

Stars Lab 2: Polytropic Stellar Models

1 Introduction

In the last lab you learned how to use numerical methods to solve various ordinary differential equations. Our aim was to finally get to the stage where we could reliably solve the Lane-Emden equation to make a simple stellar model. We did that. But we only looked at a couple of properties of the resulting solutions. Here we will look more at the interior. We will also add a luminosity equation to the integrations, and finally we will try to construct a main sequence from polytropes. We will also look briefly at the properties of white dwarfs.

2 The Lane-Emden Equation

You may recall the Lane-Emden equation which is written in dimensionless form as:

$$\frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\xi^2 \theta^n \quad (1)$$

where the following variables are used/defined:

$$r = \alpha \xi \quad (2)$$

$$\rho = \rho_c \theta^n \quad (3)$$

$$T = T_c \theta \quad (4)$$

$$P = K \rho^\gamma \quad (5)$$

$$P = P_c \theta^{n+1} \quad (6)$$

$$n = \frac{1}{\gamma - 1} \quad (7)$$

$$\alpha = \left[\frac{K(n+1)\rho_c^{1/n-1}}{4\pi G} \right]^{1/2} \quad (8)$$

$$\xi_1 = \text{first zero of } \theta(\xi) \text{ i.e. } \theta(\xi_1) = 0 \quad (9)$$

$$M = 4\pi\alpha^3 \rho_c \left(-\xi^2 \frac{d\theta}{d\xi} \right)_{\xi_1} \quad (10)$$

$$\frac{\rho_c}{\bar{\rho}} = \frac{-\xi_1}{3 \frac{d\theta}{d\xi}} \quad (11)$$

You previously used the code `le` to solve the Lane-Emden equation. This uses the Runge-Kutta-Fehlberg method to provide an answer to a specified maximum error, or tolerance. To remind yourself how to run the code run it again now.

Run `le` by typing `le` and hitting `ENTER`. The code will request

- `n`: the polytropic index (must be below 5 of course!)
- `iout`: how often to output an integrations step (use 1, for every step)
- `hmax`: the maximum step-size (try `hmax = 0.5` maybe)
- `tolerance`: the maximum error (try `1.e-5`)

The output gives you a table of $x = \xi$ (in our standard terminology), $y = \theta$, $z = \frac{d\theta}{d\xi}$ and h . It also tells you if the method had to reject a step as too large.

Once it has found the surface it prints out the other things we are interested in : ξ_1 , $\frac{\rho_c}{\bar{\rho}}$ and $-\xi^2 \frac{d\theta}{d\xi}$.

3 Structure of Polytropes

Now let's have a look at the internal structure of our polytropes. The key variable is of course θ in that it determines all of the thermodynamical variables, through equations (3)–(6).

Run the program `plotle` to see $\theta(\xi)$ and $\theta^n(\xi)$. I have written this program so that it draws a symbol (large square) at each integration point. First look at $\theta(\xi)$ which is just the scaled temperature. This is not particularly concentrated toward the centre. But the second graph (hit `ENTER` twice) shows θ^n vs ξ . Since θ^n is just ρ/ρ_c we see that the density is much more centrally concentrated than the temperature is.

Q1(a) Run `plotle` for an $n = 3$ polytrope. At what value of ξ is the temperature half of its central value? At what value of ξ is the density half of its central value?

Q1(b) Prepare a table showing the ξ values of the halfway point in temperature and density for $n = 0, 1, 1.5, 2, 3$ and 4. Just read the value from the graph as best you can, but instead of running `plotle` you should use `plotle0.5` which will guide your eye to the mid-point values.

Q1(c) What happens as we increase n ? Explain your result for $n = 0$.

3.1 Central Concentration of T and ρ

It appears that as n increases, the star concentrates its density and temperature at the centre. Run the program `multiplotle1` to view the solution for various values of n on the same axes. This code plots θ vs ξ , or the scaled temperature vs the radius.

It's a little hard to be certain of the relative concentration since the radius ξ_1 is changing for each value of n . Hence I have prepared a program `multiplotle1scaled` which plots the value of θ against ξ/ξ_1 so that all the radius is scaled to one for each star. This one clearly shows that the temperature gets more concentrated to the centre.

Now look at the distribution of ρ/ρ_c vs ξ by running `multiplotle2`. Again, it's a little hard to see what is going on with such large variations in ξ_1 . So run the version `multiplotle2scaled` which scales all radii to unity.

