Section 1: Stars

Answer up to three questions from this section.

Question 1

a) Consider the mean molecular weight μ which is defined as

$$\frac{1}{\mu} = \sum_i \frac{X_i f_i}{A_i}$$

where f_i is the number of free particles per species i, the mass fraction of species i and A_i is the mass (in atomic mass units) of species i.

i) Calculate μ for a gas composed of 50% by mass H_2 and 50% by mass of CO (which is in turn comprised of the most common isotopes of C and O, being C^{12} and O^{16}).

ii) Calculate μ for a gas composed of 50% by mass ionized H and 50% by mass of ionized C^{12}.

iii) By considering the reciprocal of μ, i.e. the number of particles per unit mass, derive a formula for μ for a fully ionized gas with H mass fraction X, He^4 mass fraction Y and a mass fraction Z of heavier species. (Eliminate Z from your final answer.)

b) Consider a perfect monatomic gas with $\gamma = 5/3$ and no radiation pressure. Show that under adiabatic changes we have

$$\nabla_{ad} = \frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = \frac{\gamma - 1}{\gamma} = 0.4.$$

c) Derive the expression $L_{Edd} = \dfrac{4\pi c G M}{\kappa}$ for the Eddington luminosity, being the maximum luminosity that can be emitted by a star in hydrostatic equilibrium.

d) Consider the Lane-Emden equation:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\xi^2 \theta^n$$

i) Find the general solution for the case $n = 1$ by making the substitution $\theta = \phi/\xi$

ii) Use the boundary conditions to evaluate the arbitrary constants

iii) Determine the dimensionless radius of an $n = 1$ polytrope.

Question 2

a) Consider a red-giant star with a hydrogen burning shell and an inactive helium-rich core. Suppose that the hydrogen shell is infinitely thin and positioned at $r = r_s$ where the density is ρ_s and the temperature is T_s. Just above the shell we have

$$\rho = \rho_s \left(\frac{r_s}{r}\right)^3 \quad (1)$$

$$T = T_s \left(\frac{r_s}{r}\right) \quad (2)$$

If the energy generation is given by

$$\epsilon = \epsilon_0 \rho X^2 T^n$$

then derive an expression for $L(r)$ of the form

$$L(r) = L_0 \left[1 - \left(\frac{r_s}{r}\right)^\alpha\right]$$

and give values of L_0 and α.

b) A good approximation to the pressure gradient in a star is given by

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{4\pi}{3} G \rho_c^2 r e^{-x^2}$$

for some scale-height a and central density ρ_c, and where $x = r/a$. Show that

$$P(r) = P_c e^{-x^2}$$

where

$$P_c = \frac{2\pi}{3} G \rho_c^2 a^2.$$

You may assume that the stellar radius $R \gg a$ and that $P(R) = 0$..

c) A red-giant of $1M_\odot$ has recently undergone first dredge-up. Before this event the mass of the H-exhausted core was $0.5M_\odot$, but afterward the value is $0.4M_\odot$.

i) If the initial composition was $X = 0.7$ and $Y = 0.28$ determine the final surface X and Y after dredge-up.

ii) If the initial CNO mass fractions are ${C^{12}}_i$, ${N^{14}}_i$ and ${O^{16}}_i$, determine the values after dredge-up.

iii) Show that the total C+N+O is conserved.

d) Briefly describe the evidence for some form of "extra-mixing" beyond what occurs during the first dredge-up. Name at least one possible mechanism that might drive such mixing.

