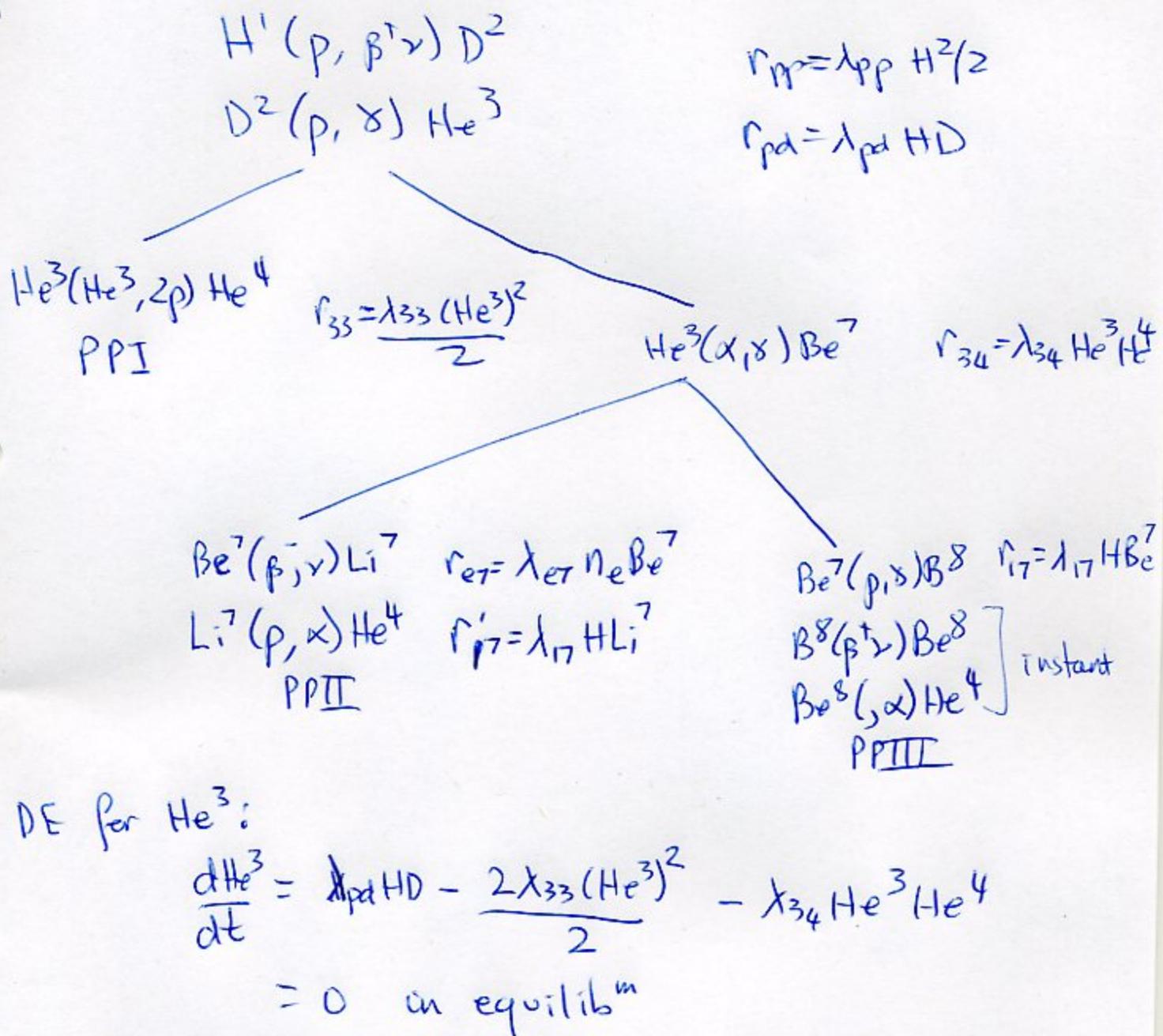


## ASP3012-Sheet 4

Q1



Thus we have a quadratic eq<sup>n</sup> for  $He^3$  abundance.  
The solution is:

$$(He^3)_e = \frac{1}{2\lambda_{33}} \left[ -\lambda_{34} He^4 + \left[ \lambda_{34} (He^4)^2 + 2\lambda_{pp}\lambda_{33} H^2 \right]^{1/2} \right]$$

\* Why take +ve sign in the solution?

