General fractal conservation laws arising from a model of detonations in gases

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Abstract

We consider a model of cellular detonations in gases. It consists in conservation laws with a non-local pseudo-differential operator whose symbol is asymptotically $|\xi|^\lambda$, where $0 < \lambda \leq 2$; it can be decomposed as the $\lambda/2$ fractional power of the Laplacian plus a convolution term. After defining the notion of entropy solution, we prove the well-posedness in the $L^\infty$ framework. In the case where $1 < \lambda \leq 2$ we also prove a regularising effect. In the appendix, we show that the assumptions made to perform the mathematical study are satisfied by the considered physical model of detonations (for which $\lambda = 1$).

Key Words: conservation law, Fourier integral operator, entropy solution, splitting method, Lévy operator. (\textsuperscript{3})

1 Introduction

This paper is concerned with the fractal conservation law

$$\partial_t u(t, x) + \text{div}(f(u))(t, x) + G[u(t, \cdot)](x) = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^N,$$  \hspace{1cm} (1.1)

supplemented with $L^\infty$ initial data

$$u(0, x) = u_0(x) \quad \text{in } \mathbb{R}^N.$$

(1.2)

Here $f : \mathbb{R} \to \mathbb{R}^N$ is locally Lipschitz-continuous and $G$ denotes the non-local operator defined through the Fourier transform by

$$\mathcal{F}(G[u(t, \cdot)])(\xi) = |\xi|^\lambda H(\xi) \mathcal{F}(u(t, \cdot))(\xi),$$

(1.3)

with $0 < \lambda \leq 2$ and $H : \mathbb{R}^N \to \mathbb{R}$.

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In the case where $H \equiv 1$ the non-local operator $G$ reduces to a positive multiple $g_\lambda$ of the fractional power $(-\Delta)^{\lambda/2}$ of order $\lambda/2$ of the Laplacian (Lévy operator), and (1.1) is well understood. More precisely, for $\lambda = 2$ it corresponds to the classical viscous conservation law (we have $G \propto -\Delta$), which is well-posed and gives rise to a unique smooth solution. The case $\lambda < 2$ has first been studied in [5], in which local-in-time well-posedness was proved (in $H^s$ Sobolev spaces, in particular) with some restrictions on $f$ or $\lambda$. For $1 < \lambda < 2$, the global well-posedness in the $L^\infty$ framework and the regularising effect of this fractal conservation law were then proved in [14]. If $0 < \lambda \leq 1$ the global well-posedness in the $L^\infty$ framework is obtained in [1] thanks to an entropy formulation. Last, if $0 < \lambda < 1$ the non regularising effect is studied in [3]: discontinuities in the initial data may persist and — even for smooth initial data — shocks may develop. Other behaviours of this equation are also known, such as asymptotic properties (see [6, 7], [4]).

Nevertheless, the physical context indicates that the case of a non-constant frequency function $H$ is quite relevant. Indeed in the context of pattern formation in detonation waves [10], [11], equation (1.1) arises with a pseudo-differential operator defined not by the symbol $|\xi|^\lambda$ but by a symbol $|\xi|^\lambda H(\xi)$ with $H(\xi) \to 1$ as $|\xi| \to \infty$ (see the physical context below for more details). This is the case we intend to consider in this paper; more precisely we assume that $H$ satisfies the following property.

**Assumption 1.** $\Pi := F^{-1}(\cdot |^\lambda (H(\cdot) - 1)) \in L^1(\mathbb{R}^N)$.

**Remark 1.1 (Generalisations).** Let us precise that a few relaxations of Assumption 1 can be handled by our analysis: $\Pi$ may “contain” Dirac masses (so that an additional linear reaction term in the equation can be treated) and may depend on the time variable. We refer to Section 7 for such generalisations.