Experiment a little with `multiplotle3a` which plots the log of the density against the radius, and `multiplotle3a` which plots the log of the density against the scaled radius. To zoom in on the central regions, run the pair of codes `multiplotle3b` and `multiplotle3bscaled`.

Q2 Summarize these experiments by selecting a couple of graphs that show what is going on with the run of density with radius and how it varies with polytropic index n .

3.2 Unscaling the Lane-Emden Equation

The dimensionless, and scaled, results we have looked at so far are very good for getting a feel for the structure of the resulting stars. But to get real numbers we have to unscale the equations and use values for real stars. How do we do that?

We start by specifying the polytropic index for the model and perform the integrations, as we have done in the past. We then write down three key outputs from the integrations:

- the dimensionless radius ξ_1
- $\frac{\rho_c}{\bar{\rho}}$ which we saw was given by $-\left. \frac{\xi}{3 \frac{d\theta}{d\xi}} \right|_{\xi_1}$
- $-\left. \xi^2 \frac{d\theta}{d\xi} \right|_{\xi_1}$

Next we specify the mass and radius of the star we are trying to model. We then follow the procedure (I hesitate to call it a recipe!) below:

1. calculate $\bar{\rho}$ from the specified M and R

2. calculate α from $R = \alpha\xi_1$
3. calculate ρ_c from equation (11)
4. calculate K from equation (8)
5. calculate T_c from the equation given in lectures: $T_c = K\rho_c^{1/n} \left(\frac{\mu}{\mathfrak{R}}\right)$ where \mathfrak{R} is the gas constant and μ is the mean molecular weight given by $\mu = \frac{4}{6X + Y + 2}$

If you do not know what X and Y are by now, then you should! Use solar values of $X = 0.7$ and $Y = 0.28$.

Thus we have everything we need to determine the actual density, temperature and pressure throughout the star.

Q3 Use the above to determine the central density and temperature for an $n = 3$ model for the Sun. How do these values compare with the accepted values for the Sun?

I have provided a program that will do this for you, so you can use this to check the value you calculated above. Run `plotlert1` to make a polytrope for an arbitrary n , M and R . Use this program to investigate how the central conditions vary as you change M and R and n .

Q4 For a given M and R . how do ρ_c and T_c vary with n ? Present your results in a table or graph (perhaps using `gnuplot`). Vary n between 0.5 and 4.0.

Code Name	You enter:	Graphical Output
<code>multiplotlerhoM</code>	$R/R_\odot, n$	1) $\log(\rho)$ vs r for various M/M_\odot 2) $\log(\rho_c)$ vs M/M_\odot
<code>multiplotleTM</code>	$R/R_\odot, n$	1) $\log(T)$ vs r for various M/M_\odot 2) $\log(T_c)$ vs M/M_\odot
<code>multiplotlerhoR</code>	$M/M_\odot, n$	1) $\log(\rho)$ vs r for various R/R_\odot 2) $\log(\rho_c)$ vs R/R_\odot
<code>multiplotleTR</code>	$M/M_\odot, n$	1) $\log(T)$ vs r for various R/R_\odot 2) $\log(T_c)$ vs R/R_\odot
<code>multiplotlerhon</code>	$R/R_\odot, M/M_\odot$	1) $\log(\rho)$ vs r for various n 2) $\log(\rho_c)$ vs n
<code>multiplotleTn</code>	$R/R_\odot, M/M_\odot$	1) $\log(T)$ vs r for various n 2) $\log(T_c)$ vs n

4 Adding Luminosity

Given that we have unscaled our equations we can now calculate a luminosity if we make the assumption that all energy generated is radiated by the layers of the star. In that case we simply add the luminosity equation in the form

$$\frac{dL(r)}{dm} = \epsilon \quad (12)$$

where ϵ is the energy generation in ergs/gm/sec. We will break this up into two contributions ϵ_{pp} and ϵ_{CNO} for the pp and CNO cycles, respectively.

From our friendly nuclear physicists we get these rates as

$$\epsilon_{pp} = \epsilon_{pp}^0 T_6^{2/3} e^{-33.8/T_6^{1/3}} \quad (13)$$

$$\epsilon_{CNO} = \epsilon_{CNO}^0 T_6^{2/3} e^{-152.28/T_6^{1/3}} \quad (14)$$

where the ϵ^0 are constants. Run the program `plotrates` to see the two rates plotted against temperature. NOte that at low temperature the pp chains dominate. But at higher values of T the CNO cycle takes over and dramatically exceeds the pp rate.

Q5(a) Why does the CNO cycle require higher temperatures to compete with the pp chains?

