ASP3012
Sheet 5

Q1: $R \sim M^{0.6}$ $\therefore \left(\frac{R_2}{R_\odot}\right) = \left(\frac{M_2}{M_\odot}\right)^{0.6}$

$$R_2 = 2^{0.6} R_\odot = 1{\cdot}52 R_\odot$$

$$R_5 = 5^{0.6} R_\odot = 2{\cdot}63 R_\odot$$

$L \sim M^{3.5}$ $\therefore L_2 = 2^{3.5} L_\odot = 11{\cdot}3 L_\odot$

$$L_5 = 5^{3.5} L_\odot = 279.5 L_\odot$$

Q2 a) $\frac{dP}{dr} = -\frac{4\pi}{3} G\rho_c^2 r e^{-x^2}$ $(x = r/a)$

$$\int_r^R dP = -\frac{4\pi}{3} G\rho_c^2 \int_r^R r e^{-x^2} dr \qquad R = \text{surface}$$

$$\underset{0}{\overset{\shortparallel}{P(R)}} - P(r) = -\frac{4\pi}{3} G\rho_c^2 a^2 \int x e^{-x^2} dx$$

$$= \frac{4\pi}{3} G\rho_c^2 a^2 \left[-\frac{1}{2} e^{-x^2}\right]_{r/a}^{R/a}$$

$$= \frac{2\pi}{3} G\rho_c^2 a^2 \left[e^{-(R/a)^2} - e^{-(r/a)^2}\right]$$

$$R >> a \quad \therefore e^{-(R/a)^2} << e^{-(r/a)^2}$$

$$\therefore P(r) = \frac{2\pi}{3} G\rho_c^2 a^2 e^{-(r/a)^2}$$

$$= P_c e^{-x^2} \quad \text{where } P_c = \frac{2\pi}{3} G\rho_c^2 a^2$$

b) $\frac{dP}{dr} = -\rho\frac{GM}{r^2}$ $\quad \frac{dM}{dr} = 4\pi r^2\rho$

$$\therefore \frac{dP}{dm} = -\frac{\cancel{\rho} GM}{r^2 \cdot 4\pi r^2 \cancel{\rho}} = -\frac{GM}{4\pi r^4}$$

$$G\int m\,dm = -4\pi\int r^4 \, dP$$

$$G\int_0^n m\,dm = -4\pi\int_0^n r^4 \cdot \frac{dP}{dr}\,dr$$

$$\frac{GM(r)^2}{2} = -4\pi \cdot \int_0^r r^4 \cdot \left(-\frac{4\pi}{3} G\rho_c^2 r e^{-x^2}\right) dr$$

$$= \frac{16\pi^2 G\rho_c^2}{3}\int_0^r r^5 e^{-x^2}\,dr$$

$x = r/a$
$dx = \frac{1}{a}dr$

$$= \frac{16\pi^2 G\rho_c^2 a^6}{3}\int_0 x^5 e^{-x^2}\,dx$$

Let $t = x^2$
$dt = 2x\,dx$

$$= \frac{(\quad)}{2}\int t^2 e^{-t}\,dt$$

Integrate by parts: $u = t^2$ $\quad dv = e^{-t}dt$
$du = 2t\,dt$ $\quad v = -e^{-t}$

$$= \left(\frac{8\pi^2 G\rho_c^2 a^6}{3}\right)\left[-t^2 e^{-t} + 2\int t e^{-t}\,dt\right]$$