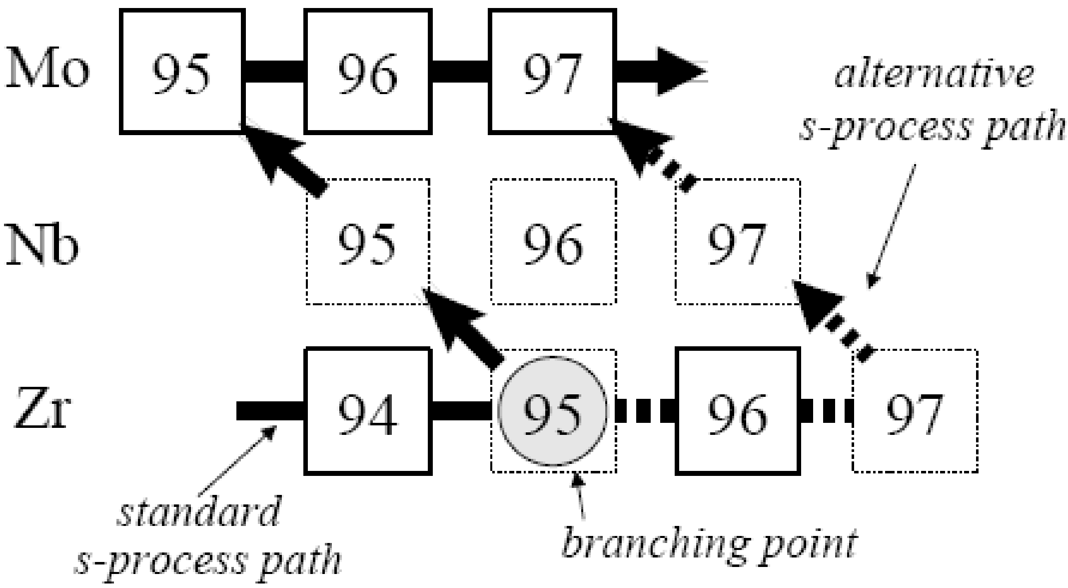
Question 3

a) Consider a region of a star that is undergoing neutron capture nucleosynthesis. Show that in equilibrium we have $N_A\langle\sigma_A\rangle = \text{constant}$ where N_A is the number density of species A, v_{th} is the thermal velocity and $\langle\sigma_A\rangle$ is defined by

$$\langle\sigma_A\rangle v_{th} = \langle\sigma_A(v)v\rangle$$

You may find it useful to use the neutron exposure τ defined by $\tau = \int_0^t N_n v_{th} dt$ where N_n is the neutron number density.

b) The diagram below shows part of the chart of the nuclides near Zr. There is a branching at Zr^{95}. Calculate the fraction of neutron captures that proceed to Zr^{96} when the neutron density is $N_n = 5 \times 10^9 \text{cm}^{-3}$. Assume that the thermal velocity is 2×10^8 cm/sec, the neutron capture cross-section for Zr^{95} is 50 mbarn and the instantaneous decay rate per Zr^{95} is $\lambda_\beta = 1.2 \times 10^{-7}\text{s}^{-1}$.



c) Measurements of the relative abundances of Uranium isotopes in rocks on the Earth reveal

$$\frac{U^{235}}{U^{238}} = 0.007$$

Given that the half-lives are

$$\begin{aligned} t_{1/2}(U^{235}) &= 7.0 \times 10^8 \text{ y} \\ t_{1/2}(U^{238}) &= 4.5 \times 10^9 \text{ y} \end{aligned}$$

calculate how long has elapsed since these species were produced by the r-process. Assume that the r-process produces the ratio $\frac{U^{235}}{U^{238}} = 1.5$.

d) Consider the pp chains for burning hydrogen (see Formulae at end). Which of the following statements is true or false (simply write T or F in your answer booklet, next to the question number).

i) The isotope D^2 comes into equilibrium very quickly, in seconds or minutes at typical temperatures for H burning.

ii) In equilibrium, the D^2 abundance is very small, always much less than the H abundance.

iii) The fate of the He^3 nucleus determines the relative strength of the pp chains.

iv) The isotope He^3 comes into equilibrium very quickly, in seconds or minutes at typical temperatures for H burning.

v) In equilibrium, the He^4 abundance is very small, always much less than the H abundance.

e) Consider the CN cycle for burning hydrogen This can be solved as an eigenvalue problem. For a particular temperature and density we obtain the following solution:

$$\begin{bmatrix} \mathrm{C}^{12} \\ \mathrm{C}^{13} \\ \mathrm{N}^{14} \end{bmatrix} = N \begin{bmatrix} 0.012 \\ 0.004 \\ \alpha \end{bmatrix} + 0.3N \begin{bmatrix} 1.0 \\ 0.3 \\ \beta \end{bmatrix} \exp\left(\delta \times 10^{-4} t\right) + 0.04N \begin{bmatrix} 1.0 \\ 200 \\ \gamma \end{bmatrix} \exp\left(\epsilon \times 10^{-4} t\right)$$

i) What is the value of α?

ii) What is the ratio of C^{12}/C^{13} when the cycle is operating in equilibrium?

iii) What is N?

iv) What is the value of β?

v) What is the value of γ?

vi) What is the sign of δ? Why?

