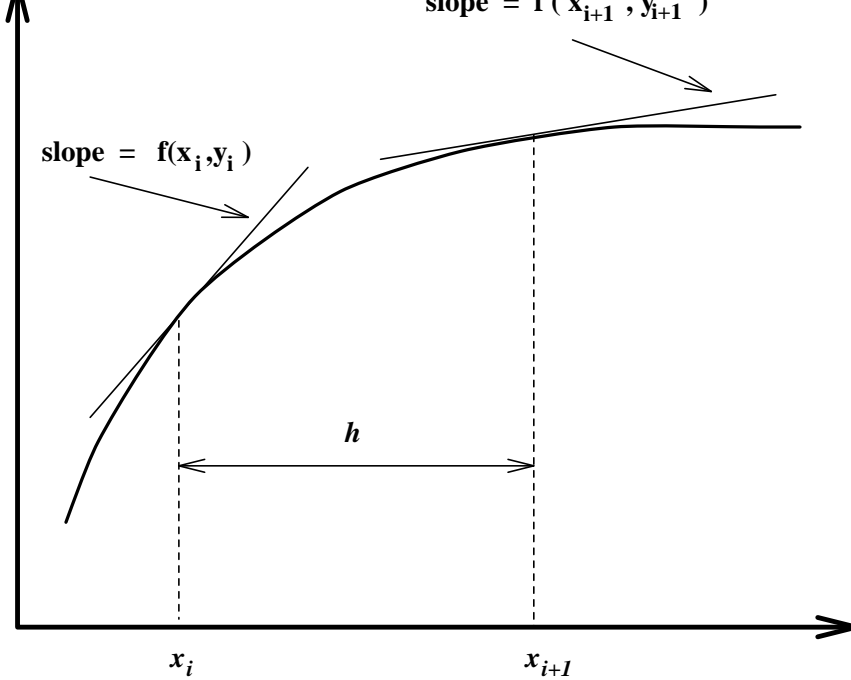*e.g.* Solve $\frac{dy}{dx} = y$ on $[0, 1]$ again, with $h = 0.1$.

137

Figure 5: The Improved-Euler Method.

Here

$$\bar{y}_{i+1} = y_i + hf(x_i, y_i) = y_i + 0.1y_i = 1.1y_i \quad \text{(the Euler method value)}$$

$$\Rightarrow \quad y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + h, \bar{y}_{i+1})\right]$$

$$= y_i + \frac{0.1}{2}\left[y_i + 1.1y_i\right]$$

$$= 1.105 y_i$$

Using this we can construct the following table:

x_i	y_i	% error
0.0	1.000	0.00
0.1	1.105	0.02
0.2	1.221	0.03
0.3	1.349	0.05
0.4	1.491	0.06
0.5	1.647	0.08
0.6	1.820	0.09
0.7	2.012	0.11
0.8	2.223	0.12
0.9	2.456	0.14
1.0	2.714	0.15

Clearly this is much more accurate than the Euler method. Figure 6 shows this solution plotted against the exact solution, and Figure 7 shows the results obtained with $n = 20$.

Let's look at the error in the Improved Euler Method. Using the Taylor's Series for 2 variables:

$$f(x_{i+1}, \bar{y}_{i+1}) = f(x_i + h, y_i + k)$$

where $k = hf(x_i, y_i)$ has been substituted to make the expansion clear. Proceeding with the expansion:

$$f(x_{i+1}, \bar{y}_{i+1}) = f(x_i, y_i) + h\frac{\partial f}{\partial x}(x_i, y_i) + k\frac{\partial f}{\partial y}(x_i, y_i) + O(h^2).$$