Integrate by parts again:

$$u = t \qquad dv = e^{-t}dt$$
$$du = dt \qquad v = -e^{-t}$$

$$= (\quad)\left\{-t^2e^{-t} + 2\left[-te^{-t} + \int e^{-t}dt\right]\right\}$$

$$= (\quad)\left[-t^2e^{-t} + 2te^{-t} - 2e^{-t}\right]_0^r$$

$$= (\quad)\left(-t^2 - 2t - 2\right)e^{-t}\Big]_0^r$$

$$= (\quad)\left[\left(-t^2 - 2t - 2\right)e^{-t} + 2\right]$$

$$\frac{Gm^2(r)}{2} = (\quad)\left[2 - e^{-x^2}(x^4 + 2x^2 + 2)\right]$$

$$\therefore m^2(r) = \frac{16\pi^2\rho_c^2a^6}{3\times3}\left[2\times3 - 3e^{-x^2}(x^4 + 2x^2 + 2)\right]$$

$$m(r) = \frac{4\pi\rho_c a^3}{3}\left[6 - 3e^{-x^2}(x^4 + 2x^2 + 2)\right]^{1/2}$$

$$= \frac{4\pi\rho_c a^3}{3}\,\Phi(x)$$

(c) $\dfrac{dm}{dr} = 4\pi r^2\rho$

$$\rho(r) = \frac{1}{4\pi r^2}\cdot\frac{dm}{dr}$$

$$= \frac{1}{4\pi r^2}\cdot\frac{4\pi\rho_c a^3}{3}\cdot\frac{d\Phi}{dr}$$

$$= \frac{1}{\cancel{4\pi} r^2}\,\frac{\cancel{4\pi}\rho_c a^3}{3}\,\frac{d\Phi}{dx}\cdot\frac{dx}{dr}$$

$$= \frac{\rho_c a^2}{3r^2}\frac{d\Phi}{dx} = \frac{\rho_c}{3x^2}\frac{d\Phi}{dx}. \quad \text{— (1)}$$

Now
$$2\Phi\frac{d\Phi}{dx} = 6x(x^4+2x^2+2)e^{-x^2} - 3e^{-x^2}(4x^3+4x)$$
$$= e^{-x^2}\left[6x^5+12x^3+12x-12x^3-12x\right]$$
$$= 6x^5e^{-x^2}$$

$$\therefore \frac{d\Phi}{dx} = \frac{3x^5e^{-x^2}}{\Phi(x)} \qquad \text{sub in (1)}$$

$$\rho(r) = \frac{\rho_c}{3x^2}\cdot\frac{3x^5 \cdot e^{-x^2}}{\Phi(x)}$$

$$= \rho_c x^3 e^{-x^2}/\Phi(x).$$

d) We will need to work to order x^4 in $\Phi(x)$ to get the required terms. ie to order x^8 in $\Phi^2(x)$.

$$\Phi^2(x) = 6 - 3(x^4+2x^2+2)e^{-x^2}$$
$$= 6 - (3x^4+6x^2+6)\left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} + \dots\right)$$
$$= \cancel{6} - \cancel{3x^4} - \cancel{6x^2} - \cancel{6} + 3x^6$$
$$+ \cancel{6x^4} + \cancel{6x^2} \qquad -\frac{3x^8}{2}$$
$$-\cancel{3x^4} \qquad -3x^6 \quad +x^8$$
$$+x^6 \quad -\frac{x^8}{4} + \dots$$
$$= x^6 + \frac{x^8}{4}(-6+4-1) + \dots$$

$$= x^6 - \frac{3}{4}x^8 + \dots$$

$$\therefore \Phi(x) \simeq (x^6 - \frac{3}{4}x^8 + \dots)^{1/2}$$

e) $\rho(r) = \rho_c x^3 e^{-x^2} / \Phi(x)$

So, near the centre, working to $O(x^2)$:

$$\rho(r) = \frac{\rho_c x^3 (1 - x^2 + \frac{x^4}{2} + \dots)}{[x^6 (1 - \frac{3}{4}x^2 + \dots)]^{1/2}}$$

$$= \frac{\rho_c \cancel{x^3}}{\cancel{x^3}} (1 - x^2 + \frac{x^4}{2} + \dots)(1 + \frac{3}{8}x^2 + \dots)$$

$$= \rho_c (1 - x^2 + \frac{3}{8}x^2 + O(x^4))$$

$$= \rho_c (1 - 5/8\, x^2 + \dots)$$

f) $P = \rho \mathcal{R} T / \mu \quad \therefore T = \frac{\mu}{\mathcal{R}} \frac{P}{\rho}$

