ASP3012 - Sheet 4

Q1

$H^1(p, \beta^+\nu) D^2$ $\qquad r_{pp} = \lambda_{pp} H^2/2$

$D^2(p, \gamma) He^3$ $\qquad r_{pd} = \lambda_{pd} HD$

$He^3(He^3, 2p) He^4$ $\qquad r_{33} = \frac{\lambda_{33}(He^3)^2}{2}$

PPI

$He^3(\alpha, \gamma) Be^7$ $\qquad r_{34} = \lambda_{34} He^3 He^4$

$Be^7(\beta^-, \nu) Li^7$ $\qquad r_{e7} = \lambda_{e7} n_e Be^7$

$Li^7(p, \alpha) He^4$ $\qquad r'_{p7} = \lambda_{17} H Li^7$

PPII

$Be^7(p, \gamma) B^8$ $\qquad r_{17} = \lambda_{17} H Be^7$

$\left.\begin{array}{l} B^8(\beta^+\nu) Be^8 \\ Be^8(, \alpha) He^4 \end{array}\right]$ instant

PPIII

DE for He^3:

$$\frac{dHe^3}{dt} = \lambda_{pd} HD - \frac{2\lambda_{33}(He^3)^2}{2} - \lambda_{34} He^3 He^4$$

$$= 0 \quad \text{in equilib}^m$$

Thus we have a quadratic eq^n for He^3 abundance. The solution is:

$$(He^3)_e = \frac{1}{2\lambda_{33}}\left[-\lambda_{34}He^4 + \left[\lambda_{34}(He^4)^2 + 2\lambda_{pp}\lambda_{33}H^2\right]^{1/2}\right]$$

* why take +ve sign in the solution?

Q2 a) PPI makes He^4 at the rate r_{33}

PPII & III rates are governed by r_{34}. These each make 2 He^4 per "34" reaction. But they also consume one He^4 at "34" itself. So in combination PPII & PPIII make one He^4 at the rate r_{34}. Hence the relative production rate is

$$\frac{PPI}{PPII + PPIII} = \frac{r_{33}}{r_{34}}$$

b) If He^3 is in eq^m then

$$\frac{r_{33}}{r_{34}} = \frac{\lambda_{33} (He^3)_e^{\cancel{2}}}{2\lambda_{34} (He^4)}$$

Insert the eq^m value from Q1:

$$\left(\frac{r_{33}}{r_{34}}\right)_e = \cancel{\lambda_{34}} \frac{1}{2\cancel{\lambda_{33}}} \cdot \frac{1}{2\lambda_{34} He^4} \cancel{\lambda_{33}} \left[-\lambda_{34} He^4 + [\lambda_{34}(He^4)^2 + 2\lambda_{pp}\lambda_{33} H^2]^{1/2} \right]$$

$$= \frac{1}{4}\left\{ -1 + \cancel{\lambda_{34} He^4} \left[\frac{\lambda_{34}(He^4)^2 + 2\lambda_{pp}\lambda_{33} H^2}{\lambda_{34}^2 (He^4)^2} \right]^{1/2} \right\}$$

$$= \frac{1}{4}\left\{ -1 + \left(1 + \frac{2}{\alpha}\right)^{1/2} \right\}$$

where $\alpha = \frac{\lambda_{34}^2}{\lambda_{pp}\lambda_{33}} \left(\frac{He^4}{H}\right)^2$

= QED.