138

Exact solution shown dashed,
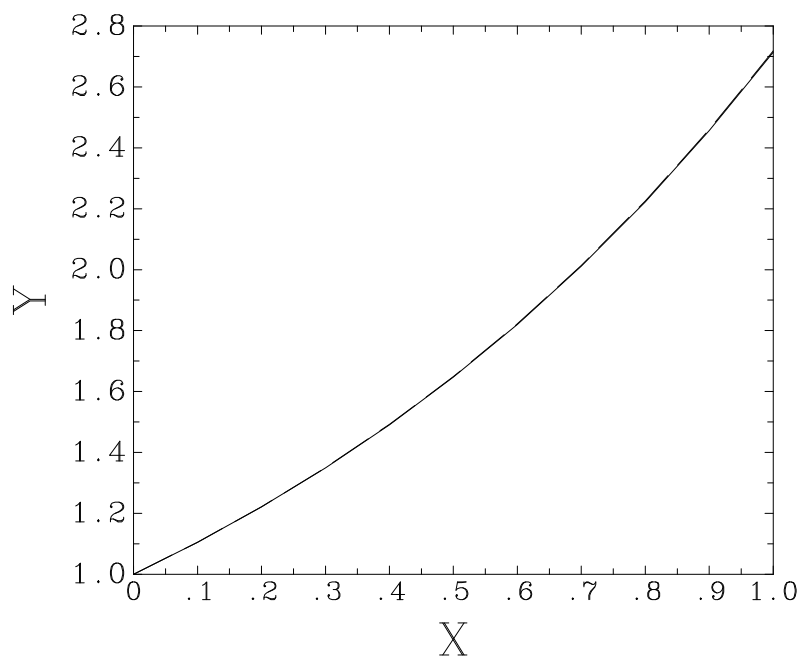Improved−E with h=0.100 shown solid.

Figure 6: Results with Improved-Euler method and $h = 0.1$.

If we now substitute in the expression for k we find

$$f(x_{i+1}, \bar{y}_{i+1}) = f(x_i, y_i) + h\frac{\partial f}{\partial x}(x_i, y_i) + (hf(x_i, y_i))\frac{\partial f}{\partial y}(x_i, y_i) + O(h^2)$$
$$= \frac{dy}{dx} + h\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right] + O(h^2)$$
$$= \frac{dy}{dx} + h\frac{df}{dx} + O(h^2).$$

But $f = \frac{dy}{dx}$ so:

$$f(x_{i+1}, \bar{y}_{i+1}) = \frac{dy}{dx} + h\frac{d^2y}{dx^2} + O(h^2).$$
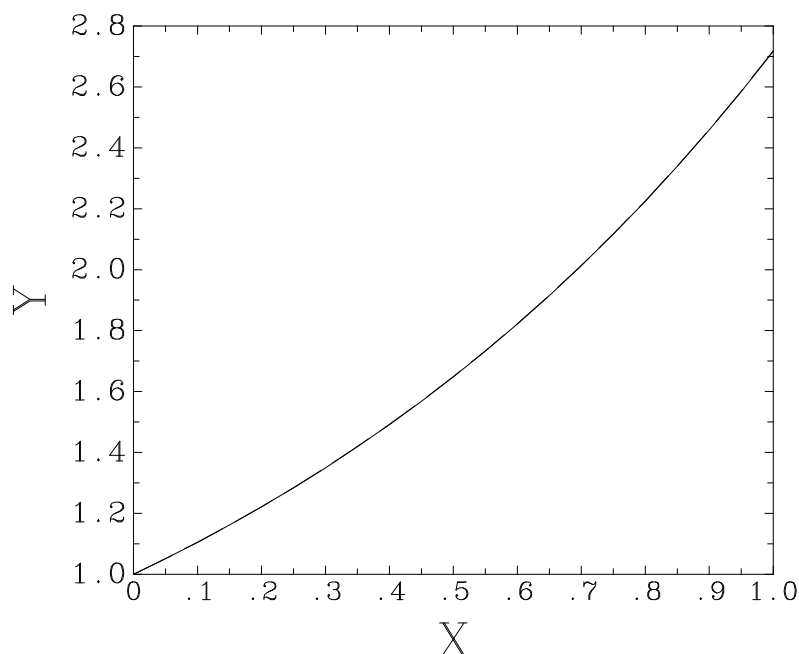
Figure 7: Results with Improved-Euler method and $h = 0.05$.

Hence, substituting this into the Improved Euler formula:

$$
\begin{aligned}
y_{i+1} &= y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i))\right] \\
&= y_i + \frac{h}{2}\left[\frac{dy}{dx} + \frac{dy}{dx} + h\frac{d^2y}{dx^2} + O(h^2)\right] \\
&= y_i + h\frac{dy}{dx} + \frac{h^2}{2}\frac{d^2y}{dx^2} + O(h^3).
\end{aligned}
$$

This is a Taylor Series to second order in h, and hence the local truncation error is $O(h^3)$. Thus the global error is $O(nh^3) \sim O(h^2)$ since $nh = b - a$.