Question 4

a) Using simple arguments (and the equations of stellar structure, see Useful Formulae at the end) explain why burning hydrogen causes a stars luminosity to rise over nuclear burning timescales.

b) By using rough scaling relations, such as

$$\frac{\mathrm{d}T}{\mathrm{d}r} \sim T/R$$

together with the equations of stellar structure and the perfect gas equation, show that a radiative star has $L \sim M^3$.

c) Briefly explain the physics behind the Schwarzschild criterion for convection.

d) Explain why a hydrostatic star cannot appear redward of the giant branch.

e) Briefly describe the evolution of a $1M_\odot$ star from the ZAMS through to the white dwarf phase. Include an HR diagram and any other sketches that help illustrate the evolution.

f) Describe the main differences between stars on the upper and lower main sequence, with diagrams as appropriate. Explain what causes this difference and the give approximate mass where the division occurs.

Section 2: Galaxies

Answer up to three questions from this section.

Question 5

A) Suppose an active galactic nucleus and its host galaxy are roughly the same luminosity. Depending on the distance of the object and spatial resolution of our observations, the host galaxy could appear unresolved (i.e. will appear point- like) and the object will look like a quasar. At what distance will this happen for

   a) ground-based imaging at the world's best site on Mauna Kea (seeing 0.6") and

   b) imagery with the Advanced Camera for Surveys on board the HST (pixel size 0.05").

   Hint: You may assume the bulk of the host galaxy's light is concentrated within 5 kpc. You will also need to use $D = cz/H_0$, where $H_0 = 70$ km/s/Mpc (the Hubble constant). Use $L_{host} = L_{AGN}$ and 1 radian = 57.295 degrees = 57.295×3600 $arcsec$ = 206265 $arcsec$.

B) a) The Sun's temperature is 6000 K. Calculate the energy per second radiated by the Sun in the form of black body radiation. Express your answer in Watts.

   b) Determine the peak wavelength of the blackbody radiation emitted by the Sun. In what region of the electromagnetic spectrum is this wavelength found?

   c) What is the average temperature of the Cosmic Microwave Background radiation? Determine the total energy density in J/m^3.

Question 6

a) Describe the interaction between an ultrahigh energy cosmic ray and the microwave background (new particles, threshold energy, energy loss distance).

b) Describe the Faber – Jackson for elliptical galaxies. Sketch this.

c) Explain how it is possible to identify neutral hydrogen (HI) in the interstellar medium (the radiative process and the instrumentation).

d) Describe two methods used to determine the masses of black holes in galactic nuclei.

e) The K line of a singly ionized ion of Calcium, which is observed in the spectrum of the Sun and spectra of other stars in our Galaxy, has a laboratory wavelength of 393.4 nm (1 nm= 10^{-9} m). In the spectrum of the galaxy NGC4839 it is observed to have a wavelength of 403.4 nm. What is the redshift of this galaxy?

Question 7

The Plummer model of a spherical system of stars is described by a simple potential with

$$\Phi = -\frac{GM}{\sqrt{r^2+b^2}},$$

where the linear scale of the system that generates this potential is set by the Plummer scale length b, while M is the system's total mass.

a) Calculate ∇^2 and use the Poisson's equation to derive the stellar density corresponding to the above potential. Use the operator:

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \tan\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}$$

b) Based upon the radial behavior in (a), analyse the density near the center of the system and at large radii.

c) Calculate the circular speed of a particle of negligible mass in a circular orbit at radius r.

d) Calculate the escape speed v_e and explain the mass dependence of this value.

Question 8

Knots in relativistic jets in some galaxies and quasars appear to be travelling faster than light.

a) Explain their superluminal motion by means of radio mapping.

b) Estimate the apparent speed of superluminal motion of the knots. Explain carefully why this speed can be expressed as

$$v_{app} = \frac{v\sin\phi}{1-\frac{v}{c}\cos\phi}$$

where v is the true space velocity of the moving component.

c) Give a lower bound to v and

d) Derive also the minimum Lorentz factor of the source.