Note that “$F^{-1}(\cdot |^\lambda (H(\cdot) - 1)) \in L^1(\mathbb{R}^N)$” is implied by “$|^\lambda (H(\cdot) - 1) \in H^s(\mathbb{R}^N)$ for some $s > N/2$” or “$|^\lambda (H(\cdot) - 1) \in W^{N+1,1}(\mathbb{R}^N)$” (see also Appendix A for less straightforward situations where a generalisation of Assumption 1 can hold).

Under the above assumption, equation (1.1) can be recast as

$$\partial_t u + \text{div}(f(u)) + g_\lambda[u] + \Pi * u = 0 \quad \text{on } (0, \infty) \times \mathbb{R}^N. \quad (1.4)$$

Our aim is to prove, for $0 < \lambda \leq 2$, the well-posedness of (1.4) in the $L^\infty$ framework and, in the case $\lambda > 1$, a regularising effect.

**The physical context**

In the framework of overdriven detonations in gases in 2D, under proper physical assumptions and simplifications (see [10], [11]), the shock wave can be represented by an equation $\zeta = \beta(\tau, \eta)$; here, $\tau$ is the (renormalised)
time, $\zeta$ and $\eta$ are the longitudinal and transverse coordinates to the shock (more precisely, transformations of these coordinates taking into account the density of the gases), and $\beta$ evolves following, at the zeroth-order (with respect to a small physical parameter), a linear wave equation.

Performing a formal expansion of $\beta$ with respect to this small physical parameter, it can be shown that its first-order term $\beta_1$ satisfies, up to a normalisation of constants, the equation

$$\frac{\partial \beta_1}{\partial \tau} + \frac{1}{2} \left( \frac{\partial \beta_1}{\partial \eta} \right)^2 + G[\beta_1] = 0.$$  \hspace{1cm} (1.5)

In this circumstance, one information of interest is the creation and evolution of cusps, abrupt changes in $u := \frac{\partial \beta_1}{\partial \eta}$. From (1.5) one sees that $u$ precisely follows (1.1) (with $t = \tau$, $N = 1$, $f(u) = \frac{1}{2} u^2$ and $x = \eta$). The operator $G$ involved here is described, after re-normalisation, by (1.3) with $\lambda = 1$ and $H(\xi) = \sqrt{1 + W(i|\xi|)}$, where $W$, defined on the imaginary axis, is regular and satisfies $W(is) \sim b/s$ as $s \to \infty$ (with $b$ constant).

Thanks to this property, we prove in the appendix that $H$ satisfies the following assumption (with $\lambda = 1$).

**Assumption 2.** There exists $c \in \mathbb{R}$ such that $\Pi := \mathcal{F}^{-1}(\cdot \cdot |^\lambda (H(\cdot) - 1)) \in c\delta_0 + L^1(\mathbb{R}^N)$, with $\delta_0$ the Dirac mass at 0.

This assumption is a generalisation of Assumption 1 (which corresponds to the case $c = 0$), and consists in adding a linear reaction term $cu$ to (1.4). In order to simplify the presentation we shall make the whole study under Assumption 1 and explain in Section 7 how to handle the more general Assumption 2. Hence our analysis covers the considered physical model.

## 2 Main results

Let us first recall that, for $0 < \lambda < 2$, the fractional Laplacian $g_\lambda$ has the following integral representation (see e.g. [15]), valid for all $r > 0$ and all $\varphi \in C_c^\infty(\mathbb{R}^N)$:

$$g_\lambda[\varphi](x) = -c_N(\lambda) \int_{|z| \geq r} \frac{\varphi(x + z) - \varphi(x)}{|z|^{N+\lambda}} \, dz$$

$$-c_N(\lambda) \int_{|z| \leq r} \frac{\varphi(x + z) - \varphi(x) - \nabla \varphi(x) \cdot z}{|z|^{N+\lambda}} \, dz, \hspace{1cm} (2.1)$$