Q3 a) $$\frac{dH}{dt} = -\frac{2\lambda_{pp}H^2}{2} - \lambda_{pd}HD + \frac{2\lambda_{33}(He^3)^2}{2} - \lambda_{17}HBe^7 - \lambda_{17}'HLi^7$$

$$\frac{dD}{dt} = \frac{\lambda_{pp}H^2}{2} - \lambda_{pd}HD$$

$$\frac{dHe^3}{dt} = \lambda_{pd}HD - \frac{2\lambda_{33}(He^3)^2}{2} - \lambda_{34}He^3He^4$$

$$\frac{dHe^4}{dt} = \frac{\lambda_{33}(He^3)^2}{2} - \lambda_{34}He^3He^4 + 2\lambda_{17}HBe^7 + 2\lambda_{17}'HLi^7$$

$$\frac{dBe^7}{dt} = \lambda_{34}He^3He^4 - \lambda_{e7}n_eBe^7 - \lambda_{17}HBe^7$$

$$\frac{dLi^7}{dt} = \lambda_{e7}n_eBe^7 - \lambda_{17}'HLi^7$$

b) $$\frac{d(Li^7+Be^7)}{dt} = \lambda_{34}He^3He^4 - \lambda_{17}HBe^7 - \lambda_{17}'HLi^7$$

$= 0$ in equilibrium.

$\therefore \lambda_{34}He^3He^4 = \lambda_{17}HBe^7 + \lambda_{17}'HLi^7$

Now $$\frac{dHe^4}{dt} = \frac{\lambda_{33}(He^3)^2}{2} - \lambda_{34}He^3He^4 + 2\lambda_{34}He^3He^4$$

$$= \frac{\lambda_{33}(He^3)^2}{2} + \lambda_{34}He^3He^4 \qquad \text{QED} \quad -1$$

c) D^2 in eqm means $\frac{dD}{dt} = 0$ ie $\frac{\lambda_{pp}H^2}{2} = \lambda_{pd}HD$

He^3 in eqm means $\frac{dHe^3}{dt} = 0$ or $\lambda_{pd}HD = \lambda_{33}(He^3)_e^2 + \lambda_{34}(He^3)_eHe^4$

Sub both of these into the DE for $\frac{dHe^4}{dt}$: $\quad -1$

$$\frac{dHe^4}{dt} = \frac{\lambda_{pd} HD - \lambda_{34}(He^3)_e He^4}{2} + \lambda_{34}(He^3)_e He^4$$

$$= \frac{\lambda_{pd} HD}{2} + \frac{\lambda_{34}(He^3)_e He^4}{2}$$

$$= \frac{\lambda_{pp} H^2}{4} + \frac{\lambda_{34}(He^3)_e He^4}{2}$$

$$= \frac{\lambda_{pp} H^2}{4}\left[1 + \frac{2\lambda_{34}(He^3)_e He^4}{2\lambda_{pp} H^2}\right]$$

d) Consider the term in [ ]

Substitute into it the $(He^3)_e$ value from Q1

The result follows some algebra :-)

Q4 $$\frac{PPI}{PPII + PPIII} = \frac{\left(1+\frac{2}{\alpha}\right)^{1/2} - 1}{4} = x \text{ (say)}$$

Note $PPI + PPII + PPIII = 1$

$\therefore PPII + PPIII = 1 - PPI$!

$$\frac{PPI}{1-PPI} = x$$

$$PPI = x - x\,PPI$$

$$PPI(1+x) = x$$

$$PPI = \frac{x}{1+x}$$

$$= \frac{\dfrac{\left(1+\frac{2}{\alpha}\right)^{1/2} - 1}{4}}{\dfrac{3 + \left(1+\frac{2}{\alpha}\right)^{1/2}}{4}}$$

$$= \left[\left(1+\frac{2}{\alpha}\right)^{1/2}-1\right]\left[\left(1+\frac{2}{\alpha}\right)^{1/2}+3\right]^{-1}$$

Q5 a) $$|\Lambda - \lambda I| = \begin{vmatrix} -(a+\lambda) & 0 & b \\ a & (-c+\lambda) & \cancel{b}0 \\ 0 & c & (-b+\lambda) \end{vmatrix}$$

where $a = 1/\tau_{12}$
$b = 1/\tau_{14}$
$c = 1/\tau_{13}$

$= -(a+\lambda)\left[(c+\lambda)(b+\lambda) + b(ac)\right]$
$=$ expand and collect terms
$= \lambda(\lambda^2 + A\lambda + B) = 0$

