# ASP3012 Sheet2

Q1 If $H_2$ exists then the temperature is not high enough to dissociate into 2H. Thus there is no ionization. He has $Z=2$ and takes even higher T to ionize than H. Thus no ionization of either H or He.

$$\frac{1}{\mu} = \sum \frac{X_Z}{A_Z} = \frac{0.8}{2.016} + \frac{0.2}{4.0028} = 0.4468$$

$$\Rightarrow \mu = 2.238$$

Q2 $$dQ = TdS = c_v dT + PdV$$

$$\text{i.e. } \underset{\text{adiabatic}}{0} = c_v dT + PdV$$

But $P = \frac{\mathcal{R}_0 T}{\mu V}$ so $c_v \frac{dT}{T} + \left(\frac{\mathcal{R}_0}{\mu}\right)\frac{dV}{V} = 0$

Also $\frac{\mathcal{R}_0}{\mu} = c_p - c_v$ so

$$\frac{dT}{T} + (\gamma - 1)\frac{dV}{V} = 0 \quad \text{—— (1)}$$

where we used $\gamma = c_p / c_v$.

Now $$dP = \frac{\mathcal{R}_0}{\mu V} dT - \frac{\mathcal{R}_0 T}{\mu V^2} dV$$

$$\therefore \frac{dP}{P} = \frac{dT}{T} - \frac{dV}{V} \quad \text{—— (2)}$$

Hence $\frac{dT}{T} = \frac{dP}{P} + \frac{dV}{V}$ Subs into (1)

$$\frac{dP}{P} + (\gamma-1)\frac{dV}{V} + \frac{dV}{V} = 0$$

$$\Rightarrow \boxed{\frac{dP}{P} + \gamma \frac{dV}{V} = 0}$$

Also from ②: $\frac{dV}{V} = \frac{dT}{T} - \frac{dP}{P}$

Subs into ①: $\frac{dT}{T} + (\gamma-1)\left(\frac{dT}{T} - \frac{dP}{P}\right) = 0$

$$\gamma \frac{dT}{T} - (\gamma-1)\frac{dP}{P} = 0$$

$$\Leftrightarrow \boxed{\frac{dP}{P} + \left(\frac{\gamma}{\gamma-1}\right)\frac{dT}{T} = 0}$$

Q3: These are the same, just more tedious because of the extra radiation terms. Try to simplify as much as you can early on.... Here are the procedures; but not every step!

$$P = \frac{\rho R T}{\mu} + \frac{1}{3}aT^4 = \frac{R T}{\mu V} + \frac{1}{3}aT^4 \quad \text{— ①}$$

$$U = \frac{1}{\gamma-1}\frac{R T}{\mu} + aT^4 V \quad \text{— ②}$$

a) From ①: $\frac{dP}{P} = (4-3\beta)\frac{dT}{T} - \beta\frac{dV}{V}$ — ③

From 1st law (which we require independent of ~~for~~ adiabatic changes): $dQ = dU + PdV = 0$ ← if adiabatic

$$0 = dU + PdV$$
$$= \left[\frac{\beta}{\gamma-1} + 12(1-\beta)\right]\frac{dT}{T} + (4-3\beta)\frac{dV}{V} \quad \text{— ④}$$

To get the required expression

$$\frac{dP}{P} + \Gamma_1 \frac{dV}{V} = 0$$

we eliminate $\frac{dT}{T}$ between eqns ③ & ④

b) Here we eliminate $\frac{dV}{V}$ between (3) & (4)

c) " " " $\frac{dP}{P}$ " " "

The definitions of the $\Gamma_i$ follow from the above.

Q4
$$\frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\frac{32-24\beta-3\beta^2}{24-21\beta}}{\frac{32-27\beta}{24-21\beta} - \frac{24-21\beta}{24-21\beta}}$$

$$= \frac{32-24\beta-3\beta^2}{8-6\beta}$$

$$= \frac{\frac{32-24\beta-3\beta^2}{24-18\beta-3\beta^2}}{\frac{8-6\beta}{24-18\beta-3\beta^2}}$$