Q5(b) What is the central temperature in an $n = 3$ model for the Sun? Which nuclear reaction network, pp or CNO, is thus dominant in the Sun?

The program `plotl1elum1` will solve the Lane-Emden equation and scale everything to M_\odot and R_\odot . You choose the polytropic index n , but for now use $n = 3$, as in the Eddington Standard Model.

Q5(c) What is the total luminosity of an $n = 3$ model for the Sun? What are the pp and CNO contributions in percentages and absolute amounts?

Q5(d) Vary the polytropic index between $n = 3$ and $n = 4$ and show how the total luminosity increases with n . Also plot (perhaps using `gnuplot`) the fraction contributions from the pp and CNO networks to this total.

The absolute value of the luminosity increases with n due to the associated increase in temperature and density, as we found earlier. But how is the distribution of luminosity affected by an increase in n ? We saw that higher n leads to more centrally condensed structure, both in t as well as ρ . We would thus expect the luminosity to be more concentrated toward the centre too. Run the program `plotl1elum2` to see if this is true. The plot here is L/L_{tot} vs r/R . Is it what you expect?

5 The Zero-Age Main Sequence

Our last task concerning luminosity is to try to construct a ZAMS from polytropes. The table below gives an approximate ZAMS from detailed stellar models. As mass increases the star needs a higher pressure to support itself against gravity. Hence the temperature increases also. But the note that the density actually decreases slightly. The overall radius increases with mass. Hence the variation of $\rho_c/\bar{\rho}$ is not monotonic, but passes through a maximum at a mass of $\sim 1.5M_\odot$.

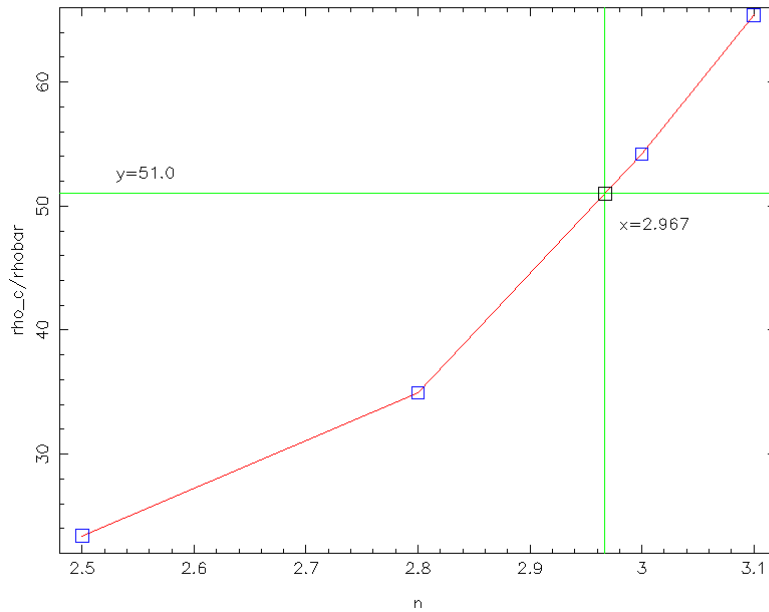
Q6 Why does the $M = M_\odot$ model in Table 1 not have the solar radius and the solar luminosity?

Hence if we were to try to simulate a ZAMS using polytropes we should start by determining an appropriate n for each mass. You can do this using the codes discussed so far. However, for simplicity I have written a code called `rhoconrhobar` which varies n in small amounts and prints out the value of $\rho_c/\bar{\rho}$. It will not give exactly the same values for $\rho_c/\bar{\rho}$ that are required, however. For this we need to interpolate within the results to find the n value appropriate for a required $\rho_c/\bar{\rho}$.

Table 1: **Table 1 - Zero Age Main Sequence**

M/M_\odot	R/R_\odot	$T_c/10^6\text{K}$	$\rho_c/\text{g/cm}^3$	$\rho_c/\bar{\rho}$	L/L_\odot
1.0	0.92	13.9	92	51	0.75
1.4	1.3	18.1	90	100	4.8
2.0	1.5	21.6	69	82	20
4.0	2.2	26.2	29	55	275
8.0	3.4	30.4	12	42	3100
10.0	3.9	31.9	9.5	40	6500

For example, look at the $M = 1$ star in the ZAMS table. It has $\rho_c/\bar{\rho} = 51$. Run `rhoconrhobar`. There is no value of n for exactly 51. We see that the required n must be between 2.8 (which has $\rho_c/\bar{\rho} = 34.95$) and 3.0 (which has $\rho_c/\bar{\rho} = 54.18$). To determine the value of n corresponding to $\rho_c/\bar{\rho} = 51$ we “interpolate” between the two bracketing values. We can use linear interpolation, which simply assumes a straight line joins the two data points. In this case we have the situation shown in Figure 1.