$$T(x) = \frac{\mu}{\mathcal{R}} \frac{P(x)}{\rho(x)} = \frac{\mu}{\mathcal{R}} \frac{P_c \cancel{e^{-x^2}}}{\rho_c x^3 \cancel{e^{-x^2}} / \Phi(x)}$$

$$= \left(\frac{\mu P_c}{\mathcal{R} \rho_c}\right) \frac{\Phi(x)}{x^3}$$

$T_c \rightarrow$

$$= \frac{T_c [x^6 - 3/4\, x^8 + \dots]^{1/2}}{x^3}$$

$$= T_c [1 - 3/4\, x^2 + \dots]^{1/2}$$

$$= T_c (1 - 3/8\, x^2 + \dots)$$

On

$$T_c = \frac{\mu}{R} \frac{P_c (1 - x^2 + x^4/2 + ---)}{\rho_c (1 - 5/8\, x^2 + ---)}$$

$$= T_c \left(1 - x^2 + \frac{x^4}{2}\right)(1 + 5/8\, x^2 + ---)$$

$$= T_c (1 - x^2 + 5/8\, x^2 + ---)$$

$$= T_c (1 - 3/8\, x^2 + ---)$$

9) $$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$$

$$= 4\pi a^2 x^2 \rho\, \varepsilon_0 \rho X^2 T^n$$

$$= 4\pi a^2 \varepsilon_0 X^2 \rho_c^2 T_c^n x^2 \left(1 - \frac{5}{8}x^2 + ..\right)^2 \left(1 - \frac{3}{8}x^2 + ..\right)^n$$

$$= (\qquad) x^2 \left(1 - \frac{5}{4}x^2 - \frac{3nx^2}{8} + ...\right)$$

$$= (\qquad) x^2 \left(1 - \frac{(10+3n)}{8} x^2 + --\right)$$

$$= (\qquad) \left(x^2 - \frac{(10+3n)}{8} x^4 + ---\right)$$

$$\int_0^r dL = (\qquad) a \int_0^r \left(x^2 - \frac{(10+3n)}{8} x^4 + ---\right) dx$$

$$L(r) - \underset{0}{\underset{\parallel}{L(0)}} = (\qquad) \left[\frac{x^3}{3} - \frac{(10+3n)}{8\times 5} x^5 + ..\right]_0^x$$

$$L(r) = 4\pi a^3 \varepsilon_0 \rho_c^2 T_c^n X^2 \left[\frac{x^3}{3} - \frac{(10+3n)}{40} x^5 + ...\right]$$

h) Integrate to $r = 0.1 R_\odot$ with $a = R_\odot / 5$

ie to $x = \dfrac{0.1 R_\odot}{R_\odot / 5} = 0.5$

$$L(0.5) = \frac{4.\pi.R_\odot^3}{5^3} . 4\times10^{-26} \times (150)^2 (1.5\times10^7)^{3.5} (0.4)^2$$

$$\times \left[\frac{0.5^3}{3} - \frac{(10+10.5)}{40} 0.5^5 \right]$$

$$= 6.38\times10^{34} \times 0.0257 \quad \text{(all in cgs)}$$

$$\simeq 0.4 L_\odot . \quad \text{Not too bad!}$$

i) Integrate to $r = 0.05 R_\odot$ with $a = R_\odot / 5$

ie to $x = \dfrac{0.05 R_\odot}{R_\odot / 5} = 0.25$

$$L(0.25) = \frac{4.\pi.R_\odot^3}{5^3} . 4\times10^{-130} (0.5)^2 (100)^2 (3\times10^7)^{18}$$

$$\times \left[\left(\frac{1}{4}\right)^3 - \left(\frac{10+54}{40}\right) \left(\frac{1}{4}\right)^5 \right]$$