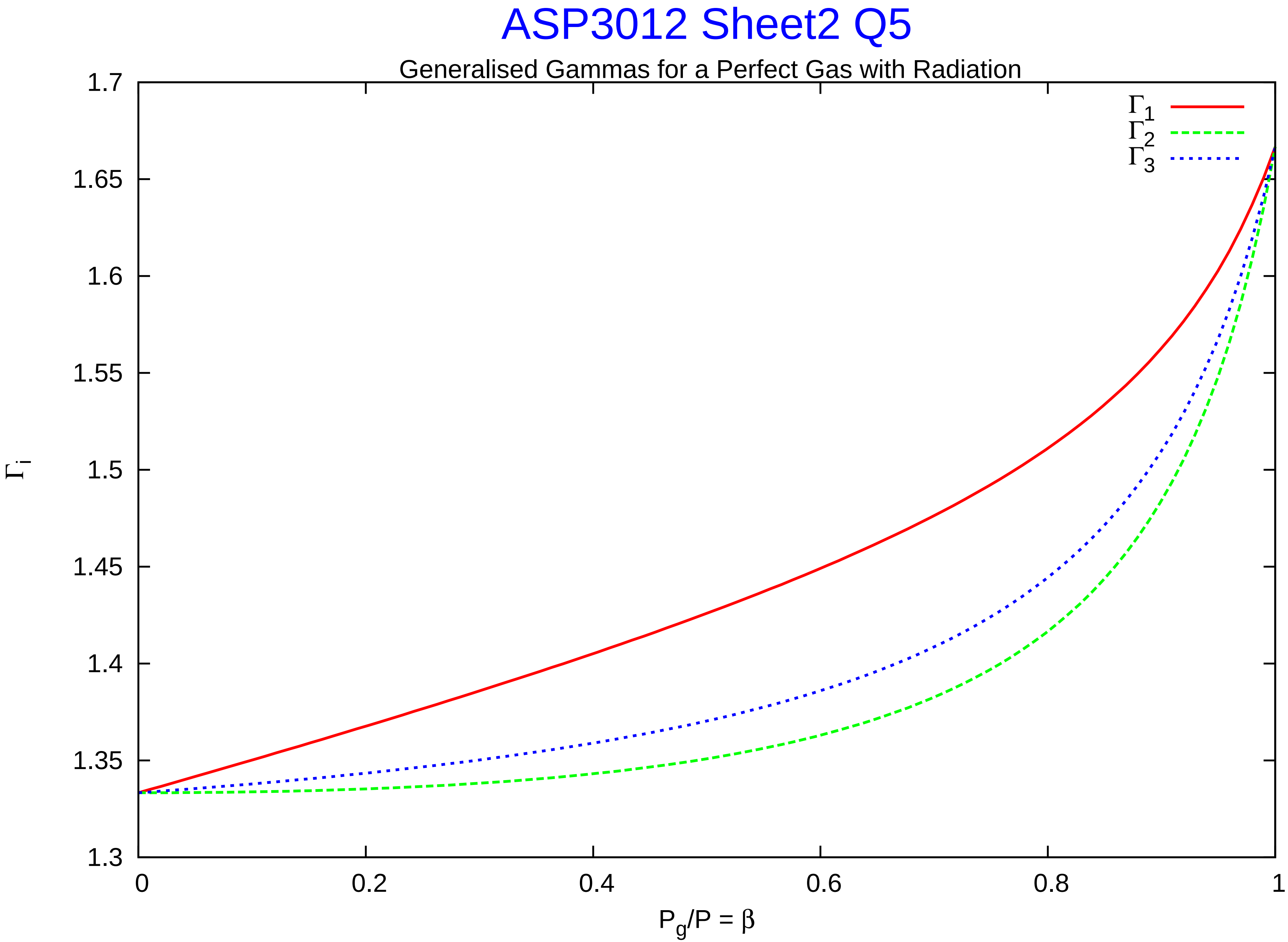
$$= \frac{\Gamma_2}{(8-6\beta)/(24-18\beta-3\beta^2)}$$

Let $\frac{8-6\beta}{24-18\beta-3\beta^2} = x-1$ for some $x$.

$$\Rightarrow x = \frac{8-6\beta}{24-18\beta-3\beta^2} + 1$$

$$= \frac{32-24\beta-3\beta^2}{24-18\beta-3\beta^2} = \Gamma_2$$

$$\therefore \frac{\Gamma_1}{\Gamma_3 - 1} = \frac{\Gamma_2}{\Gamma_2 - 1}$$



ASP3012 Sheet2 Q5
Generalised Gammas for a Perfect Gas with Radiation
Γ1
Γ2
Γ3
1.7
1.65
1.6
1.55
1.5
1.45
1.4
1.35
1.3
0
0.2
0.4
0.6
0.8
1
Γi
Pg/P = β