In our previous example

$$
\begin{aligned}
y(1) &= 2.7172 \text{ with } n = 10 : \text{ so the global error } = 0.15\% \\
y(1) &= 2.7140 \text{ with } n = 20 : \text{ so the global error } = 0.04\%
\end{aligned}
$$

So with h decreasing by a factor of 2 the error decreased by a factor of 4, as expected. Note that to achieve an error of 0.1% at $x = 1$ would require

— $h = \frac{1}{480}$ with the Euler Method.

Thus there are two important advantages of the Improved Euler over the Euler:

— the error is smaller for a given h

— decreasing h increases the accuracy more quickly in the Improved Euler than in the Euler.

Finally, consider the case where

$$\frac{dy}{dx} = f(x) \tag{1}$$

i.e. the right-hand-side does not include y. Then the predictor step is not required and the corrector equation is

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i) + f(x_{i+1})\right].$$

We now demonstrate that this is equivalent to the Trapezoidal Rule. From (1):

$$\int_{y_i}^{y_{i+1}} dy = \int_{x_i}^{x_{i+1}} f(x)dx$$

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} f(x)dx$$

$$\text{or } y_{i+1} = y_i + \int_{x_i}^{x_{i+1}} f(x)dx.$$

Now, using the Trapezoidal Rule to evaluate the integral:

$$\int_{x_i}^{x_{i+1}} f(x)dx = \left(\frac{x_{i+1} - x_i}{2}\right)\left(f(x_{i+1}) + f(x_i)\right).$$

Substituting:

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_{i+1}) + f(x_i)\right],$$

as required. Thus the Improved Euler method is equivalent to the Trapezoidal Rule.

A goodway to estimate the relative amount of computer time required to solve a differential equation is to count the number of calls to the function routine. The overheads are essentially proportional to this figure, so twice as many calls will mean twice as long to run, in general. Note that the Improved Euler method uses two function calls per step, which is twice as many as the Euler method. So for the same amount of computer time (or actual dollar cost, if one is paying for time on a major computer) one can use steps which are half as small in the Euler Method than in the Improved Euler method. The advantage lies in the fact that the Improved Euler method with step $2h$ is usually more accurate than the Euler with step h, as we saw above.

Another simple modification to the Euler Method is to use the slope of the mid-point of $[x_i, x_{i+1}]$:

$$y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}),$$

where $x_{i+\frac{1}{2}} = x_i + h/2$ and for $y_{i+\frac{1}{2}}$ we use the Euler method:

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2}f(x_i, y_i).$$

Like the Improved Euler method, the Modified Euler Method has a global error $O(h^2)$, and uses two function calls per step.

4. Fourth-Order Runge-Kutta Method

The Euler, Improved Euler and Modified Euler are particular examples of a class of techniques known as "Runge-Kutta" methods, which have the general form

$$y_{i+1} = y_i + h\phi$$

where ϕ is some approximation to the slope. For example:

a) $\phi = k_1$ where $k_1 = f(x_i, y_i)$ is the Euler Method.

b) $\phi = k_2$ where $k_2 = f(x_i + \frac{1}{2}h, y_i + h\frac{k_1}{2})$ is the Modified Euler Method.

c) $\phi = \frac{1}{2}(k_1 + k_2^\star)$ where $k_2^\star = f(x_i + h, y_i + hk_1)$ is the Improved Euler Method.

The most popular method of this class is the 4th Order Runge-Kutta Method, which has a global error of order h^4 which means that the local truncation error is $O(h^5)$. In this method we take:

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2})$$
$$k_3 = f(x_i + \frac{h}{2}, y_i + \frac{hk_2}{2})$$
$$k_4 = f(x_i + h, y_i + hk_3)$$

and

$$y_{i+1} = y_i + h\phi$$

where

$$\phi = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

e.g. Solve $\frac{dy}{dx} = y$ (again !) with $y(0) = 1$ but with only TWO subdivisions of $h = \frac{1}{2}$.