$$= 1.31\times10^{40} \times 0.014$$

$$= 47,800 L_\odot$$

Q3 $\rho(r) = \rho_c(1 - r/R)$

a) $m(r) = \int_0^r 4\pi r^2 \rho \, dr$

$= 4\pi \int_0^r r^2 \rho_c (1 - r/R) \, dr$

$= 4\pi\rho_c \int (r^2 - r^3/R) \, dr$

$= 4\pi\rho_c \left(r^3/3 - \frac{r^4}{4R}\right)$

$= 4\pi\rho_c \left(r_s^3/3 - \frac{r_s^4}{4R}\right)$

$= \frac{4\pi\rho_c r_s^3}{3}\left[1 - \frac{3r_s}{4R}\right]$ $\quad r_s << R$

$\simeq 4\pi\rho_c r_s^3/3$

b) $\frac{dP}{dr} = -\frac{\rho G M}{r^2}$

$= -\rho_c(1 - \frac{r}{R}) G \underbrace{\left(\frac{4\pi}{3}\rho_c r^3\right)}_{m(r) \text{ for small } r} \cdot \frac{1}{r^2}$

$= -4\pi G\rho_c^2 (1 - \frac{r}{R}) \cdot \left(\frac{r}{3}\right)$

$\int_{core}^{shell} dP = -4\pi G\rho_c^2 \int_{core}^{shell} \left(\frac{r}{3} - \frac{r^2}{3R}\right) dr$

$P(s) - P(c) = -4\pi G\rho_c^2 \left(\frac{r^2}{6} - \frac{r^3}{9R}\right)\Big|_{core}^{shell}$

$$= -\frac{4\pi G \rho_c^2 r_s^3}{6}\left(1 - \frac{2}{3}\frac{r_s}{R}\right)$$

$r_s << R$

$$\simeq -\frac{2\pi G \rho_c^2 r_s^2}{3}$$

$$\therefore P_c = P_{shell} + \frac{2\pi G \rho_c^2 r_s^2}{3}$$

c) $P_c >> P_s \Rightarrow P_c \simeq 2\pi G \rho_c^2 r_s^2 / 3 = \dfrac{\rho_c R_0 T}{\mu}$

$$\therefore T_c = \frac{2\pi G \mu \rho_c r_s^2}{3 R_0}$$

d) From (a): $M_c = \frac{4\pi}{3} \rho_c r_s^3$

$$\left(\frac{M_c}{M_\odot}\right) M_\odot = \frac{4\pi}{3}\left(\frac{\rho_c}{1000}\right) 1000 \left(\frac{r_s}{R_\odot}\right)^3 \cdot R_\odot^3$$

$$\left(\frac{M_c}{M_\odot}\right) = \frac{4000\pi}{3 M_\odot}(R_\odot)^3 \left(\frac{r_s}{R_\odot}\right)^3 \left(\frac{\rho_c}{1000}\right)$$

$$= 706 \left(\frac{r_s}{R_\odot}\right)^3 \left(\frac{\rho_c}{1000}\right)$$

$$\left(\frac{r_s}{R_\odot}\right)^3 = \left(\frac{M_c}{M_\odot}\right)\left(\frac{\rho_c}{1000}\right)^{-1} \frac{1}{706}$$

and hence $\left(\dfrac{r_s}{R_\odot}\right) = 0.11 \left(\dfrac{M_c}{M_\odot}\right)^{1/3} \left(\dfrac{\rho_c}{1000}\right)^{-1/3}$

e) $$T_c = \frac{2\pi G \mu \rho_c r_s^2}{3 R_0}$$ from (c)

$$= \frac{2\pi G\mu}{3R}\left(\frac{\rho_c}{1000}\right) 1000 \left(\frac{r_s}{R_\odot}\right)^2 R_\odot^2$$

$$= \frac{2000\pi G}{3R} R_\odot^2 \mu\left(\frac{\rho_c}{1000}\right)(0.11)^2\left(\frac{M_c}{M_\odot}\right)^{2/3}\left(\frac{\rho_c}{1000}\right)^{-2/3}$$

$$= 9.8\times10^7 \mu \left(\frac{M_c}{M_\odot}\right)^{2/3}\left(\frac{\rho_c}{1000}\right)^{1/3}$$