Q6 a) $P = \frac{\mathcal{R}T}{\mu V} \Rightarrow \frac{dP}{P} = \frac{dT}{T} - \frac{dV}{V}$

$U = \frac{1}{\gamma-1}\frac{\mathcal{R}T}{\mu} \Rightarrow dQ = \frac{1}{\gamma-1}\frac{\mathcal{R}}{\mu}dT + PdV = 0$ at constant entropy

$$\therefore \frac{1}{\gamma-1}\frac{\mathcal{R}}{\mu}\frac{dT}{T} + \frac{PV}{T}\frac{dV}{V} = 0$$

$\frac{\mathcal{R}}{\mu}$

$$\frac{1}{\gamma-1}\frac{dT}{T} + \frac{dV}{V} = 0$$

$$\frac{1}{\gamma-1}\frac{dT}{T} + \left(\frac{dT}{T} - \frac{dP}{P}\right) = 0$$

$$\frac{dT}{T}\left(\frac{1}{\gamma-1} + 1\right) = \frac{dP}{P}$$

$$\therefore \nabla_{ad} = \left.\frac{d\ln T}{d\ln P}\right|_s = \frac{\gamma-1}{\gamma} = \frac{5/3-1}{5/3} = 2/5 \quad \text{QED}$$

b) Repeat, but with $P_{rad}$ & $U_{rad}$ included.

OR From Q3 $\quad \frac{dP}{P} + \frac{\Gamma_2}{1-\Gamma_2}\frac{dT}{T} = 0$

$$\therefore \left.\frac{d\ln T}{d\ln P}\right|_s = \nabla_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2} \qquad \text{QED}$$

Q7 ~~Q5~~ $\quad P = P_g + P_r \Rightarrow P_r = P - P_g = P(1-\beta)$

$$\frac{1}{3}aT^4 = P(1-\beta) = P_g(1-\beta)/\beta = \frac{\rho\mathcal{R}T}{\mu}\left(\frac{1-\beta}{\beta}\right)$$

$$\Rightarrow T = \left(\frac{3\mathcal{R}}{a\mu}\cdot\frac{1-\beta}{\beta}\right)^{1/3}\rho^{1/3}$$

But $P = \frac{P_g}{\beta} = \frac{\rho\mathcal{R}T}{\mu}$

$$= \left[ \left(\frac{k_B}{\mu}\right)^4 \left(\frac{3}{a}\right) \frac{1-\beta}{\beta^4} \right]^{1/3} \rho^{4/3}$$

Hence note that if $\beta$ = constant throughout the star, then we have

$$P = K\rho^{4/3}$$

ie. a polytropic equation of state with $\gamma = \frac{c_p}{c_v} = 1 + \frac{1}{n}$

$$= 4/3$$

$$\therefore n = 3.$$

This is the "Eddington Standard Model".

Q8 a) $dQ = TdS = dU + PdV$

$$\therefore dS = \frac{dU}{T} + \frac{P}{T}dV$$

$$= \frac{1}{T}\left[ \frac{\partial U}{\partial T}\Big|_V dV + \frac{\partial U}{\partial V}\Big|_T dT \right] + \frac{P}{T}dV$$

$$= \frac{1}{T}\left[ \frac{\partial U}{\partial V}\Big|_T + P \right] dV + \frac{1}{T}\frac{\partial U}{\partial T}\Big|_V dT$$

But also

$$dS = \frac{\partial S}{\partial T}\Big|_V dT + \frac{\partial S}{\partial V}\Big|_T dV$$

$$\therefore \frac{\partial S}{\partial T}\Big|_V = \frac{1}{T}\frac{\partial U}{\partial T}\Big|_V \quad \text{and} \quad \frac{\partial S}{\partial V}\Big|_T = \frac{1}{T}\left[ \frac{\partial U}{\partial V}\Big|_T + P \right]$$

b) $dS$ must be a perfect differential.