The first step is

$$k_1 = y_0 = 1$$
$$k_2 = y_0 + \frac{hk_1}{2} = 1.25$$
$$k_3 = y_0 + \frac{hk_2}{2} = 1.3125$$
$$k_4 = y_0 + hk_3 = 1.65625$$

And hence

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.64844.$$

And the second step is

$$k_1 = y_1 = 1.64844$$
$$k_2 = y_1 + \frac{hk_1}{2} = 1.25y_1$$
$$k_3 = y_1 + \frac{hk_2}{2} = 1.3125y_1$$
$$k_4 = y_1 + hk_3 = 1.65625y_1$$

And hence

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.71735.$$

Thus we have a global error of only 0.03% with $n = 2$! This is more accurate than the Improved Euler method with $n = 20$!

Concerning computational efficiency, we note that each step of the 4th order Runge-Kutta method take 4 function evaluations (one per k value). So $n = 2$ uses 8 evaluations, whereas the Improved Euler with $n = 20$ uses 40 function evaluations ! So the RK4 method gave a better answer with less work. Figure 8 shows the log of the errors for this problem with $n = 10$ and each of the Euler, Improved Euler, and RK4 methods.
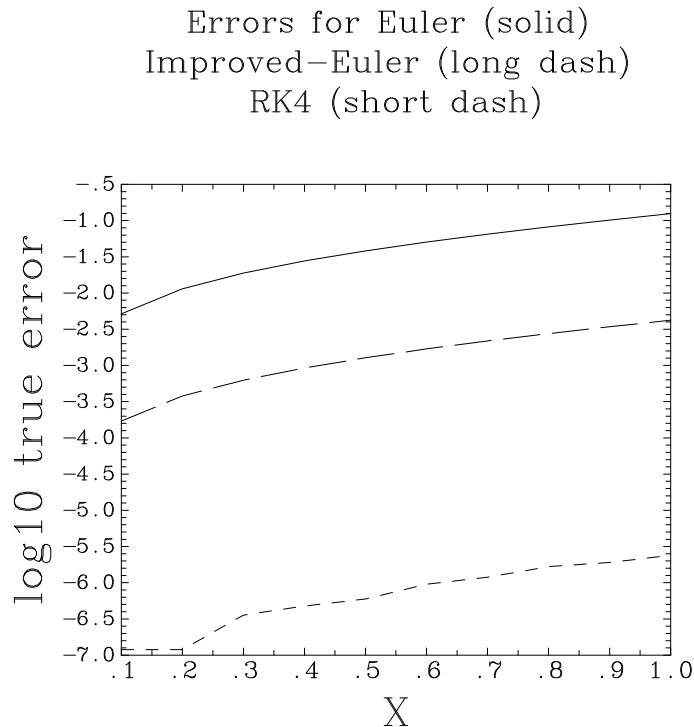

Figure 8: Comparison of errors with the different methods.

143

5. Errors

We'll now look at an example, but keep the number of function evaluations constant, so that each method takes the same amount of computer time. Consider now the d.e.

$$\frac{dy}{dx} = -y + 1$$

over the interval $[0, 0.5]$ with $y(0) = 0$. Suppose we solve this with RK4 using $h = 0.1$, so that $n = 5$. This is 20 function evaluations. If the Improved Euler method is to step across this interval with 20 function evaluations, then we need 10 steps of $h = 0.05$ because each step takes two function evaluations. And for the Euler Method, as each step needs one function evaluation, then we take 20 steps across the interval, or $h = 0.025$. The table below shows the results:

x	E ($h = 0.025$)	IE ($h = 0.05$)	RK4 ($h = 0.1$)	Exact
0.1	0.096312	0.095120	0.09516250	0.09516258
0.2	0.183348	0.181198	0.18126910	0.18126925
0.3	0.262001	0.259085	0.25918158	0.25918178
0.4	0.333079	0.329085	0.32967971	0.32967995
0.5	0.397312	0.393337	0.39346906	0.39346934

This shows quite clearly that even for the same amount of work. the RK4 method is significantly better, and hence the method of choice.

