ASP3012
~~ASP3012/ASP3012~~ Sheet ~~4~~ 3 solutions

Q1. $\frac{d}{d\xi}(\xi^2\theta') = -\xi^2\theta$ for $n=1$.

Let $\theta = \phi/\xi$ so $\theta' = \frac{\xi\phi' - \phi}{\xi^2}$

$$\therefore \frac{d}{d\xi}\left(\xi^2 \cdot \frac{\xi\phi'-\phi}{\xi^2}\right) = -\xi^2\phi/\xi$$

$$\xi\phi'' - \phi' + \phi' = -\xi\phi$$

$$\phi'' + \phi = 0$$

$$\therefore \phi = A\sin\xi + B\cos\xi$$

$$\Rightarrow \theta = \frac{A\sin\xi}{\xi} + \frac{B\cos\xi}{\xi}$$

Initial conditions: $\theta(0)=1$ $\lim_{\xi\to 0}\left(\frac{\sin\xi}{\xi}\right) = \lim_{\xi\to 0}\left(\frac{\xi - \xi^3/6 + \cdots}{\xi}\right)$

$$= \lim_{\xi\to 0}\left(1 - \xi^2/6 + \cdots\right)$$

$$= 1$$

$$\lim_{\xi\to 0}\left(\frac{\cos\xi}{\xi}\right) = \lim_{\xi\to 0}\left(\frac{1-\xi^2/2+\cdots}{\xi}\right)$$

$$= \lim_{\xi\to 0}\left(\frac{1}{\xi}\right) \to \infty.$$

$\therefore$ $B=0$ and $A=1$.

ie $\theta = \frac{\sin\xi}{\xi}$.

Also $\theta'(\xi) = \frac{\xi\cos\xi - \sin\xi}{\xi^2} = \frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}$

$$\lim_{\xi\to 0}(\theta'(\xi)) = \lim_{\xi\to 0}\left[\frac{1-\xi^2/2+\cdots}{\xi} - \frac{\xi - \xi^3/6 + \cdots}{\xi^2}\right]$$

$$= \lim_{\xi\to 0}\left[\cancel{\frac{1}{\xi}} - \xi/2 + \cdots - \cancel{\frac{1}{\xi}} - \xi/6 + \cdots\right]$$

$$= \lim_{\xi\to 0}\left[-\frac{2\xi}{3} + \cdots\right] = 0 \quad \text{as required.}$$

Q2

$$\frac{d}{d\xi}\left(\xi^2\theta'\right) = -\xi^2\theta^n$$

$$\therefore \xi^2\theta' = -\int \xi^2\theta^n d\xi$$

$\theta(0) = 1$ so near centre $\theta \simeq 1$.
Let $\xi_1 \ll \xi$ so $\theta(\xi_1) = 1 - \Delta\theta$ and $\Delta\theta \ll 1$.

Then
$$\xi_1^2\theta'(\xi_1) = -\int_0^{\xi_1} \xi^2\theta^n d\xi$$

$$= -\int_0^{\xi_1} \xi^2(1-\Delta\theta)^n d\xi$$

$$\simeq -\int_0^{\xi_1} \xi^2(1-n\Delta\theta) d\xi$$

Now $n \simeq 1-5$ $\therefore n\Delta\theta \ll 1$

$$\simeq -\int_0^{\xi_1} \xi^2 d\xi$$

$$\simeq -\xi_1^3/3$$

$$\therefore \theta'(\xi_1) = -\xi_1/3$$

$$\therefore \theta'(0) = \lim_{\xi_1 \to 0} \theta'(\xi_1) = 0. \qquad \text{QED.}$$

Q3

a) $LHS = \frac{d}{d(-\xi)}\left[(-\xi)^2 \frac{d\theta(-\xi)}{d(-\xi)}\right] = +\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)$

$RHS = -(-\xi)^2\theta^n = -\xi^2\theta^n$

$\therefore$ $\theta(\xi)$ and $\theta(-\xi)$ are solutions.
$\therefore$ no odd powers of ξ.