where $c_N(\lambda)$ is a (known) positive constant. From this representation, [1] defines a notion of entropy solution to $\partial_t u + \text{div}(f(u)) + g_\lambda[u] = 0$ with initial data $u_0 \in L^\infty(\mathbb{R}^N)$: for all $r > 0$, all entropy pair $(\eta, \Phi)$ and all non-negative
ϕ ∈ C^∞_c([0, ∞) × R^N),
\[
\int_0^\infty \int_{\mathbb{R}^N} (\eta(t) \partial_t \varphi + \Phi(u) \cdot \nabla \varphi) dt + \int_{\mathbb{R}^N} \eta(u_0) \varphi(0, \cdot) ≥ 0,
\]
(2.2)
where, here and in the following,
\[
G_{\lambda,r}[u, \eta, \varphi](t) := c_N(\lambda) \int_{\mathbb{R}^N} \int_{|z|≥ r} \eta(u(t,x)) \frac{u(t,x+z) - u(t,x)}{|z|^{N+\lambda}} \varphi(t,x) dz dx
\]
+ c_N(\lambda) \int_{\mathbb{R}^N} \int_{|z|≤ r} \eta(u(t,x)) \frac{\varphi(t,x+z) - \varphi(t,x) - \nabla \varphi(t,x) \cdot z}{|z|^{N+\lambda}} dz dx.

This notion of entropy solution ensures the well-posedness in the L^∞ framework of the equation \(\partial_t u + \text{div}(f(u)) + g_\lambda[u] = 0\).

If \(\lambda = 2\), \(g_2[u] = -c_N(2)\Delta u\) and the definition of \(G_{\lambda,r}\) must naturally be changed into
\[
G_{2,r}[u, \eta, \varphi](t) := c_N(2) \int_{\mathbb{R}^N} \eta(u) \Delta \varphi.
\]

Our definition of entropy solution to ((1.4),(1.2)) is a straightforward extension of this definition from [1].

**Definition 2.1** (Entropy solution). An entropy solution to (1.4) with initial condition \(u_0 \in L^\infty(\mathbb{R}^N)\) is a function \(u\) belonging to \(L^\infty((0,T) \times \mathbb{R}^N)\) for all \(T > 0\) and such that, for all \(r > 0\), all non-negative \(\varphi \in C^\infty_c([0, \infty) \times \mathbb{R}^N)\), all convex function \(\eta \in C^1(\mathbb{R})\) and all function \(\Phi : \mathbb{R} \to \mathbb{R}^N\) such that \(\nabla \Phi = \eta' \nabla f\), we have
\[
\int_0^\infty \int_{\mathbb{R}^N} (\eta(t) \partial_t \varphi + \Phi(u) \cdot \nabla \varphi) dt + \int_0^\infty G_{\lambda,r}[u, \eta, \varphi](t) dt
\]
- \(\int_0^\infty \int_{\mathbb{R}^N} \eta'(u) \varphi (\Pi * u) + \int_{\mathbb{R}^N} \eta(u_0) \varphi(0, \cdot) ≥ 0\).

**Remark 2.2.** Note that, as in the case of pure conservation laws, one can replace the smooth pairs \((\eta, \Phi)\) in this definition by Kruzhkov’s entropy pairs [16] without changing the notion of entropy solution. For a given Kruzhkov entropy \(\eta(s) = |s - \kappa|\), the value of \(\eta'\) at \(s = \kappa\) to be considered in (2.3) can be any element of the sub-differential \([-1,1]\) of \(\eta\) at \(s = \kappa\).

Thanks to this definition, we will prove the well-posedness of the considered equation.

**Theorem 2.3** (Well-posedness). Let \(0 < \lambda ≤ 2\) and \(u_0 \in L^\infty(\mathbb{R}^N)\). Let Assumption 1 be satisfied. Then there exists a unique entropy solution \(u\) to ((1.4),(1.2)). Moreover, \(u\) is continuous \([0, \infty) \to L^1_{\text{loc}}(\mathbb{R}^N)\).
Remark 2.4. Note that our analysis also covers the elementary situation \( \lambda = 0 \), in which case \( g_0[u] = u \) and \( G_{0,r}[u, \eta, \varphi] = -\int_{\mathbb{R}^N} \eta'(u) u \varphi \).

