Section 1: Stars

Answer up to three questions from this section.

Question 1 [Total Mark: 25 points]

a) By integrating an approximate equation of motion, as done in lectures, show that the free-fall time for a star of mass M and radius R is

$$t_{ff} = \left(\frac{R^3}{GM} \right)^{1/2}.$$

Assume the average value of r is $R/2$ and the average value of $m(r)$ is $M/2$.

b) Given that the gravitational energy Ω can be written as

$$\Omega = -\int \frac{Gm(r)}{r} dm$$

show that an approximate expression for Ω for this star is $\Omega = -GM^2/R$. Again, assume the average value of r is $R/2$ and the average value of $m(r)$ is $M/2$.

c) If the star has a luminosity L, give the definition of the Kelvin-Helmholtz timescale t_{KH} in words, and write down an expression for t_{KH}.

d) Write the free-fall timescale as

$$t_{ff} = a \left(\frac{R}{R_\odot} \right)^{3/2} \left(\frac{M}{M_\odot} \right)^{-1/2} \text{ seconds.}$$

Find the value of the co-efficient a, assuming $G = 6.7 \times 10^{-11} \text{m}^3/\text{kg}/\text{sec}^2$.

Question 2 [Total Mark: 25 points]

a) Derive a differential equation for dm/dr in a spherical self-gravitating gas of density $\rho(r)$.

b) Derive the differential equation for hydrostatic equilibrium dP/dr for a spherical self-gravitating star.

c) Combine the two equations from (a) and (b) into a single, second order equation.

d) Discuss the behaviour of the hydrostatic support equation near the centre of the star. Is the equation well behaved? Why/why not?

e) The equation derived in (c) can be turned into the Lane-Emden equation by applying the scaling $r = \alpha\xi$ and $\rho = \rho_c\theta^n$ for an appropriate value of α:

$$\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\xi^2\theta^n.$$

DO NOT ATTEMPT TO DERIVE THIS EQUATION.

What is the value of $\theta(0)$ and why?

f) Determine the solution for $n = 0$, using the central boundary condition to remove any arbitrary constants.

g) Similarly, find the solution to the Lane-Emden equation for the case with $n = 1$, by making the substitution $\theta = \phi/\xi$. Again, remove any arbitrary constants.

Question 3 [Total Mark: 25 points]

a) Show that a homogeneous star with pressure due to a perfect gas with radiation can be modelled as a polytrope of index 3 if the ratio of the gas to total pressure β is constant throughout the star.

b) Consider a perfect monatomic gas with $\gamma = 5/3$ and no radiation pressure. Show that under adiabatic changes we have

$$\nabla_{ad} = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = \frac{\gamma - 1}{\gamma} = 0.4.$$

c) Consider the CN cycle for burning hydrogen. This can be solved as an eigenvalue problem. For a particular temperature and density we obtain the following solution:

$$\begin{bmatrix} ^{12}\mathrm{C} \\ ^{13}\mathrm{C} \\ ^{14}\mathrm{N} \end{bmatrix} = N \begin{bmatrix} 0.012 \\ 0.004 \\ \alpha \end{bmatrix} + 0.3N \begin{bmatrix} 1.0 \\ 0.3 \\ \beta \end{bmatrix} \exp(\delta \times 10^{-4} t) + 0.04N \begin{bmatrix} 1.0 \\ 200 \\ \gamma \end{bmatrix} \exp(\epsilon \times 10^{-4} t)$$

i) What is the value of α?

ii) What is the value of $^{12}\mathrm{C}/^{13}\mathrm{C}$ when the cycle is operating in equilibrium?

iii) What is N?

iv) What is the value of β?

v) What is the value of γ?

vi) What is the sign of δ and why?

vii) What is the sign of ϵ and why?