where $A = a+b+c$
$= \frac{1}{\tau_{12}} + \frac{1}{\tau_{13}} + \frac{1}{\tau_{14}} = \Sigma$

and $B = ab + ac + bc$

So $\lambda = 0$ or $\lambda = \dfrac{-A \pm \sqrt{A^2 - 4B}}{2\cancel{A}}$

Note $A^2 - 4B = \Sigma^2 - 4\left(\frac{1}{\tau_{12}\tau_{13}} + \frac{1}{\tau_{12}\tau_{14}} + \frac{1}{\tau_{13}\tau_{14}}\right)$
$= \Delta^2$ as given in notes.

ie $\lambda = 0$, $\dfrac{-\Sigma+\Delta}{2}$ or $\dfrac{-\Sigma-\Delta}{2}$

b) $\underset{\sim}{\Lambda}\underset{\sim}{U}_1 = \lambda_1 \underset{\sim}{U}_1 = \underset{\sim}{0}$

Let $\underset{\sim}{U}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Then $\begin{pmatrix} -a & 0 & b \\ a & -c & 0 \\ 0 & c & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow -ax + bz = 0$ or $x = \frac{b}{a} z$ (z arbitrary)

and $ax - cy = 0$ $\quad y = \frac{a}{c} x$

$= \frac{a}{c} \times \frac{b}{a} z$

$= \frac{b}{c} z$

and $cy - bz = 0 \Rightarrow y = \frac{b}{c} z$ ✓

So $\underset{\sim}{U}_1 = (b/a \quad b/c \quad 1)^T$

Since $\underset{\sim}{U}_1$ is an eigen~~value~~ vector, $f\underset{\sim}{U}_1$ is also an eigen~~value~~vector for any $f \neq 0$. So multiply each term by $1/b$:

$\underset{\sim}{U}_1 = (1/a \quad 1/c \quad 1/b)^T$

$= (\tau_{12} \quad \tau_{13} \quad \tau_{14})^T$ QED

Q6 $\frac{dx}{dt} = a - bx$

$x = N_{13}(t)$

$a = c^{12}/\tau_{12}$

$b = 1/\tau_{\beta(13)}$

$$\frac{dx}{a-bx} = dt$$

$$\frac{1}{b}\int\frac{d(bx)}{a-bx} = \int dt$$

$$\int\frac{d(bx)}{a-bx} = b\int dt$$

$$-\ln(a-bx) = bt + K$$

$$\ln(a-bx) = -bt + k$$

At $t=0$ $x = N^{13}(0) = 0$ $\therefore k = \ln a$.

$$\ln(a-bx) = -bt + \ln a$$

$$a - bx = ae^{-bt}$$

$$bx = a(1 - e^{-bt})$$

$$x = \frac{a}{b}(1 - e^{-bt})$$

$$\text{ie } N^{13}(t) = \frac{c^{12}}{\tau_{12}\tau_{\beta(13)}}\left[1 - e^{-t/\tau_{\beta(13)}}\right]$$

Q7 H burning: $4H^1 \rightarrow He^4$

$\Delta E = \Delta m \times c^2 = 26.7\text{MeV}$

Per 4H burned.

ie 26.7 MeV per 4.03128 amu of H burned

$\Rightarrow \frac{26.7}{4.03128}$ MeV / amu of H burned

$= 6.62$ MeV / amu of H

But for He burning: $3He^4 \rightarrow C^{12}$

$\Delta m = 3 \times 4.002603 - 12.00$

$= 0.007809$ amu

$\Delta E = \Delta m \times c^2 = 0.007809 \text{ amu} \times 931 \frac{\text{MeV}}{\text{amu}}$

$= 7.27$ MeV

This is per $3He^4$ burned = mass of 12.0078 amu

ie energy release is

7.27 MeV per 12.0078 amu He^4

$= \frac{7.27}{12.0078}$ MeV / amu He^4

$= 0.605$ MeV / amu of He^4

$\simeq \frac{1}{10}$ rate from H burning

(per unit mass burned!)