In figure 9 we apply the three methods we have developed to the solution of the d.e.

$$\frac{dy}{dx} = 5x^4$$

with $y(-1) = -1$, and $n = 10$. In figure 10 we repeat this, but with $n = 20$. For the Euler Method the log of the error decreases from about -0.3 to -0.6, which is a factor of 2 smaller when h was halved. For the Improved Euler method the log of the error decreases from -1.3 to -1.8, which is about a factor of 4 smaller. And, for RK4 the log of the error goes from about -4 or -5 to -5.2 or -6.5, which is about a factor of 16 smaller. Thus the errors decrease as

— h for the Euler method,

— h^2 for the Improved Euler method,

— h^4 for the RK4 method.

Finally, Figure 11 shows the solution to

$$\frac{dy}{dx} = y + x$$

with $y(0) = 0$ and the three methods discussed so far. Again, we have adjusted n in each case so that the amount of work done in each case is identical. The solid line is the analytical solution, and we see that even though we used only $n = 5$ for the RK4 method, the solution is very good, and well ahead of both the Improved Euler (with $n = 10$) and the Euler (with $n = 20$).
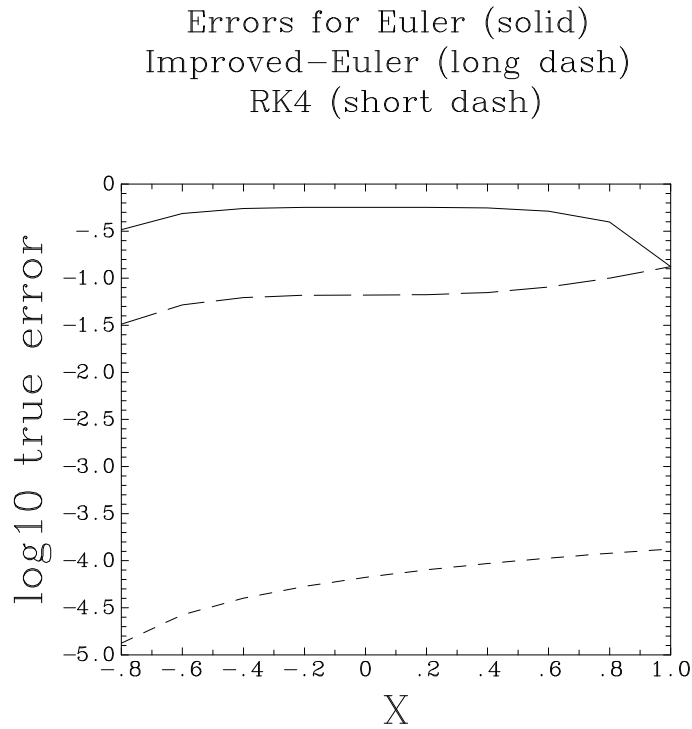
Errors for Euler (solid)
Improved−Euler (long dash)
RK4 (short dash)

Figure 9: Comparison of errors in solving $\frac{dy}{dx} = 5x^4$ with $n = 10$.

Errors for Euler (solid)
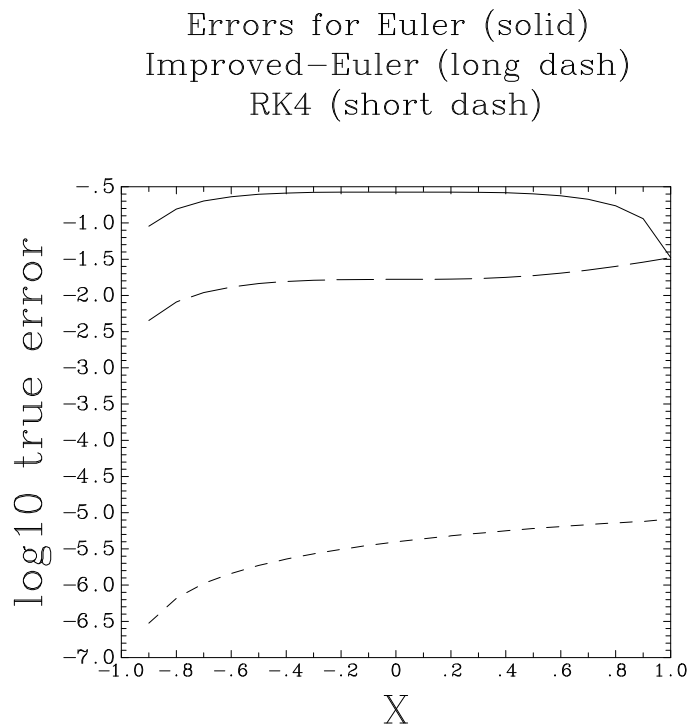Improved−Euler (long dash)
RK4 (short dash)

Figure 10: Comparison of errors in solving $\frac{dy}{dx} = 5x^4$ with $n = 20$.