Q2 a) PPI makes  $\text{He}^4$  at the rate  $r_{33}$

PPI & III rates are governed by  $r_{34}$ . These each make 2  $\text{He}^4$  per "34" reaction. But they also consume one  $\text{He}^4$  at "34" itself. So

In combination PPI & PPIII make one  $\text{He}^4$  at the rate  $r_{34}$ . Hence the relative production rate is

$$\frac{\text{PPI}}{\text{PPII} + \text{PPIII}} = \frac{r_{33}}{r_{34}}$$

b) If  $\text{He}^3$  is un eq<sup>m</sup> then

$$\frac{r_{33}}{r_{34}} = \frac{\lambda_{33} (\text{He}^3)_e}{2\lambda_{34} (\text{He}^4)}$$

Insert the eq<sup>m</sup> value from Q1:

$$\left(\frac{r_{33}}{r_{34}}\right)_e = \cancel{\lambda_{34}} \cancel{\alpha} \cdot \frac{1}{2\lambda_{33}} \frac{\lambda_{33}}{2\lambda_{34}\text{He}^4} \left[ -\lambda_{34}\text{He}^4 + [\lambda_{34}(\text{He}^4)^2 + 2\lambda_{pp}\lambda_{33}\text{H}^2]^{1/2} \right]$$

$$= \frac{1}{4} \left\{ -1 + \cancel{\lambda_{34}\text{He}^4} \left[ \frac{\lambda_{34}(\text{He}^4)^2 + 2\lambda_{pp}\lambda_{33}\text{H}^2}{\lambda_{34}^2(\text{He}^4)^2} \right]^{1/2} \right\}$$

$$= \frac{1}{4} \left\{ -1 + \left( 1 + \frac{\alpha}{2} \right)^{1/2} \right\}$$

$$\text{where } \alpha = \frac{\lambda_{34}^2}{\lambda_{pp}\lambda_{33}} \left( \frac{\text{He}^4}{\text{H}} \right)^2$$

= QED.

$$Q3 a) \frac{dH}{dt} = -\frac{\lambda_{pp} H^2}{2} - \lambda_{pd} HD + \frac{2\lambda_{33}(He^3)^2}{2} - \lambda_n H Be^7 - \lambda_{n'} H Li^7$$

$$\frac{dD}{dt} = \frac{\lambda_{pp} H^2}{2} - \lambda_{pd} HD$$

$$\frac{dHe^3}{dt} = \lambda_{pd} HD - \frac{2\lambda_{33}(He^3)^2}{2} - \lambda_{34} He^3 He^4$$

$$\frac{dHe^4}{dt} = \frac{\lambda_{33}(He^3)^2}{2} - \lambda_{34} He^3 He^4 + 2\lambda_{n'} H Be^7 + 2\lambda_{n'}' H Li^7$$

$$\frac{dBe^7}{dt} = \lambda_{34} He^3 He^4 - \lambda_{e7} Ne Be^7 - \lambda_n H Be^7$$

$$\frac{dLi^7}{dt} = \lambda_{e7} Ne Be^7 - \lambda_{n'}' H Li^7$$

$$b) \frac{d(Li^7 + Be^7)}{dt} = \lambda_{34} He^3 He^4 - \lambda_n H Be^7 - \lambda_{n'}' H Li^7 \\ = 0 \text{ in equilibrium.}$$