The simple way to do this is the recall that the equation for a straight line passing through two point (x_1, y_1) and (x_2, y_2) is:

$$\frac{(y - y_1)}{x - x_1} = \frac{(y_2 - y_1)}{(x_2 - x_1)} \quad (15)$$

So in our case we specify the required value of $y=51$ and then use the closest points either side of that value as given by `rhoconrhobar`.

Q7(a) Verify the values given in Figure 1 by using equation (15).

I do not expect you to do that for each value in the table. Rather, I have written a code that does it for you. You will need to enter the bracketing values of $\rho_c/\bar{\rho}$ and n . Use the program `lininterp`.

Q7(b) Present a table of values of n for the masses in the ZAMS of Table 1.

We have calculated the luminosity for a polytropic model for the Sun with $n = 3$. We saw that the luminosity was not quite right, and this is of course due to the approximations in the polytropic equation of state. We know the Sun has higher T and ρ than the polytrope, so we must get a lower L also. To allow for this we will multiply the burning rates by a constant factor. We will determine this factor by assuming that the $M = 1$ model is powered entirely by pp reactions. We will then scale our pp rate so that we get the L given in Table 1 for the $M = 1$ star. Similarly, the $M = 8$ model is entirely powered by CNO so we will scale the CNO rates by the factor required to get the luminosity right for the $M = 8$ model in Table 1.

Run `lelum1` and input values for the $M = 1$ case, and use the n determined above to be 2.967 for this case. It will prompt for values of the correction factors f_{pp} and f_{CNO} . Use 1 for each initially to see what the uncorrected values are. The code will output the resulting L for the values you input. This L is likely to be less than the value in the ZAMS table, so now determine the value of f_{pp} needed to get the L value in the table. It is a simple linear scaling.

Q7(c) List the values given by the code for no correction factors and explain how you determined the value of f_{pp} .

Q7(d) Repeat the procedure for the $8M_{\odot}$ case and hence determine f_{CNO} .

Q7(e) Using these values of f_{pp} and f_{CNO} , together with the n value you determined for each mass (in Q7(b)), present a table of luminosities determined by the polytropic models. Compare with the ZAMS table.

Q7(f) Finally, present your ZAMS in the HR diagram. You have L from the previous part. Determine $\log(T_e)$ from $L = 4\pi R^2 \sigma T_e^4$. Compare with the ZAMS in Table 1. ($\sigma = 5.67 \times 10^{-5} \text{ erg cm}^{-2} \text{ K}^4 \text{ s}^{-1}$)

6 White Dwarfs

Our last task is to look at white dwarfs, since these are very well approximated by polytropes. We saw in lectures that the maximum mass for a White Dwarf is about $1.44M_{\odot}$, corresponding to an $n = 3$ model with a completely degenerate, super-relativistic equation of state. But let's look at some less extreme examples! The most common white dwarf is one with a mass close to $0.6M_{\odot}$.

We will take our equation of state to be

$$P = K\rho^{7/5} \quad (16)$$

with $K = 1. \times 10^{14}$ in cgs units, and $n = 2.5$.

Q8(a) Using the information above, and the codes we have used earlier, determine the radius of the $n = 2.5$ white-dwarf of mass $0.6M_{\odot}$. Express the radius in Earth radii, $R_{\oplus} = 6.38E8\text{cm}$. This involves using the Lane-Emden solution and then unscaling the equations, but in a different way to what you did before. Here we know K but do not know R . You may have to think a little!

Degenerate matter behaves strangely. A white-dwarf is supported entirely by the degenerate pressure of the electrons. There is no nuclear energy source in white dwarfs, so they cool. As they cool, however, they lose thermal energy and they become increasingly degenerate. The density increases and the radius decreases. A curious fact is that increasing the mass of a white dwarf leads to higher degeneracy, higher density and hence a lower radius!

Run the program `wd` to determine the central density and radius of white dwarfs of different masses and polytropic indices n .

Q8(b) Using `wd` to generate values, show how the white-dwarf radius decreases with increasing mass. Present your graphs as a table, or preferably a graph. Use at least two values of n .

You can check your results by running `plotwd`.