b) $\theta = c_0 + c_2\xi^2 + \ldots$
$\theta(0) = 1 = c_0$ $\therefore$ $c_0 = 1$.

c) $\frac{d}{d\xi}(\xi^2\theta') = -\xi^2\theta^n$

$\Rightarrow \theta'' + \frac{2}{\xi}\theta' = -\theta^n$

Now $\theta^n = (1 + c_2\xi^2 + c_4\xi^4 + c_6\xi^6 + \ldots)^n$

$= (1+A)^n$ where $A = c_2\xi^2 + B$

and $B = c_4\xi^4 + c_6\xi^6 + \ldots$

$= 1 + nA + \frac{n(n-1)}{2}A^2 + \ldots$

$= 1 + nA + \frac{n(n-1)}{2}(c_2\xi^2 + B)^2 + \ldots$

$= 1 + nA + \frac{n(n-1)}{2}(c_2^2\xi^4 + 2c_2\xi^2 B + \underset{=\text{ order } \xi^8}{B^4}) + \ldots$

$= 1 + nc_2\xi^2 + nc_4\xi^4 + nc_6\xi^6 + \ldots$

$\quad + \frac{n(n-1)}{2}(c_2^2\xi^4 + 2c_2\xi^2 c_4\xi^4 + \ldots)$

$\quad + \ldots$

$= 1 + nc_2\xi^2 + (nc_4 + \frac{n(n-1)}{2}c_2^2)\xi^4 + O(\xi^6)$

~~LHS~~ Now $\theta = 1 + c_2\xi^2 + c_4\xi^4 + c_6\xi^6 + \ldots$

$\Rightarrow \theta' = 2c_2\xi + 4c_4\xi^3 + 6c_6\xi^5 + \ldots$

$\theta'' = 2c_2 + 12c_4\xi^2 + 30c_6\xi^4 + \ldots$

$\therefore LHS = 6c_2 + 20c_4\xi^2 + 42c_6\xi^4 + \ldots$

Equating co-efficients $\quad 6c_2 = -1 \Rightarrow c_2 = -1/6$

$20c_4 = -nc_2 = -\frac{n}{6} \Rightarrow c_4 = \frac{n}{120}$

$-42c_6 = n\left(c_4 + \frac{(n-1)c_2^2}{2}\right)$

$= n\left(\frac{3n + 5n - 5}{360}\right) = \frac{(8n-5)n}{360}$

$\Rightarrow c_6 = -n(8n-5)/42 \times 360$ QED

Q4. a) $n=0$.

$$\theta = 1 - \frac{1}{6}\xi^2 \qquad \theta(0)=1$$
$$\theta' = -\frac{1}{3}\xi \qquad \theta'(0)=0$$
$$\theta'' = -\frac{1}{3} \qquad \theta''(0) = -1/3$$

all other $\theta^{(n)}=0$.

$$\therefore \theta(\xi) = \theta(0) + \xi\theta'(0) + \frac{\xi^2}{2}\theta''(0) + \dots$$
$$= 1 + \xi\times 0 + \frac{\xi^2}{2}\left(-\frac{1}{3}\right) + 0 + \dots$$
$$= 1 - \frac{1}{6}\xi^2 \quad \text{as expected.}$$

b) $n=1$ $\quad \theta = \sin\xi/\xi$

$$\theta' = \frac{\xi\cos\xi - \sin\xi}{\xi} = -\frac{1}{3}\xi + \frac{1}{30}\xi^3 + \dots$$