$$\therefore \lambda_{34} He^3 He^4 = \lambda_n H Be^7 + \lambda_{n'}' H Li^7$$

$$\text{Now } \frac{dHe^4}{dt} = \frac{\lambda_{33}(He^3)^2}{2} - \lambda_{34} He^3 He^4 + 2\lambda_{34} He^3 He^4 \\ = \frac{\lambda_{33}(He^3)^2}{2} + \lambda_{34} He^3 He^4 \quad \text{QED} \quad \textcircled{1}$$

$$c) D^2 \text{ in eqm means } \frac{dD}{dt} = 0 \text{ ie } \frac{\lambda_{pp} H^2}{2} = \lambda_{pd} HD$$

$$He^3 \text{ in eqm means } \frac{dHe^3}{dt} = 0 \text{ or } \lambda_{pd} HD = \frac{\lambda_{33}(He^3)^2}{2} + \lambda_{34}(He^3)He^4$$

$$\text{Sub both of these into the DE for } \frac{dHe^4}{dt}: \quad \textcircled{1}$$

$$\begin{aligned}
 \frac{dHe^4}{dt} &= \frac{\lambda_{pd} HD - \lambda_{34}(He^3)_e He^4}{2} + \lambda_{34}(He^3)_e He^4 \\
 &= \frac{\lambda_{pd} HD}{2} + \frac{\lambda_{34}(He^3)_e He^4}{2} \\
 &= \frac{\lambda_{pp} H^2}{4} + \frac{\lambda_{34}(He^3)_e He^4}{2} \\
 &= \frac{\lambda_{pp} H^2}{4} \left[ 1 + \frac{2\lambda_{34}(He^3)_e He^4}{2\lambda_{pp} H^2} \right]
 \end{aligned}$$

d) Consider the term in [ ]

Substitute into it the  $(He^3)_e$  value from Q1

The result follows some algebra :-

$$Q4 \quad \frac{PPI}{PPI + PPII} = \frac{(1 + \frac{2}{\alpha})^{1/2} - 1}{4} = x \text{ (say)}$$

$$\begin{aligned}
 \text{Note } PPI + PPII + PPIII &= 1 \\
 \therefore PPII + PPIII &= 1 - PPI
 \end{aligned}$$

$$\frac{PPI}{1-PPI} = x$$

$$PPI = x - xPPI$$

$$PPI(1+x) = x$$

$$PPI = \frac{x}{1+x}$$

$$= (1 + \frac{2}{\alpha})^{1/2} - 1$$

$$\frac{}{4}$$

$$\frac{}{3 + (1 + \frac{2}{\alpha})^{1/2}}$$

$$= \left[ \left( 1 + \frac{z}{\alpha} \right)^{1/2} - 1 \right] \left[ \left( 1 + \frac{z}{\alpha} \right)^{1/2} + 3 \right]^{-1}$$

Q5 a)  $|A - \lambda I| = \begin{vmatrix} -(a+\lambda) & 0 & b \\ a & (-c+\lambda) & 0 \\ 0 & c & (-b+\lambda) \end{vmatrix}$

$$\text{where } a = 1/C_{12}$$

$$b = 1/C_{14}$$

$$c = 1/C_{13}$$

$$= -(a+\lambda) [ (c+\lambda)(b+\lambda) + b(ac) ]$$

= expand and collect terms

$$= \lambda(\lambda^2 + A\lambda + B) = 0$$

$$\text{where } A = a+b+c$$

$$= \frac{1}{C_{12}} + \frac{1}{C_{13}} + \frac{1}{C_{14}} = \sum$$

$$\text{and } B = ab + ac + bc$$

$$\therefore \lambda = 0 \quad \text{or} \quad \lambda = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

$$\text{Note } A^2 - 4B = \sum^2 - 4 \left( \frac{1}{C_{12}C_{13}} + \frac{1}{C_{12}C_{14}} + \frac{1}{C_{13}C_{14}} \right)$$

$$= \Delta^2 \text{ as given in notes.}$$

$$\text{i.e. } \lambda = 0, -\frac{\sum + \Delta}{2} \quad \text{or} \quad -\frac{\sum - \Delta}{2}$$