Remark 2.5. The use of an entropy formulation is mandatory. Indeed, it has been proved in [2] that, even for the simplest case where \( \Pi = 0 \), the notion of weak solution is not strong enough to provide uniqueness if \( \lambda < 1 \).

We will also obtain, for \( \lambda > 1 \), a regularising effect.

**Theorem 2.6 (Regularising effect).** Let \( 1 < \lambda \leq 2 \) and \( u_0 \in L^\infty(\mathbb{R}^N) \). Let Assumption 1 be satisfied. Then the entropy solution \( u \) to ((1.4), (1.2)) is smooth for \( t > 0 \); more precisely, for all \( 0 < a < T \), \( u \in C_0^\infty((a,T) \times \mathbb{R}^N) \).

**Remark 2.7.** As mentioned in the introduction, it is known that for \( \lambda < 1 \) the regularising effect does not occur. In fact, in this case, shocks can occur [9] even with smooth initial data [3], although these shocks can sometimes disappear if \( \Pi = 0 \) (i.e. \( G = g_\lambda \)), the initial data belongs to \( L^2 \) and the exponent \( \lambda \) is not too far from 1.

For \( \lambda = 1 \) and \( f(u) = u^2 \), it is proved in [8] that if \( \Pi = 0 \) and if the initial data belongs to \( L^2 \) then the regularising effect occurs. However, the situation with a merely bounded initial data or with \( \Pi \neq 0 \) is not clear, the techniques in [8] being strongly based on a scaling that is only true for the pure fractal Burgers equation. In particular, for the physical context described in the introduction (which corresponds to \( \lambda = 1 \) and \( \Pi \neq 0 \)), the regularity or loss of regularity is still an open question.

The organisation of the paper is as follows. In Section 3 we introduce notations and useful preliminary results. By using a splitting method we construct an entropy solution in Section 4. Uniqueness of the solution is proved via a “finite speed propagation property” in Section 5. In Section 6, by taking advantage of a Duhamel’s formula for \( 1 < \lambda \leq 2 \) we prove Theorem 2.6. A few generalisations are discussed in Section 7. Last, the consistency with the physical context is proved in Appendix A.

### 3 Notations and preliminary remarks

Before proving our results, we introduce some notations. Let

\[
K(t) := \mathcal{F}^{-1}(e^{-t|\cdot|^\lambda}).
\]

The (unique bounded) solution to \( \partial_t u + g_\lambda[u] = 0 \) with initial condition \( u_0 \in L^\infty(\mathbb{R}^N) \) is given by \( u(t) = K(t) * u_0 \).

For any integrable function \( \alpha \), we define

\[
S_{-\alpha}(t) := \delta_0 + \sum_{n \geq 1} \frac{t^n}{n!} (-\alpha)^n.
\]
The functions \( \delta_0 \) is the Dirac mass at 0 and \((-\alpha)^{(n)} := (-\alpha) * \cdots * (-\alpha)\) is the convolution of \(-\alpha\) with itself \(n - 1\) times. The (unique) bounded solution to \( \partial_t u + \alpha * u = 0 \) with initial condition \( u_0 \in L^\infty (\mathbb{R}^N) \) is given by \( u(t) = S_{-\alpha}(t) * u_0 \) \(^4\).

In several proofs to come, we denote

\[
K[2](t) := K(2t) \quad \text{and} \quad S_{-\alpha}[2](t) := S_{-\alpha}(2t),
\]

namely the semi-groups associated with \( \partial_t u + 2g(u) = 0 \) and \( \partial_t u + 2\alpha * u = 0 \).

Let us state the main properties of \( K \) and \( S_{-\alpha} \).