Thus

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T}$$

$$\Rightarrow \frac{\partial}{\partial V}\left( \frac{1}{T}\frac{\partial U}{\partial T}\Big|_V \right) = \frac{\partial}{\partial T}\left( \frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{P}{T} \right)_V$$

$$\cancel{\frac{1}{T}\frac{\partial^2 U}{\partial T \partial V}} = \cancel{\frac{1}{T}\frac{\partial^2 U}{\partial V \partial T}} - \frac{1}{T^2}\left(\frac{\partial U}{\partial V}\Big|_T + P\right) + \frac{1}{T}\frac{\partial P}{\partial T}\Big|_V$$

$$\therefore \frac{\partial P}{\partial T}\Big|_V = \frac{1}{T}\left(\frac{\partial U}{\partial V}\Big|_T + P\right)$$

$$\Rightarrow \frac{\partial U}{\partial V}\Big|_T = T\frac{\partial P}{\partial T}\Big|_V - P \quad \text{QED.}$$

Q9. $$dU = \frac{\partial U}{\partial V}\Big|_T dV + \frac{\partial U}{\partial T}\Big|_V dT \qquad — (1)$$

$$\therefore \frac{dU}{dT} = \frac{\partial U}{\partial V}\Big|_T \frac{\partial V}{\partial T} + \frac{\partial U}{\partial T}\Big|_V$$

But also from (1) $$\frac{\partial U}{\partial T}\Big|_P = \frac{\partial U}{\partial V}\Big|_T \frac{\partial V}{\partial T}\Big|_P + \frac{\partial U}{\partial T}\Big|_V$$

$$= \frac{\partial U}{\partial T}\Big|_V + \frac{\partial V}{\partial T}\Big|_P\left(T\frac{\partial P}{\partial T}\Big|_V - P\right) \quad (2)$$

Now $$C_P - C_V = \frac{dQ}{dT}\Big|_P - \frac{dQ}{dT}\Big|_V$$

$$= \frac{\partial U}{\partial T}\Big|_P + P\frac{\partial V}{\partial T}\Big|_P - \frac{\partial U}{\partial T}\Big|_V$$

$$= P\frac{\partial V}{\partial T}\Big|_P + \frac{\partial V}{\partial T}\Big|_P\left(T\frac{\partial P}{\partial T}\Big|_V - P\right) \quad \text{from (2)}$$

$$= T\frac{\partial P}{\partial T}\Big|_V \frac{\partial V}{\partial T}\Big|_P \qquad — (3)$$

Q10 $$\alpha = \frac{\partial \ln \rho}{\partial \ln P}\Big|_T \qquad P = \frac{\rho \mathcal{R} T}{\mu} + \frac{1}{3}aT^4$$

If T is constant then

$$\frac{dP}{d\rho}\Big|_T = \frac{\mathcal{R}T}{\mu} = \frac{P_g}{\rho}$$

$$\frac{\rho}{P}\frac{dP}{d\rho}\bigg|_T = \frac{P_g}{P} = \beta$$

$$\therefore \quad \alpha = 1/\beta .$$

b) $\delta = -\frac{\partial \ln\rho}{\partial \ln T}\bigg|_P$

$$\therefore dP = 0 = \frac{\rho \mathcal{R}}{\mu}dT + \frac{\mathcal{R}T}{\mu}d\rho + \frac{4}{3}aT^3dT$$

$$= \left(\frac{\rho\mathcal{R}}{\mu} + \frac{4}{3}aT^3\right)dT + \frac{\mathcal{R}T}{\mu}d\rho$$

$$= (P_g + 4P_r)\,d\ln T + P_g\, d\ln\rho = 0$$

$$\therefore \frac{d\ln\rho}{d\ln T}\bigg|_P = \frac{P_g + 4P_r}{P_g} = \frac{\beta + 4(1-\beta)}{\beta}$$

$$= \frac{4-3\beta}{\beta} .$$

c) From calculus: $\frac{\partial f}{\partial g}\Big|_h \frac{\partial g}{\partial h}\Big|_f \frac{\partial h}{\partial f}\Big|_g = -1.$