6. Error Control: The Runge-Kutta-Fehlberg Method

Ideally, we wish to specify the maximum error we will accept in our numerical solution, and then expend the minimum effort necessary to achieve this. In general this is not consistent with a constant step-length. So we now look at a method which uses an adaptive step-length h.

Solution with Euler (o),
Improved−E (+) and RK4 (*).
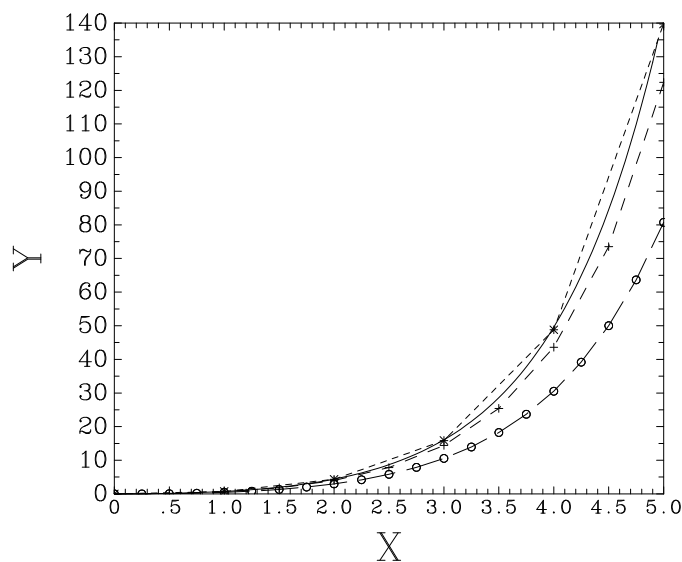Analytic solution is solid line

Figure 11: Comparison with different n but same amount of CPU time.

One technique, as we saw in Richardson Extrapolation, is to do the calculation once with h and then again with $h/2$, enabling us to estimate the errors. We shall use a related, but more advanced technique here.

Suppse $Y(x)$ is an exact solution of

$$\frac{dy}{dx} = f(x, y).$$

Let y be a solution from a method with global error $O(h^4)$, such as the RK4 method. Let z be a solution from a method with global error $O(h^5)$ (e.g. a different RK scheme, say RK5). Then the local truncation error is

$$y_i = Y_i + \alpha h^5,$$
$$z_i = Y_i + \beta h^6.$$

Thus

$$|y_i - z_i| = Y_i + \alpha h^5 - (Y_i + \beta h^6)$$
$$= \alpha h^5 + O(h^6). \tag{1}$$

Suppose we now want to make the same step but with $h' = sh$. Then the solution $\bar{y}_i$ has truncation error:

$$\bar{y}_i - Y_i = \alpha(sh)^5 = \alpha s^5 h^5.$$

If we limit this error to the value T then

$$\alpha s^5 h^5 = T,$$
$$\Rightarrow \alpha h^5 = \frac{T}{s^5}.$$

146

Substituting this into (1):

$$|y_i - z_i| = \frac{T}{s^5}$$

$$\Rightarrow s = \left(\frac{T}{|y_i - z_i|}\right)^{1/5}.$$

This determines s, given y_i and z_i, and hence tells us the new step to use so that our error is below the specified tolerance. To be sure, one usually replaces T by $T/2$ so that

$$s = \left(\frac{T}{2|y_i - z_i|}\right)^{1/5}.$$

Alternatively, one can look at the global error:

$$y_i = Y_i + \alpha(sh)^4$$

and hence

$$|y_i - Y_i| = \alpha s^4 h^4 = T \quad \Rightarrow \alpha h^4 = \frac{T}{s^4}.$$

Substituting this into equation (1):

$$\frac{hT}{s^4} = |y_i - z_i|$$

and

$$\Rightarrow s = \left(\frac{hT}{|y_i - z_i|}\right)^{1/4}.$$

Note that this is a factor of $h^{1/4}$ stricter than our previous criterion. Again, it is common to let $T \to T/2$:

$$s = \left(\frac{hT}{2|y_i - z_i|}\right)^{1/4}. \tag{2}$$

The problem that remains is to find y and z !