Thus $\theta'(0)=0$

$$\theta'' = -\theta - \frac{2}{\xi}\theta'$$

$$\therefore \theta''(0) = -1 - \frac{2}{\xi}\left(-\frac{1}{3}\xi + \frac{1}{30}\xi^3 + \dots\right)$$
$$= -\frac{1}{3} \quad \text{as } \xi \to 0.$$

$$\theta''' = \frac{d}{d\xi}(\theta'') = \frac{d}{d\xi}\left(-\theta - \frac{2}{\xi}\theta'\right)$$
$$= -\theta' - \frac{2}{\xi}\theta'' + \frac{2}{\xi^2}\theta'$$
$$= \frac{2\theta}{\xi} + \frac{6}{\xi^2}\theta' - \theta'$$

$$\therefore \theta'''(0) = \frac{2}{\xi}\left(1 - \frac{\xi^2}{6} + \dots\right) + \left(\frac{6}{\xi^2} - 1\right)\left(-\frac{1}{3}\xi + \dots\right)$$
$$= O(\xi) \qquad \therefore \theta'''(0) = 0$$

$$\theta^{iv}(0) = \theta - \frac{8\theta}{\xi^2} + \frac{4\theta'}{\xi} - \frac{24\theta'}{\xi^3}$$

so $\theta^{iv}(0) = \frac{1}{5}$ after some algebra!

Thus $\theta(\xi) = \theta(0) + \xi\theta'(0) + \frac{\xi^2\theta''(0)}{2} + \frac{\xi^3\theta'''(0)}{6} + \dots$

$= 1 + \xi \times 0 + \frac{\xi^2}{2}\left(-\frac{1}{3}\right) + \frac{\xi^3}{3} \times 0 + \frac{\xi^4}{24} \times \left(\frac{1}{5}\right) + \dots$

$= 1 - \frac{\xi^2}{6} + \frac{\xi^4}{120} + \dots$

= general series for $n = 1$.

c) $n=5$.

$\theta = \left(1 + \frac{1}{3}\xi^2\right)^{-1/2}$ $\therefore \theta(0) = 1$

$\theta' = -\frac{1}{3}\xi\left(1 + \xi^2/3\right)^{-3/2}$

$= -\frac{\xi}{3}\theta^3$ $\therefore \theta'(0) = 0$

$\theta'' = -\frac{1}{3}\left[-\xi^2\theta^5 + \theta^3\right]$ $\theta''(0) = -\frac{1}{3}$

$\theta''' = -\frac{1}{3}\left[\frac{5}{3}\xi^3\theta^7 - 3\xi\theta^5\right]$ $\therefore \theta'''(0) = 0$

$\theta^{iv} = -\frac{1}{3}\left[\frac{35}{9}\xi^4\theta^9 + 10\xi^2\theta^7 - 3\theta^5\right]$

$\Rightarrow \theta^{iv}(0) = -\frac{1}{3}(0 + 0 - 3)$

$= 1.$

Thus $\theta(\xi) = \theta(0) + \xi\theta'(0) + \frac{\xi^2\theta''}{2} + \dots$

$= 1 + \xi \times 0 + \frac{\xi^2}{2}\left(-\frac{1}{3}\right) + 0 + \frac{\xi^4}{24} \times 1$

$= 1 - \frac{1}{6}\xi^2 + \frac{\xi^4}{24} + \dots$

= general solution with $n=5$!

Q5 $n=0$ and 1 are trivial

$n=5$: $\theta = (1+\xi^2/3)^{-1/2}$

$\theta' = -\frac{1}{2}(1+\xi^2/3)^{-3/2} \cdot \frac{2}{3}\xi = -\frac{\xi}{3}\theta^3$

$\theta'' = \frac{\xi^2\theta^5}{3} - \frac{\theta^3}{3}$

Lane-Emden equation: $\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\theta') = -\theta^5$

$LHS = \frac{1}{\xi^2}(\xi^2\theta'' + 2\xi\theta')$

$= \theta'' + \frac{2}{\xi}\theta'$

$= \frac{\xi^2\theta^5}{3} - \frac{\theta^3}{3} - \frac{2}{3}\theta^3$

$= \frac{\xi^2\theta^5}{3} - \theta^3$

$= RHS = -\theta^5$ only if $\theta^3 = \theta^5(1+\frac{\xi^2}{3})$

ie $\theta = (1+\frac{\xi^2}{3})^{-1/2}$ ✓

Q6. $\theta = (1+\xi^2/3)^{-1/2}$

$\theta' = -\frac{1}{2}(1+\xi^2/3)^{-3/2} 2.\xi/3$

$= -\frac{\xi}{3}(1+\xi^2/3)^{-3/2}$

$\therefore \xi^2\theta' = -\xi^3/3\,(1+\xi^2/3)^{-3/2}$

$-\xi^2\theta' = \frac{\xi^3}{3}\theta^{-3} = \frac{1}{3}\left(\frac{\xi}{\theta}\right)^3$