**Proposition 3.1** (Properties of the kernels). For all \( 0 < \lambda \leq 2 \) and all \( \alpha \in L^1(\mathbb{R}^N) \), the kernels \( K \) and \( S_{-\alpha} \) satisfy the following properties.

(i) \( K \) is positive and, for all \( t > 0 \), \( K(t) \in L^1(\mathbb{R}^N) \), \( \|K(t)\|_{L^1(\mathbb{R}^N)} = 1 \) and, for all \( x \in \mathbb{R}^N \), \( K(t, x) = t^{N/\lambda} K(1, t^{-1/\lambda} x) \).

(ii) \( K \in C^\infty((a, \infty) \times \mathbb{R}^N) \) for all \( a > 0 \), and there exists \( C > 0 \) such that, for all \( t > 0 \), \( \|\nabla K(t)\|_{L^1(\mathbb{R}^N)} \leq C t^{-1/\lambda} \).

(iii) For all \( t, s > 0 \), \( K(t) * K(s) = K(t + s) \) and \( (\nabla K(t)) * K(s) = \nabla K(t + s) \).

(iv) The functions \( t \in (0, \infty) \mapsto K(t) \in L^1(\mathbb{R}^N) \) and \( t \in (0, \infty) \mapsto \nabla K(t) \in L^1(\mathbb{R}^N)^N \) are continuous.

(v) For all \( t, s > 0 \), \( S_{-\alpha}(t) * S_{-\alpha}(s) = S_{-\alpha}(t + s) \).

(vi) The function \( t \in [0, \infty) \mapsto S_{-\alpha}(t) - \delta_0 \in L^1(\mathbb{R}^N) \) is continuous.

(vii) For all \( t > 0 \), the functions \( K(t) * S_{-\alpha}(t) \) and \( \nabla K(t) * S_{-\alpha}(t) \) belong to \( C^\infty_b(\mathbb{R}^N) \).

(viii) The functions \( (t, s) \in (0, \infty)^2 \mapsto K(t) * S_{-\alpha}(s) \in L^1(\mathbb{R}^N) \) and \( (t, s) \in (0, \infty)^2 \mapsto \nabla K(t) * S_{-\alpha}(s) \in L^1(\mathbb{R}^N)^N \) are continuous. Moreover, there exists \( C > 0 \) such that, for all \( t, s > 0 \), \( \|K(t) * S_{-\alpha}(s)\|_{L^1(\mathbb{R}^N)} \leq C e^{||\alpha||_1 s t^{-1/\lambda}} \) and \( \||\nabla K(t) * S_{-\alpha}(s)\|_{L^1(\mathbb{R}^N)} \leq C e^{||\alpha||_1 s t^{-1/\lambda}} \).

**Proof.** The properties on \( K \) are quite classical and, aside from its positivity, can be deduced straightforwardly from its definition (see also [14], [15]); the positivity of \( K \) can be found in [17], [14].

Property (v) is the expression of the fact that \( S_{-\alpha} \) is a semi-group (in fact, a group...), and property (vi) is a consequence of the normal convergence, in \( C([0, T]; L^1(\mathbb{R}^N)) \), of the series \( S_{-\alpha}(t) - \delta_0 = \sum_{n \geq 1} \frac{t^n}{n!} (-\alpha)^{(n)} \).

Finally, properties (vii) and (viii) come from the writing \( X * S_{-\alpha}(s) = \)

\(^4\)Obviously, though the convolution of a Dirac mass by an \( L^\infty \) function is not pointwise well defined, we let \( \delta_0 * u_0 = u_0 \).
Lemma 3.2. Let \( n \) and, for all \( \lambda > 0 \) with \( \lambda > N \) and \( \int \), from items (ii), (iv), (vi) and from the estimate \( \|S_\alpha(s) - \delta_0\|_{L^1(\mathbb{R}^N)} \leq \sum_{n \geq 1} \frac{\alpha^n}{n!} \leq e^{\|\alpha\|_1^s} \).

We will also need the following estimate on \( g_\lambda \).