Well, we already have a method which has global error $O(h^4)$—this is the RK4 method. If we can devise a 5th order method it will, in general, have it's own 5 k's, giving a total of 9 k's to be evaluated at each step. This is very expensive in computer time. But these k's are not unique, and Fehlberg (in 1966) found a set of 6 k's which would allow a 4th order **and** a 5th order solution ! This is only one function evaluation more than any 5th order method. This is the so-called "Runge-Kutta-Fehlberg" method (RKF). It uses

$$y_{i+1} = y_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5.$$
$$z_{i+1} = z_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 + \frac{9}{50}k_5 + \frac{2}{55}k_6.$$

147

$$k_1 = hf(x_i, y_i)$$

$$k_2 = hf(x_i + \frac{h}{4}, y_i + \frac{1}{4}k_1)$$

$$k_3 = hf(x_i + \frac{3h}{8}, y_i + \frac{3}{32}k_1 + \frac{9}{32}k_2)$$

$$k_4 = hf(x_i + \frac{12h}{13}, y_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3)$$

$$k_5 = hf(x_i + h, y_i + \frac{439}{210}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4)$$

$$k_6 = hf(x_i + \frac{1}{2}h, y_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2566}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5)$$

Then we adjust $h \to sh$ according to

$$s = \left(\frac{hT}{2|y_i - z_i|} \right)^{1/4}.$$

Typically we restrict the rate of change of h with something like:

```
if (s > 2) then
        Set  h = 2h
else if (s < 0.5) then
        Set  h = 0.5h
else
        Set  h = sh
endif
if (h < h_min) set  h = h_min
if (h > h_max) set  h = h_max
```

It is important to note neither of our formulae for s will actually work in practice. The reason is because we are using the *current* values of y and z, and these include all accumulated differences from the first step. In reality, we need the difference in the addition to y and z at the current step. So the denominator in the equations for s should not really be $|y_i - z_i|$ but $|y_{(i} - y_{i-1}) - z_{(i} - z_{i-1})|$. This is purely because we derived the formula assuming that we started with the exact valuye Y, and looked at the change. If we accumulate the change, however, then the errors begin to accumulate also. Hence we must look only at the difference in the increments. Think about this, and it should all become clear (eventually).

7. Higher-Order Differential Equations

The earlier methods can be easily extended to 2nd or higher order differential equations. Here we seek a solution of

$$\frac{d^2y}{dx^2} = F(x, y, \frac{dy}{dx}),$$

where F is given and $y(x_0) = y_0$ and $\frac{dy}{dx}(x_0) = z_0$, say. Note that we have two (because it is a second order equation) **initial** conditions, and hence this is an **initial** value problem. If we have $y(x_0) = y_0$ and $y(x_n) = y_n$ then we have **boundary** conditions and we say the problem is a **boundary** value problem. These are more subtle, and we will discuss them briefly later.

The basis of our approach is to split the 2nd order d.e. into two first order d.e.s by defining a new dependent variable:

$$z = \frac{dy}{dx}.$$

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = F(x, y, z).$$

This is a special case of the coupled pair of equations

$$\frac{dy}{dx} = G(x, y, z)$$

$$\frac{dz}{dx} = F(x, y, z)$$

where $G = z$. But the principle is the same for both: we simply apply our chosen method to the two first order d.e.s (noting that the z_i and y_i resulting from $\frac{dz}{dx}$ and $\frac{dy}{dx}$ are then used in each d.e. Note that this principle can be generalised to any number of coupled d.e.s, and hence to any d.e. of any order.

So, for a second order d.e. solved by the Euler Method we have:

$$y_{i+1} = y_i + hz_i$$

$$z_{i+1} = z_i + hF(x_i, y_i, z_i)$$

for $i = 0, 1, \ldots, n$. Note that these equations are coupled: we need both y_i and z_i before we can get y_{i+1} or z_{i+1}, We cannot solve for all the y_i and then the z_i (or vice versa).