$= \frac{\xi^3}{3[(1+\xi^2/3)^{3/2}]}$

$= \frac{\xi^3}{3[\xi^2/3\,(1+3/\xi^2)]^{3/2}}$

$$= \frac{\cancel{\xi^3} \cdot 3^{3/2}}{3\, \cancel{\xi^3}} \frac{1}{[1 + 3/\xi^2]^{3/2}}$$

For large ξ:

$$-\xi^2\theta' = \sqrt{3}\,(1 + 3/\xi^2)^{-3/2}$$
$$= \sqrt{3}\,(1 - \tfrac{3}{2} \times \tfrac{3}{\xi^2} + \ldots)$$
$$= \sqrt{3}\,(1 - 0 + \ldots)$$
$$\simeq \sqrt{3}$$

$$\therefore \lim_{\xi \to \infty} (-\xi^2\theta') = \sqrt{3}.$$

Q7. $P = \frac{\rho k T}{\mu} + \frac{1}{3} a T^4$ $\qquad P_r = \frac{1}{3} a T^4 = (1-\beta) P$

$$\therefore T = \left(\frac{3}{a}\right)^{1/4} (1-\beta)^{1/4} P^{1/4}$$

$$P_g = \frac{\rho k T}{\mu} = \beta P$$

$$\Rightarrow P = \frac{\rho k}{\mu} \left(\frac{3}{a}\right)^{1/4} \frac{(1-\beta)^{1/4}}{\beta} P^{1/4}$$

$$= \left[\frac{3(1-\beta)}{a\beta^4}\right]^{1/3} \left(\frac{k}{\mu}\right)^{4/3} \rho^{4/3}$$

$$= K\rho^{4/3} \quad \text{for } K = \left[\frac{3(1-\beta)}{a\beta^4}\right]^{1/3} \left(\frac{k}{\mu}\right)^{4/3}$$

and $n = 3$

Now $\alpha^2 = \frac{K(3+1)}{4\pi G} \rho_c^{-2/3} = \frac{3(1-\beta)}{a\beta^4}^{1/3} \left(\frac{k}{\mu}\right)^{4/3} \frac{\rho_c^{-2/3}}{\pi G}$

Thus

$$M = 4\pi\alpha^3 \rho_c (-\xi^2\theta')_{\xi_*}$$

$$= 4\pi\left[\frac{3(1-\beta)}{a\beta^4}\right]^{1/2}\left(\frac{k}{\mu}\right)^2 \frac{\rho_c^{-1}}{(\pi G)^{3/2}}\rho_c(2.081)$$

$$= \text{constant}\ \frac{\sqrt{1-\beta}}{\mu^2\beta^2}$$

$$= \frac{18\sqrt{1-\beta}}{\mu^2\beta^2} M_\odot$$

NB For the Eddington Standard Model β = constant.
Thus M determines β.
For the Sun with $X=0.68$, $Z=0.02$, $\mu=0.63$
and we get

$$1 = \frac{18\sqrt{1-\beta}}{(0.63)^2\beta^2}$$

$$0.15\beta^4 = 324(1-\beta)$$

Numerical solution yields

$$\beta = 0.99954.$$

For a $30M_\odot$ star

$$30 = \frac{18\sqrt{1-\beta}}{(0.63)^2\beta^2}$$

$$\Rightarrow \beta = 0.81085.$$

Q8 $U = \frac{r}{m}\frac{dm}{dr}$ $V = -\frac{r}{P}\frac{dP}{dr}$

a) $U = \frac{r}{m} \times 4\pi r^2 \rho = \frac{4\pi r^3 \rho}{m}$

So near centre take $\rho \simeq$ constant $\Rightarrow m \simeq \frac{4}{3}\pi r^3 \rho$

$\therefore U \to \frac{4\pi r^3 \rho}{\frac{4}{3}\pi r^3 \rho} \to 3$.