Question 4 [Total Mark: 25 points]

a) Consider a red-giant star with a hydrogen burning shell and an inactive helium-rich core. Suppose that the hydrogen shell is infinitely thin and positioned at $r = r_s$ where the density is ρ_s and the temperature is T_s. Just above the shell we have

$$\rho = \rho_s \left(\frac{r_s}{r}\right)^3 \tag{1}$$

$$T = T_s \left(\frac{r_s}{r}\right) \tag{2}$$

If the energy generation is given by

$$\epsilon = \epsilon_0 \rho X^2 T^n$$

then derive an expression for $L(r)$ of the form

$$L(r) = L_0 \left[1 - \left(\frac{r_s}{r}\right)^\alpha\right]$$

and give values of L_0 and α.

b) A good approximation to the pressure gradient in a star is given by

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{4\pi}{3} G \rho_c^2 r e^{-x^2}$$

for some scale-height a and central density ρ_c, and where $x = r/a$. Show that

$$P(r) = P_c e^{-x^2}$$

where

$$P_c = \frac{2\pi}{3} G \rho_c^2 a^2.$$

You may assume that the stellar radius $R \gg a$ and that $P(R) = 0$.

c) Briefly describe the evidence for some form of "extra-mixing" beyond what occurs during the first dredge-up. Name at least one possible mechanism that might drive such mixing.

d) Describe how neutron capture can produce elements with different atomic number Z.

Section 2: Galaxies

Answer up to three questions from this section.

Question 5 [Total Mark: 25 points]

The rotation curve of the galaxy NGC2742 is shown in the Figure below.

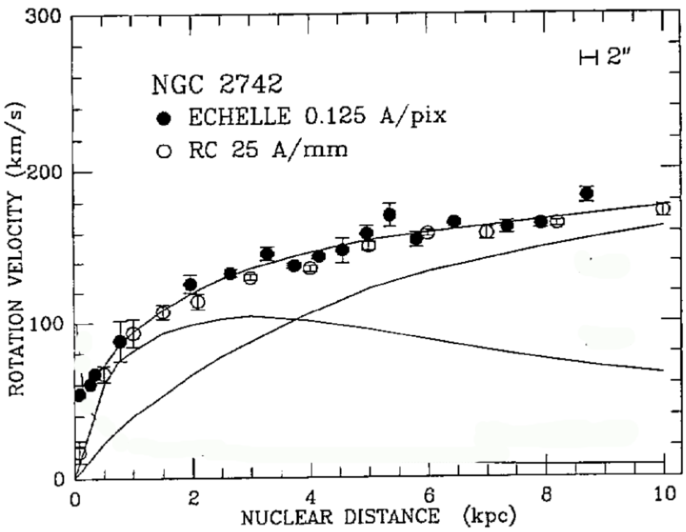

Figure 1:

a) Based on what you have learned, can you say what type of galaxy is NGC2742?

b) Explain what are the continuous curves shown in the Figure. Why do they have that shape? Which component of the galaxy is represented by each curve?

c) For a radius of about 6 kpc of the galaxy, estimate the total mass inside this radius. Use Newton's law of gravity and law of motion in a circular orbit. The gravitational constant is $G = 4.3 \times 10^{-6}$ kpc $\times$ km$^2/M_{\rm sun}/s^2$. Express the mass in units of solar masses ($M_{\rm sun}$).

d) True or False? Explain why it is true or false:
Galaxies are mostly made of stars and star dust.

e) True or False? Explain why it is true or false:
The dark matter is distributed in the plane of the galaxy and its density peaks in the center of the galaxy.

Question 6 [Total Mark: 25 points]

a) Some galaxy spectra are characterized by strong absorption lines, some are characterized by strong emission lines. Which of the spectra in Figure 2 corresponds to an elliptical galaxy? Which one is emitted by a spiral galaxy? On the x-axis you read the wavelength of observation.