**Lemma 3.2.** Let \( \lambda \in (0, 2] \). There exists \( C_\lambda > 0 \) such that, for all \( \varphi \in S(\mathbb{R}^N) \),

\[
\|g_\lambda[\varphi]\|_{L^1(\mathbb{R}^N)} \leq C_\lambda \|\varphi\|_{W^{2,1}(\mathbb{R}^N)}.
\]

In particular, \( g_\lambda \) can be extended into a linear continuous operator from \( W^{2,1}(\mathbb{R}^N) \) into \( L^1(\mathbb{R}^N) \).

**Proof.** The property for \( \lambda = 2 \) is obvious (since, up to a multiplicative constant, \( g_\lambda \) is the Laplace operator). We thus consider that \( \lambda < 2 \) and we use the integral representation (2.1) of \( g_\lambda \) with \( r = 1 \) and a Taylor expansion to write \( |g_\lambda[\varphi](x)| \leq T_1[\varphi](x) + T_2[\varphi](x) \) with

\[
T_1[\varphi](x) = c_N(\lambda) \int_{|z| \geq 1} \frac{|\varphi(x+z) + |\varphi(x)\|}{|z|^{N+\lambda}} \, dz,
\]

and

\[
T_2[\varphi](x) = c_N(\lambda) \int_{|z| \leq 1} \frac{\int_0^1 \frac{1}{2} |D^2\varphi(x+sz)| \, |z|^2 \, ds}{|z|^{N+\lambda}} \, dz,
\]

where \( |D^2\varphi| \) is the Euclidean matrix norm of \( D^2\varphi \). Then, using Fubini-Tonelli’s theorem and linear changes of variable, we find

\[
\int_{\mathbb{R}^N} T_1[\varphi](x) \, dx = \frac{c_N(\lambda)}{N+\lambda} \int_{|z| \geq 1} \frac{\int_{\mathbb{R}^N} |\varphi(x+z)| \, dx}{|z|^{N+\lambda}} \, dz
\]

\[
= 2c_N(\lambda) \|\varphi\|_{L^1(\mathbb{R}^N)} \int_{|z| \geq 1} \frac{dz}{|z|^{N+\lambda}},
\]

with \( N + \lambda > N \), and

\[
\int_{\mathbb{R}^N} T_2[\varphi](x) \, dx = \frac{c_N(\lambda)}{2} \int_{|z| \leq 1} \frac{\left(\int_{\mathbb{R}^N} |D^2\varphi(x+sz)| \, dx\right) \, ds}{|z|^{N+\lambda-2}} \, dz
\]

\[
= \frac{c_N(\lambda)}{2} \|D^2\varphi\|_{L^1(\mathbb{R}^N)} \int_{|z| \leq 1} \frac{dz}{|z|^{N+\lambda-2}},
\]

with \( N + \lambda - 2 < N \). The proof is complete. \( \blacksquare \)

4 Existence of an entropy solution

By using the splitting method developed in [14] and later in [1] we construct an entropy solution to \((1.4),(1.2))\).

For \( \delta > 0 \) we define \( u^\delta : [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R} \) as follows. Let \( u^\delta(0, \cdot) := u_0 \) and, for all \( n \geq 0 \), define by induction
• \( u^\delta \) on \((2n\delta, (2n+1)\delta) \times \mathbb{R}^N \) as the (entropy) solution to
\[ \partial_t u + 2 \text{div}(f(u)) + 2 g_\lambda[u] = 0, \tag{4.1} \]
supplemented with the initial data \( u^\delta(2n\delta, \cdot) \).

• \( u^\delta \) on \(((2n+1)\delta, (2n+2)\delta) \times \mathbb{R}^N \) as the (unique bounded) solution to
\[ \partial_t u + 2 \Pi * u = 0, \tag{4.2} \]
supplemented with the initial data \( u^\delta((2n+1)\delta, \cdot) \).