And $V = -\left(\frac{r}{P}\right)\left(\frac{-\rho G m}{r^2}\right) = \frac{\rho}{P}\frac{Gm}{r} \sim \frac{m}{r} \sim r^2 \to 0$.

b) $U = \frac{\alpha \xi \quad 4\pi\alpha^2 \xi^2 \rho_c \theta^n}{-4\pi\alpha^3 \rho_c^2 \xi^2 \theta'} = -\frac{\xi\theta^n}{\theta'}$ ✓

$V = \left(\frac{-\alpha\xi}{K\rho_c^{1+\frac{1}{n}}\theta^{n+1}}\right)\left(\frac{-\rho_c\theta^n G(-4\pi\alpha^2\rho_c)\xi^2\theta'}{\alpha^2\xi^2}\right)$

$= -\frac{\alpha^2 \xi \theta^n G 4\pi \rho_c \theta'}{K\rho_c^{1/n}\theta^{n+1}}$

$= \frac{-K(n+1)\rho_c^{\frac{1}{n}-1}}{4\pi G} \times \frac{\xi\theta^n G 4\pi \rho_c \theta'}{K\rho_c^{\frac{1}{n}}\theta^{n+1}}$

$= -\frac{(n+1)\xi\theta'}{\theta}$ ✓

c) Now $U = -\xi\theta^n/\xi'$

$\Rightarrow \frac{1}{U}\frac{dU}{d\xi} = \frac{1}{\xi} + \frac{n}{\theta}\theta' - \frac{\theta''}{\theta'}$

From Lane-Emden eq^n: $\theta'' = -\theta^n - \frac{2}{\xi}\theta'$.

Thus $\frac{1}{U}\frac{dU}{d\xi} = \frac{1}{\xi}\left[3 + \frac{n\xi\theta'}{\theta} + \frac{\xi\theta''}{\theta'}\right]$

$= \frac{1}{\xi}\left[3 - \frac{nV}{n+1} - U\right] = \frac{d\ln U}{d\xi}$.

Likewise $\frac{1}{V}\frac{dV}{d\xi} = \frac{d\ln V}{d\xi} = \frac{1}{\xi}\left[U + \frac{V}{n+1} - 1\right]$

$\therefore \frac{d\ln V}{d\ln U} = \frac{d\ln V}{d\xi} \Big/ \frac{d\ln U}{d\xi} = \frac{1 - U - \frac{1}{n+1}V}{U + \frac{n}{n+1}V - 3}$ ✓

Q9. a) Combine $\frac{dP}{dr} = -\frac{\rho G m}{r^2}$ and $\frac{dm}{dr} = 4\pi r^2\rho$.

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} = -4\pi r^2\rho G$$

Put $P = \frac{\rho k T}{\mu}$ $\quad \frac{d}{dr}\left[\frac{r^2}{\rho}\left(\frac{kT}{\mu}\right)\frac{d\rho}{dr}\right] = -4\pi r^2\rho G$

b) $r = \alpha\xi$, $\rho = \rho_c e^{-\psi}$

$$\frac{1}{\alpha}\frac{d}{d\xi}\left[\frac{\cancel{\alpha^2}\xi^2}{\rho_c \cancel{e^{-\psi}}}\frac{kT}{\mu}\frac{\cancel{\rho_c}}{\alpha}e^{-\psi}\frac{d\psi}{d\xi}\right] = -4\pi\alpha^2\xi^2\cancel{\rho_c}e^{-\psi}$$

$$\frac{1}{4\pi\alpha^2\rho_c G}\frac{kT}{\mu}\frac{d}{d\xi}\left[\xi^2\frac{d\psi}{d\xi}\right] = -\xi^2 e^{-\psi}$$