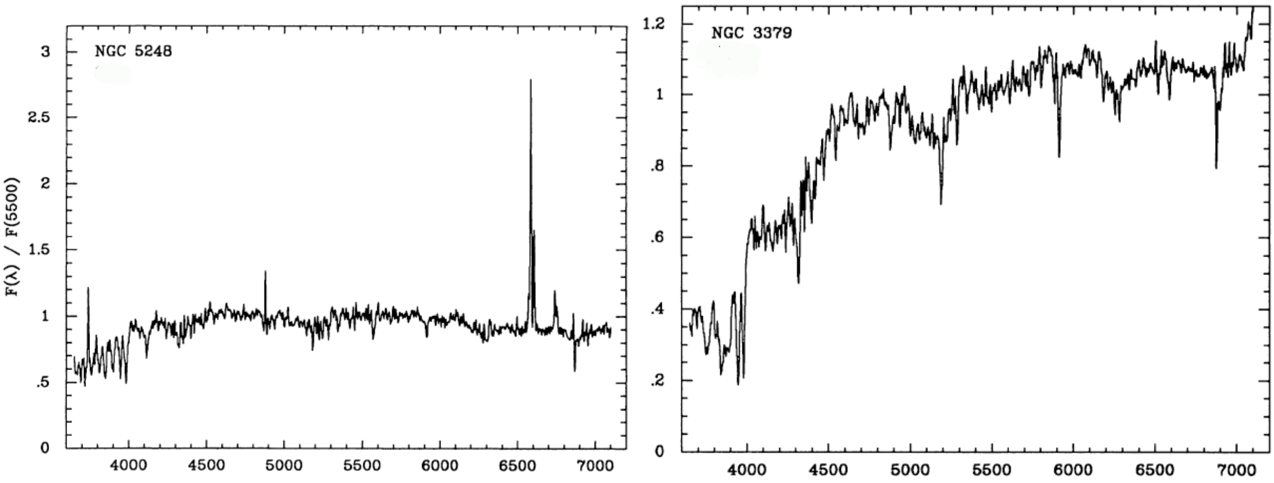

Figure 2:

b) In which type of galaxy are there essentially no young stars and no gas?

c) Figure 3 shows an empirical relationship between the intrinsic luminosity (proportional to the stellar mass) of a spiral galaxy and its velocity width (the amplitude of its rotation curve). Identify the name of the relation and write down the mathematical form of this relation, explaining all terms.

d) Does the relation in Figure 3 apply to elliptical galaxies? If so, explain why. If not, explain why not.

e) Figure 4 shows stars and interstellar gas in an Sc galaxy. Identify which of the maps shows the surface density of HI gas. Explain how the HI gas emits its radiation. What is the wavelength at which astronomical instruments should measure this emission?

f) Identify in the Figure 4 the map showing the optical emission from stars in the Sc galaxy. Describe what you see.

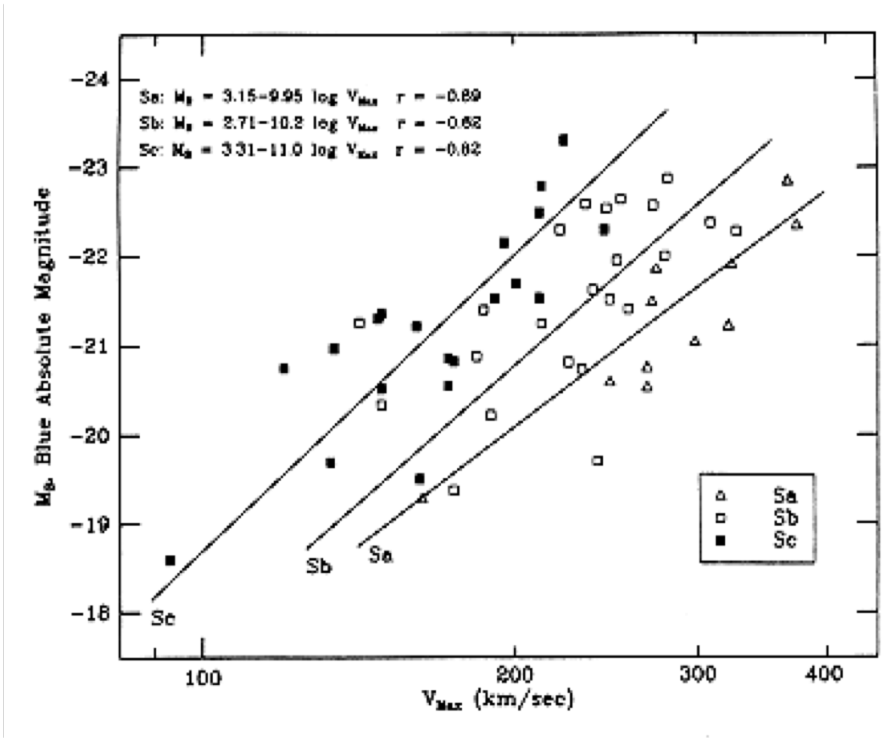

Figure 3:

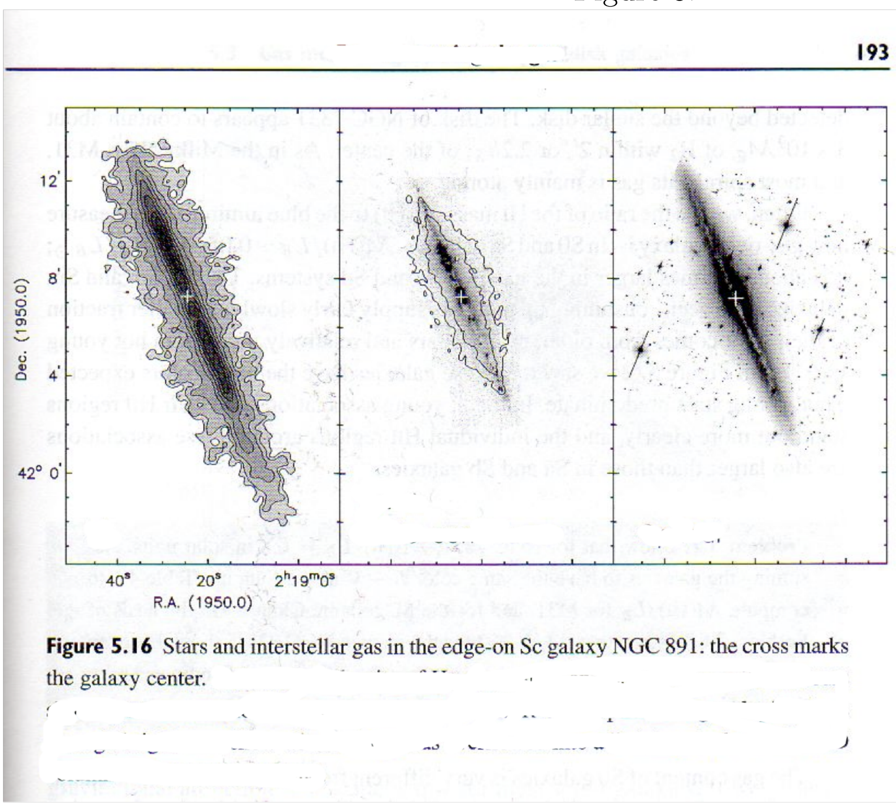

Figure 5.16 Stars and interstellar gas in the edge-on Sc galaxy NGC 891: the cross marks the galaxy center.

Figure 4:

Question 7 [Total Mark: 25 points]

Knots in relativistic jets in some galaxies and quasars appear to be travelling faster than light.

a) Explain their superluminal motion by means of radio mapping.

b) Estimate the apparent speed of superluminal motion of the knots as a function of v the true space velocity of the moving component and φ the angle between the line of sight and the direction of the radio knot.

c) Write down five true sentences about the relativistic jets in active galactic nuclei.

d) What can you say about the magnetic field in our galaxy, based on the polarization measurement shown in Figure 5. Explain the features that appear in the map at the galactic longitude 30^o and the galactic latitude larger than 30^o. What is the meaning of each small line in the map?

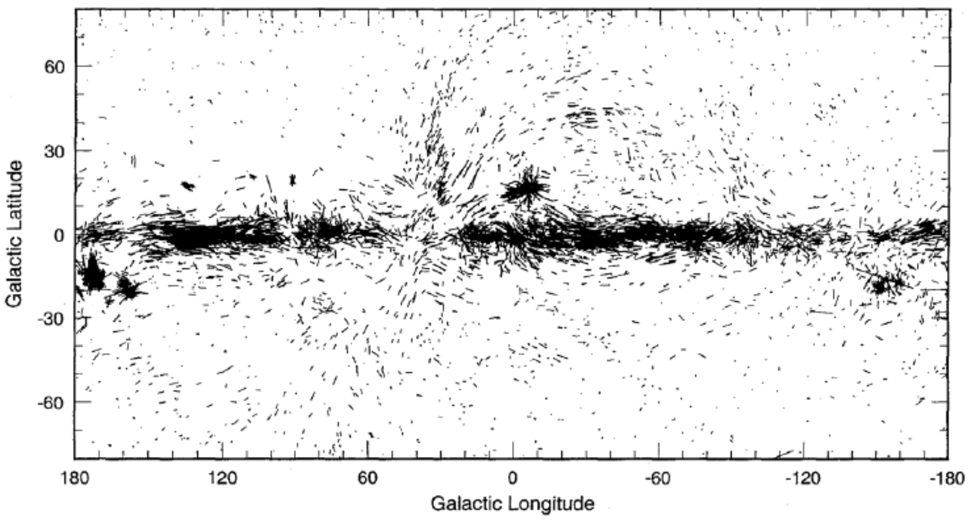

Figure 5:

e) Light from which object in the galaxy is polarized by the interstellar dust? Expand on your answer.