For the improved Euler method we have a predictor and a corrector step for each variable:

Predict:
$$\bar{y}_{i+1} = y_i + hz_i$$
$$\bar{z}_{i+1} = z_i + hF(x_i, y_i, z_i)$$

Correct:
$$y_{i+1} = y_i + \frac{1}{2}h(z_i + \bar{z}_{i+1})$$
$$z_{i+1} = z_i + \frac{1}{2}h\left(F(x_i, y_i, z_i) + F(x_{i+1}, \bar{y}_{i+1}, \bar{z}_{i+1})\right)$$

e.g. Consider the d.e.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 1,$$

with $y(0) = 0$ and $\frac{dy}{dx}(0) = 0$. Here we set $\frac{dy}{dx} = z$ and thus

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} = 1 - 2\frac{dy}{dx} - y.$$

We have $y_0 = 0$ and $z_0 = 0$. Using 2 steps of the Improved Euler Method yields:

$$\bar{y}_1 = y_0 + hz_0 = 0 \qquad \bar{z}_1 = z_0 + h(1 - y_0 - 2z_0) = \frac{1}{2}$$

$$y_1 = y_0 + \frac{1}{2}h(z_0 + \bar{z}_1) \qquad z_1 = z_0 + \frac{1}{2}h\left[(1 - y_0 - 2z_0) + (1 - \bar{y}_1 - 2\bar{z}_1)\right]$$

$$= \frac{1}{8} \qquad\qquad\qquad = \frac{1}{4}$$

$$\bar{y}_2 = y_1 + hz_1 = \frac{1}{4} \qquad \bar{z}_2 = z_1 + h(1 - y_1 - 2z_1) = \frac{7}{16}$$

$$y_2 = y_1 + \frac{1}{2}h(z_1 + \bar{z}_2) \qquad z_2 = z_1 + \frac{1}{2}h\left[(1 - y_1 - 2z_1) + (1 - \bar{y}_2 - 2\bar{z}_2)\right]$$

$$= \frac{19}{64} \qquad\qquad = \frac{5}{16}$$

Thus we estimate $y(1) = 19/64 \simeq 0.297$. Using $h = \frac{1}{4}$ yields $y(1) \simeq 0.270$. The exact solution is

$$y(x) = 1 - (x+1)e^{-x}$$

and hence $y(1) = 0.2642$. Of course, one can now use Richardson Extrapolation on our two 2nd order approximations to obtain:

$$y(1) \simeq \frac{4}{3} \times 0.270 - \frac{1}{3} 0.297 \simeq 0.261.$$

Alternatively, of course, we could use the RK4 scheme. In this case we would have

$$
\begin{aligned}
y_{i+1} &= y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\
z_{i+1} &= z_i + \frac{h}{6}\left(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4\right)
\end{aligned}
$$

where

$$
\begin{aligned}
k_1 &= f(x_i, y_i, z_i) = z_i \\
k_2 &= f(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2}, z_i + \frac{h\ell_1}{2}) = (z_i + \frac{h\ell_1}{2}) \\
k_3 &= f(x_i + \frac{h}{2}, y_i + \frac{hk_2}{2}, z_i + \frac{h\ell_2}{2}) = (z_i + \frac{h\ell_2}{2}) \\
k_4 &= f(x_i + h, y_i + hk_3, z_i + h\ell_3) = (z_i + h\ell_3) \\[6pt]
\ell_1 &= g(x_i, y_i, z_i) \\
\ell_2 &= g(x_i + \frac{h}{2}, y_i + \frac{hk_1}{2}, z_i + \frac{h\ell_1}{2}) \\
\ell_3 &= g(x_i + \frac{h}{2}, y_i + \frac{hk_2}{2}, z_i + \frac{h\ell_2}{2}) \\
\ell_4 &= g(x_i + h, y_i + hk_3, z_i + h\ell_3)
\end{aligned}
$$

One could also use the RKF method, etc etc. And, as said earlier, this technique can be used to solve any system of o.d.e.s of any order.

8. Boundary Value Problems

We limit our discussion in this section to 2nd order differential equations. In this case we do not know $\frac{dy}{dx}$ initially, but rather we know $y(a) = A$ and $y(b) = B$. There are two main techniques for solving these problems:

    i) the shooting method

    ii) finite-difference methods.