Choose $\alpha^2 = \left[\frac{kT}{4\pi\rho_c G\mu}\right]^{1/2}$

and hence

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\psi}{d\xi}\right] = -e^{-\psi} \qquad \text{QED}$$

c) i) $\rho = \rho_c e^{-\psi}$

$$\therefore \rho(0) = \rho_c e^{-\psi(0)} \qquad \therefore \psi(0) = 0$$

ii) Let $\xi_1 \ll 1$ and $\psi_1 = \psi(\xi_1) \ll 1$

Then
$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = 1$$

$$\xi^2\frac{d\psi}{d\xi}\Big|_{\xi_1} = \int_0^{\xi_1}\xi^2 d\xi = \frac{\xi_1^3}{3}$$

$$\therefore \frac{d\psi}{d\xi} = \xi_1/3$$

$$\therefore \frac{d\psi}{d\xi}\Big|_0 = \lim_{\xi_1\to 0}\frac{d\psi}{d\xi}\Big|_{\xi_1} = 0.$$

d) As in Lane-Emden case, $-\xi$ is also a solution if ξ is a solution. Thus no odd powers of ξ. And since $\psi(0)=0$ we must have

$$\psi = c_2\xi^2 + c_4\xi^4 + c_6\xi^6 + \dots$$

$$\Rightarrow \psi' = 2c_2\xi + 4c_4\xi^3 + 6c_6\xi^5 + \dots$$

$$\xi^2\psi' = 2c_2\xi^3 + 4c_4\xi^5 + 6c_6\xi^7 + \dots$$

$$\frac{d}{d\xi}(\xi^2\psi') = 6c_2\xi^2 + 20c_4\xi^4 + 42c_6\xi^6 + \dots$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\psi') = 6c_2 + 20c_4\xi^2 + 42c_6\xi^4$$

$$\text{RHS} = e^{-\psi} = 1 - \psi + \psi^2/2 - \psi^3/6 + \dots$$

$$= 1 - (c_2\xi^2 + c_4\xi^4 + c_6\xi^6 + \dots)$$

$$+\frac{1}{2}(c_2\xi^2 + c_4\xi^4 + \dots)^2 - \frac{1}{6}(c_2\xi^2 + \dots)^3 + \dots$$

$$= 1 - c_2\xi^2 - c_4\xi^4 + c_6\xi^6 + \dots$$

$$+\frac{1}{2}(c_2\xi^2 + A)^2 - \frac{1}{6}c_2^3\xi^6 + \text{higher powers}$$

$$A = c_4\xi^4 + \dots$$

$$= 1 - c_2\xi^2 - c_4\xi^4 - c_6\xi^6 + \dots$$

$$+\frac{1}{2}(c_2^2\xi^4 + 2c_2\xi^2 A + O(\xi^8)) + \dots$$

$$= 1 - c_2\xi^2 - c_4\xi^4 - c_6\xi^6 + \dots$$

$$+\frac{1}{2}c_2^2\xi^4 - 2\xi^2 c_2 c_4\xi^6 + \cdot\, O(\xi^8)$$

$$= 1 - c_2\xi^2 + \xi^4\left(\frac{1}{2}c_2^2 - c_4\right) + \xi^6(-2c_2c_4 - c_6) + \dots$$

Equating co-efficients:

$$6c_2 = 1 \quad \therefore \quad c_2 = \frac{1}{6}$$

$$20c_4 = -c_2 = -\frac{1}{6} \quad \text{or} \quad c_4 = \frac{-1}{120}$$

$$42c_6 \;\cancel{= -2c_2c_4 - c_6}$$

$$= \frac{1}{2}c_2^2 - c_4$$

$$= \frac{1}{72} + \frac{1}{120} = \frac{8}{360}$$

$$\therefore c_6 = \frac{1}{1890}$$

QED.