Note that equation (4.1) does not increase the \( L^\infty \) norm and that its solutions are continuous with values in \( L^1_{\text{loc}}(\mathbb{R}^N) \) (see [1] for instance). On the other hand, the representation \( u(t) = S_{-2t}(t-s) * u(s) \) of the solutions to (4.2) show that they satisfy \( \| u(t) \|_{\infty} \leq e^{2\|f\|_1(t-s)} \| u(s) \|_{\infty} \) for \( t \geq s \), and also that they are continuous with values in \( L^1_{\text{loc}}(\mathbb{R}^N) \). In particular, at each step the functions \( u^\delta(2n\delta, \cdot) \) and \( u^\delta((2n+1)\delta, \cdot) \) are bounded and thus suitable initial data for the considered equations.

Therefore we are equipped with \( u^\delta \in C([0, \infty); L^1_{\text{loc}}(\mathbb{R}^N)) \) such that
\[ \| u^\delta(t) \|_{\infty} \leq e^{\|f\|_1 t} \| u_0 \|_{\infty}. \tag{4.3} \]

By Arzéla-Ascoli’s theorem, we first prove the relative compactness of \( \{ u^\delta : 0 < \delta < T \} \) in \( C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)) \). Then by extraction of a subsequence as \( \delta \to 0 \) we construct an entropy solution to ((1.4),(1.2)).

### 4.1 Relative compactness in \( C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)) \)

**Step 1.** We fix \( T \geq 0 \) and prove that \( \{ u^\delta(t) : 0 < \delta < T, \ t \in [0,T] \} \) is relatively compact in \( L^1_{\text{loc}}(\mathbb{R}^N) \).

For a given \( u \) we define \( T_h u \) the associated translated function of \( u \) by \( T_h u(t, x) := u(t, x+h) \). Note that \( T_h u^\delta \) solves (4.1) and (4.2) on the intervals where \( u^\delta \) solves these equations.

We recall that the kernel associated to equation \( \partial_t u + 2 g_\lambda[u] = 0 \) is nothing else but \( K(2t) := K^{[2]}(t) \), and quote [1, Theorem 3.2] — which can be seen as a finite speed propagation property for equation (4.1):

**Lemma 4.1.** Let \( u \) and \( v \) be the entropy solutions to (4.1) with initial conditions \( u_0 \) and \( v_0 \) in \( L^\infty \). Then, for all \( x_0 \in \mathbb{R}^N \), all \( t > 0 \), all \( R > 0 \),
\[ \int_{B(x_0,R)} | u - v|(t) \leq \int_{B(x_0,R+2Lt)} K^{[2]}(t) * | u_0 - v_0|, \]
where \( L \) is a Lipschitz constant of \( f \) on \( \{ s \in \mathbb{R} : |s| \leq \max(\|u_0\|_\infty, \|v_0\|_\infty) \} \) and \( B(x_0, R) \) is the ball in \( \mathbb{R}^N \) of center \( x_0 \) and radius \( R \).
In view of (4.3), by selecting $L$ as a Lipschitz constant of $f$ on the interval $[-e^{\|\Pi\|T}\|u_0\|_{\infty}, e^{\|\Pi\|T}\|u_0\|_{\infty}]$, we can apply the above lemma, with $(u, v) = (u^\delta, T_h u^\delta)$, on all intervals of $[0, T]$ where $u^\delta$ (and so $T_h u^\delta$) solves (4.1).