$$b) \quad \lambda \tilde{U}_1 = \lambda_1 U_1 = \underline{0}$$

Let  $\tilde{U}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Then  $\begin{pmatrix} -a & 0 & b \\ a & -c & 0 \\ 0 & c & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow -ax + bz = 0 \quad \text{or} \quad x = \frac{b}{a}z \quad (z \text{ arbitrary})$$

$$\text{and } ax - cy = 0 \quad y = \frac{a}{c}x \\ = \frac{a}{c} \times \frac{b}{a}z \\ = \frac{b}{c}z$$

$$\text{and } cy - bz = 0 \Rightarrow y = \frac{b}{c}z \checkmark$$

$$\text{so } \tilde{U}_1 = \begin{pmatrix} b/a & b/c & 1 \end{pmatrix}^T$$

Since  $U_1$  is an eigenvector,  $f \tilde{U}_1$  is also an eigenvector for any  $f \neq 0$ . So multiply each term by  $1/b$ :

$$\begin{aligned} \tilde{U}_1 &= \begin{pmatrix} 1/a & 1/c & 1 \end{pmatrix}^T (b)^T \\ &= (v_{12} \quad v_{13} \quad v_{14})^T \text{ QED} \end{aligned}$$

$$Q6 \quad \frac{dx}{dt} = a - bx \quad x = N_B(t)$$

$$a = C^{12}/C_{12}$$

$$b = 1/C_B(13)$$

$$\frac{dx}{a-bx} = dt$$

$$\frac{1}{b} \int \frac{d(bx)}{a-bx} = \int dt$$

$$\int \frac{d(bx)}{a-bx} = b \int dt$$

$$-\ln(a-bx) = bt + K$$

$$\ln(a-bx) = -bt + k$$

$$\text{At } t=0 \quad x = N^B(0) = 0 \quad \therefore k = \ln a.$$

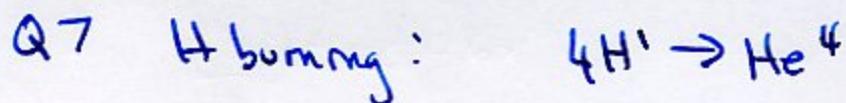
$$\ln(a-bx) = -bt + \ln a$$

$$a-bx = ae^{-bt}$$

$$bx = a(1 - e^{-bt})$$

$$x = \frac{a}{b}(1 - e^{-bt})$$

$$\therefore N^B(t) = \frac{C^{12}}{C_{12}C_B(13)} \left[ 1 - e^{-t/C_B(13)} \right]$$



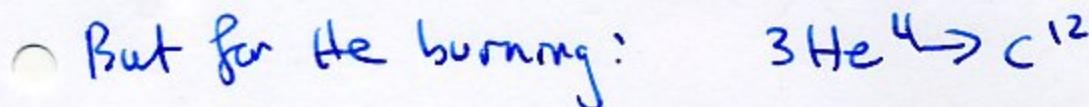
$$\Delta E = \Delta m \times c^2 = 26.7 \text{ MeV}$$

Per 4 H burned.

ie 26.7 MeV per 4.03128 amu of H burned

$$\Rightarrow \frac{26.7}{4.03128} \text{ MeV / amu of H burned}$$

$$= 6.62 \text{ MeV / amu of H}$$



$$\Delta m = 3 \times 4.002603 - 12.00$$

$$= 0.007809 \text{ amu}$$

$$\Delta E = \Delta m \times c^2 = 0.007809 \text{ amu} \times \frac{931 \text{ MeV}}{\text{amu}}$$

$$= 7.27 \text{ MeV}$$

This is per 3 He<sup>4</sup> burned = mass of 12.0078 amu  
ie energy released is

$$7.27 \text{ MeV per } 12.0078 \text{ amu He}^4$$

$$= \frac{7.27}{12.0078} \text{ MeV / amu He}^4$$

$$= 0.605 \text{ MeV / amu of He}^4$$

$\approx \frac{1}{10}$  rate from H burning

(per unit mass  
burned.)