USEFUL FORMULAE AND CONSTANTS

Physical Constants:

$$\begin{aligned}
\Re &= \text{Universal Gas Constant} &&= 8.314 \times 10^7 \text{erg/K/g} \\
k &= \text{Boltzmann's constant} &&= m_u \Re = 1.38 \times 10^{-16} \text{erg/K} \\
a &= \frac{4\sigma_B}{c} &&= \text{radiation density constant} = 7.56 \times 10^{-16} \text{J/m}^3\text{/K}^4 \\
\sigma_B &= 5.67 \times 10^{-8} \text{W/m}^2\text{K}^4 && \\
c &= \text{speed of light} &&= 2.998 \times 10^{10} \text{cm/sec} \\
h_{\text{Planck}} &= 6.62 \times 10^{-34} && \\
m_u &= \text{atomic mass unit} &&= 1.66053 \times 10^{-24} \text{ g} \\
& \text{mass of H nucleus} &&= 1.0073 m_u \\
& \text{mass of H}_2 \text{ molecule} &&= 2.0160 m_u \\
& \text{mass of He}^4 \text{ nucleus} &&= 4.0014 m_u \\
& \text{mass of He}^4 \text{ atom} &&= 4.0028 m_u \\
& \text{mass of C}^{12} \text{ nucleus} &&= 12.0000 m_u
\end{aligned}$$

Physical Formulae:

The blackbody Planck function: $B_\nu(T) = \dfrac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}$ erg/cm^2/s/Hz/steradians

Poisson's equation: $\nabla^2 \phi = 4\pi G \rho$

Astronomical Constants

$$\begin{aligned}
R_\odot &= 6.96 \times 10^{10} \text{cm} \\
M_\odot &= 1.989 \times 10^{33} \text{g} \\
L_\odot &= 3.86 \times 10^{33} \text{erg/sec} \\
1 \text{ year} &= 3.156 \times 10^7 \text{seconds} \\
1 \text{ eV} &= 1.6 \times 10^{-19} \text{J} \\
1 \text{ J} &= 10^7 \text{erg} \\
\text{Chandrasekhar Mass :} &= 1.46 M_\odot
\end{aligned}$$

Astronomical Formulae

$$\begin{aligned}
M_{BOL} &= -2.5 \log\left(\tfrac{L}{L_\odot}\right) + 4.72 && \\
m - M &= 5 \log d - 5 && \text{without extinction} \\
m - M &= 5 \log d - 5 + A && \text{with extinction of } A \text{ magnitudes} \\
A &= (2.5 \log_{10} e)\tau && \\
\frac{1}{\mu} &= \sum \frac{X_Z f_Z}{A_Z} &&
\end{aligned}$$

Useful Mathematical Identities:

$$\int_{-\infty}^{+\infty} e^{-\alpha u^2} du = \sqrt{\frac{\pi}{\alpha}}$$

$$\sin \Phi = \sqrt{\frac{1}{1 + \cot^2 \Phi}}$$

$$\cos \Phi = \sqrt{\frac{\cot^2 \Phi}{1 + \cot^2 \Phi}}$$

Miscellaneous:

$$\text{Flux} = \frac{\text{Luminosity}}{4\pi\text{Distance}^2}$$

The velocity for the stable orbit can be derived as: $E_{\text{kin}} = E_{\text{potential}}$

For a perfect monatomic gas: $\gamma = c_P/c_V = 5/3$

$$\frac{\Re}{\mu} = c_P - c_V$$

$$u = c_V T$$

The energy density of a photon gas: $u = aT^4$

EQUATIONS OF STELLAR STRUCTURE:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho\frac{GM(r)}{r^2}$$

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2\rho\epsilon$$

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\rho$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3\kappa\rho}{16\pi acr^2}\frac{L(r)}{T^3} \quad \text{if radiative}$$

PPI Chain:

$$^1\text{H} + ^1\text{H} = {}^2\text{D} + \beta^+ + \nu \quad (1)$$

$$^1\text{H} + ^2\text{D} = {}^3\text{He} + \gamma \quad (2)$$

$$^3\text{He} + ^3\text{He} = {}^4\text{He} + 2p \quad (3)$$

PPII Chain:

$$^3\text{He} + ^4\text{He} = {}^7\text{Be} + \gamma \quad (4)$$

$$^7\text{Be} + \beta^- = {}^7\text{Li} + \gamma \quad (5)$$

$$^7\text{Li} + ^1\text{H} = 2\ {}^4\text{He} \quad (6)$$

PPIII Chain:

$$^3\text{He} + ^4\text{He} = {}^7\text{Be} + \gamma \quad (7)$$

$$^7\text{Be} + ^1\text{H} = 2\ {}^4\text{He} \quad (8)$$