Question 8 [Total Mark: 25 points]

a) Explain what is the dynamical friction (draw a picture).

b) Copy the sentence in your exambook and fill in the dots in the following text (underline your contribution):

The cooled radiation of the hot big bang that fills the entire .... and can be observed today with an average temperature of about .... kelvin is called: ....

c) Describe the importance of the WMAP image of the CMB temperature anisotropy shown below. What are the features seen in the Figure 6?

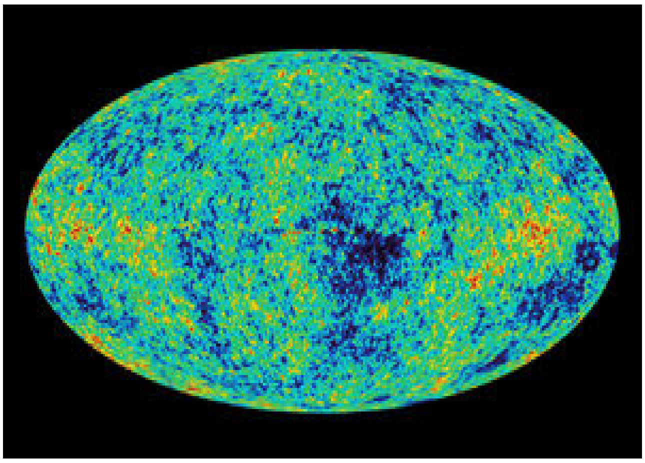

Figure 6:

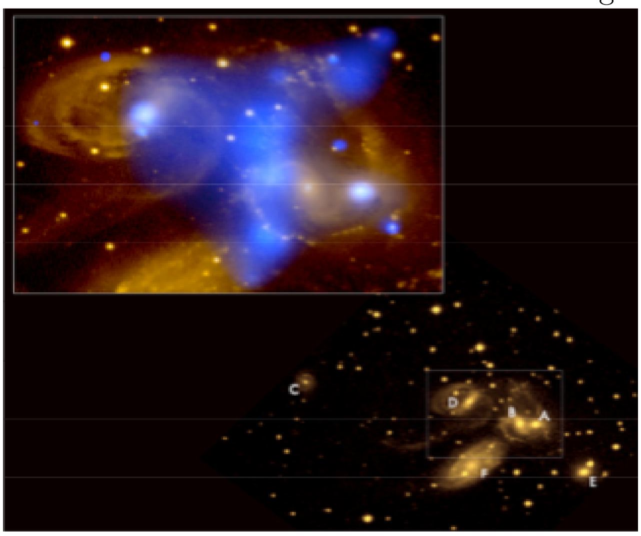

Figure 7:

d) Calculate the peak wavelength (in nm) of the blackbody radiation emitted by a blackbody at a temperature of 1000 K. In what region of the electromagnetic spectrum is this wavelength found?

e) The first identified compact galaxy group, Stephan's Quintet is featured in this stunning image from the Hubble Space Telescope (Figure 7). The compact group of galaxies, was discovered about 130 years ago, about 280 million light years from Earth. The image also shows Chandra observations of the intergalactic medium of the compact galaxy group. Explain what did the Chandra satellite observe (see the upper left corner of the Figure 7). How hot would it have the plasma to be to emit the observed Chandra radiation? (k_B=1.38 $\times 10^{-23}$ m^2 kg s^{-2} K^{-1})

END OF EXAM QUESTIONS

USEFUL FORMULAE AND CONSTANTS

Physical Constants:

$$
\begin{aligned}
\Re &= \text{Universal Gas Constant} &&= 8.314 \times 10^{7}\text{erg/K/g} \\
k &= \text{Boltzmann's constant} &&= m_u \Re = 1.38 \times 10^{-16}\text{erg/K} \\
a &= \frac{4\sigma_B}{c} &&= \text{radiation density constant} = 7.56 \times 10^{-16}\text{J/m}^3/\text{K}^4 \\
\sigma_B &= 5.67 \times 10^{-8}\text{W/m}^2\text{K}^4 && \\
c &= \text{speed of light} &&= 2.998 \times 10^{10}\text{cm/sec} \\
h_{\text{Planck}} &= 6.62 \times 10^{-34} && \\
m_u &= \text{atomic mass unit} &&= 1.66053 \times 10^{-24}\ \text{g} \\
& \quad \text{mass of H nucleus} &&= 1.0073 m_u \\
& \quad \text{mass of } H_2 \text{ molecule} &&= 2.0160 m_u \\
& \quad \text{mass of } He^4 \text{ nucleus} &&= 4.0014 m_u \\
& \quad \text{mass of } He^4 \text{ atom} &&= 4.0028 m_u \\
& \quad \text{mass of } C^{12} \text{ nucleus} &&= 12.0000 m_u
\end{aligned}
$$

Physical Formulae:

The blackbody Planck function: $B_\nu(T) = \dfrac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$ erg/cm^2/s/Hz/steradians

Poisson's equation: $\nabla^2 \phi = 4\pi G \rho$

Astronomical Constants

$$
\begin{aligned}
R_\odot &= 6.96 \times 10^{10}\text{cm} \\
M_\odot &= 1.989 \times 10^{33}\text{g} \\
L_\odot &= 3.86 \times 10^{33}\text{erg/sec} \\
1\ \text{year} &= 3.156 \times 10^{7}\text{seconds} \\
1\ \text{eV} &= 1.6 \times 10^{-19}\text{J} \\
1\ \text{J} &= 10^{7}\text{erg} \\
\text{Chandrasekhar Mass :} &= 1.46 M_\odot
\end{aligned}
$$

Astronomical Formulae

$$
\begin{aligned}
M_{BOL} &= -2.5 \log\left(\tfrac{L}{L_\odot}\right) + 4.72 && \\
m - M &= 5 \log d - 5 && \text{without extinction} \\
m - M &= 5 \log d - 5 + A && \text{with extinction of } A \text{ magnitudes} \\
A &= (2.5 \log_{10} e)\tau && \\
\frac{1}{\mu} &= \sum \frac{X_Z f_Z}{A_Z} &&
\end{aligned}
$$

Useful Mathematical Identities:

$$\int_{-\infty}^{+\infty} e^{-\alpha u^2} du = \sqrt{\frac{\pi}{\alpha}}$$

$$\sin \Phi = \sqrt{\frac{1}{1 + \cot^2 \Phi}}$$

$$\cos \Phi = \sqrt{\frac{\cot^2 \Phi}{1 + \cot^2 \Phi}}$$

Miscellaneous:

$$\text{Flux} = \frac{\text{Luminosity}}{4\pi \text{Distance}^2}$$

The velocity for the stable orbit can be derived as: $E_{\text{kin}} = E_{\text{potential}}$

For a perfect monatomic gas: $\gamma = c_P/c_V = 5/3$

$$\frac{\Re}{\mu} = c_P - c_V$$

$$u = c_V T$$

The energy density of a photon gas: $u = aT^4$

EQUATIONS OF STELLAR STRUCTURE:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{GM(r)}{r^2}$$

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho \epsilon$$

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2 \rho$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\kappa\rho}{16\pi a c r^2} \frac{L(r)}{T^3} \quad \text{if radiative}$$

PPI Chain:

$$^1\mathrm{H} + ^1\mathrm{H} = \ ^2\mathrm{D} + \beta^+ + \nu \quad (1)$$

$$^1\mathrm{H} + ^2\mathrm{D} = \ ^3\mathrm{He} + \gamma \quad (2)$$

$$^3\mathrm{He} + ^3\mathrm{He} = \ ^4\mathrm{He} + 2p \quad (3)$$

PPII Chain:

$$^3\mathrm{He} + ^4\mathrm{He} = \ ^7\mathrm{Be} + \gamma \quad (4)$$

$$^7\mathrm{Be} + \beta^- = \ ^7\mathrm{Li} + \gamma \quad (5)$$

$$^7\mathrm{Li} + ^1\mathrm{H} = 2\ ^4\mathrm{He} \quad (6)$$

PPIII Chain:

$$^3\mathrm{He} + ^4\mathrm{He} = \ ^7\mathrm{Be} + \gamma \quad (7)$$

$$^7\mathrm{Be} + ^1\mathrm{H} = 2\ ^4\mathrm{He} \quad (8)$$

END OF EXAM PAPER