Thus $\frac{\partial P}{\partial T}\Big|_V \frac{\partial T}{\partial V}\Big|_P \frac{\partial V}{\partial P}\Big|_T = -1$

$$\Rightarrow \frac{\partial P}{\partial T}\bigg|_V = -\frac{\partial V}{\partial T}\bigg|_P \cdot \frac{\partial P}{\partial V}\bigg|_T$$

$$= -\frac{1}{T}\frac{\partial V}{\partial \ln T}\bigg|_P \cdot P \cdot \frac{\partial \ln P}{\partial V}\bigg|_T$$

$$= -\frac{P}{T}\frac{\partial \ln V}{\partial \ln T}\bigg|_P \cdot \frac{\partial \ln P}{\partial \ln V}\bigg|_T$$

Now $V = \frac{1}{\rho}$ $\therefore \ln V = -\ln\rho$

$$\therefore \frac{\partial P}{\partial T}\bigg|_V = \frac{-P}{T}\frac{\partial \ln\rho}{\partial \ln T}\bigg|_P \bigg/ \frac{\partial \ln\rho}{\partial \ln P}\bigg|_T = \frac{P}{T}\frac{\delta}{\alpha}$$ 4

Sub this into equation (3) (Q9)

$$c_p - c_v = T\left(\frac{P}{T}\frac{\delta}{\alpha}\right)\frac{\partial V}{\partial T}\bigg|_P$$

$$= \frac{P}{T}\frac{\delta}{\alpha}\frac{\partial V}{\partial \ln T}\bigg|_P = \frac{VP}{T}\frac{\delta}{\alpha}\frac{\partial \ln V}{\partial \ln T}\bigg|_P$$

$$= \frac{P}{\rho T}\frac{\delta^2}{\alpha}$$

Q11

$$dQ = dU + PdV = \frac{\partial U}{\partial T}\bigg|_V dT + \left(\frac{\partial U}{\partial V}\bigg|_T + P\right)dV$$

$$= \frac{\partial U}{\partial T}\bigg|_V dT + T\frac{dP}{dT}\bigg|_V dV$$

from Q6(b).

Now $c_v = \frac{dQ}{dT}\Big|_V = \frac{dU}{dT}\Big|_V$

So $$dQ = c_v dT - \frac{T}{\rho^2}\frac{\partial P}{\partial T}\bigg|_V d\rho$$

$$= c_v dT - \frac{T}{\rho^2}\frac{P}{T}\frac{\delta}{\alpha}d\rho \qquad \text{from eqn (4).}$$

$$= c_v dT - \frac{P\delta}{\alpha\rho^2}d\rho$$

Now from the definitions of $\alpha$ and $\delta$:

$$\frac{d\rho}{\rho} = \alpha\frac{dP}{P} - \delta\frac{dT}{T}$$

Eliminating $\frac{\delta\rho}{\rho}$ gives

$$dQ = c_v dT - \frac{P\delta}{\alpha\rho}\left(\alpha\frac{dP}{P} - \delta\frac{dT}{T}\right)$$

$$= \left( c_v + \frac{P\delta^2}{\alpha\rho T} \right) dT - \frac{\delta}{\rho} dP$$

But from Q8 $\quad \frac{P\delta^2}{\rho T\alpha} = c_p - c_v$

So $\quad dQ = c_p dT - \frac{\delta}{\rho} dP$

If adiabatic then $dS=0$ $\therefore dQ = TdS = 0$

$$\Rightarrow c_p dT = \frac{\delta}{\rho} dP$$

$$\text{or } \nabla_{ad} = \frac{P\delta}{\rho c_p T}$$

QED

Q12 $\quad P = P_g / \beta$

Thus $\quad \nabla_{ad} = \frac{P_g \delta}{\beta\rho c_p T} = \frac{\cancel{\rho} \mathcal{R} \cancel{T}}{\mu} \cdot \frac{\delta}{\beta \cancel{\rho} c_p \cancel{T}}$

$$= \frac{\mathcal{R}\delta}{\mu\beta C_p}$$

QED

Now $\delta = \frac{4-3\beta}{\beta}$ and $\frac{\mathcal{R}}{\mu} = c_p - c_v$

So

$$\nabla_{ad} = \left(\frac{c_p - c_v}{c_p}\right)\left(\frac{4-3\beta}{\beta}\right) \cdot \frac{1}{\beta}$$

$$= \left(1 - \frac{1}{\gamma}\right)\left(\frac{4-3\beta}{\beta^2}\right)$$

$$= \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{4-3\beta}{\beta^2}\right).$$