Let $t \in [0, T]$. Assume that $2n\delta < t \leq (2n + 1)\delta$, for some $n \geq 0$. Then it follows from Lemma 4.1 that, denoting $B(R) = B(0, R)$,

$$\int_{B(R)} |u^\delta - T_h u^\delta|(t) \leq \int_{B(R+2L(t-2n\delta))} K^{[2]}(t-2n\delta) * |u^\delta - T_h u^\delta|(2n\delta) \leq \int_{B(R+2L\delta)} K^{[2]}(t-2n\delta) * |u^\delta - T_h u^\delta|(2n\delta),$$

thanks to the positivity of the kernel $K$. Now, if $n \neq 0$ we go further in the past. Since

$$\partial_t(u^\delta - T_h u^\delta) + 2 (\Pi - T_h \Pi) * u^\delta = 0 \text{ on } ((2n - 1)\delta, 2n\delta],$$

we have, on the above time interval,

$$\|\partial_t(u^\delta - T_h u^\delta)(t)\|_{\infty} \leq 2\|\Pi - T_h \Pi\|_1\|u^\delta(t)\|_{\infty} \leq 2\|\Pi - T_h \Pi\|_1 e^{\|\Pi\|T}\|u_0\|_{\infty} =: \omega_T(h),$$

with $\omega_T(h)$ not depending on $\delta$ and $\omega_T(h) \to 0$ as $h \to 0$. It follows that, for all $x \in \mathbb{R}^N$,

$$|u^\delta - T_h u^\delta|(2n\delta, x) \leq \omega_T(h)\delta + |u^\delta - T_h u^\delta|((2n - 1)\delta, x).$$

(4.5)

By plugging this into (4.4), using $\|K(t)\|_1 = 1$ and $B(R + 2L\delta) \subset B(R + 2LT)$, we find that

$$\int_{B(R)} |u^\delta - T_h u^\delta|(t) \leq \int_{B(R+2L\delta)} K^{[2]}(t-2n\delta) * |u^\delta - T_h u^\delta|((2n-1)\delta) + \omega_T(h)\delta |B(R + 2LT)|.$$

(4.6)

In order to estimate the first term in the right hand side member we notice that $u^\delta$ and $T_h u^\delta$ solve (4.1) on $((2n - 2)\delta, (2n - 1)\delta]$ and thus, applying
Lemma 4.1, we find:

\[
\int_{B(R+2L\delta)} K^{[2]}(t-2n\delta) * |u^\delta - \mathcal{T}_h u^\delta|(2n-1)\delta \\
= \int_{\mathbb{R}^N} K^{[2]}(t-2n\delta, y) \int_{B(R+2L\delta)} |u^\delta - \mathcal{T}_h u^\delta|(2n-1)\delta, x-y) \, dx \, dy \\
\leq \int_{\mathbb{R}^N} K^{[2]}(t-2n\delta, y) \\
\int_{B(R+4L\delta)} [K^{[2]}(\delta, \cdot) * |u^\delta - \mathcal{T}_h u^\delta|(2n-2)\delta, \cdot)](x-y) \, dx \, dy \\
\leq \int_{B(R+4L\delta)} K^{[2]}(t-(2n-1)\delta) * |u^\delta - \mathcal{T}_h u^\delta|(2n-2)\delta, \\

\]

We plug this into (4.6) to get

\[
\int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) \\
\leq \int_{B(R+4L\delta)} K^{[2]}(t-(2n-1)\delta) * |u^\delta - \mathcal{T}_h u^\delta|(2n-2)\delta \\
+ \omega_T(h)|B(R+2LT)|. \quad (4.7)
\]

By repeating \(n-1\) more times the procedure from (4.5) to (4.7), we discover that

\[
\int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) \\
\leq \int_{B(R+2L(n+1)\delta)} K^{[2]}(t-n\delta) * |u_0 - \mathcal{T}_h u_0| + \omega_T(h)n\delta|B(R+2LT)| \\
\leq \sup_{0 \leq s \leq T} \int_{B(R+2LT)} K^{[2]}(s) * |u_0 - \mathcal{T}_h u_0| + \omega_T(h)s|B(R+2LT)|, \quad (4.8)
\]

the last line following from \(0 \leq t-n\delta \leq (n+1)\delta \leq 2n\delta \leq t \leq T\).

Assume that \((2n+1)\delta < t \leq (2n+2)\delta\), for some \(n \geq 0\). By using similar arguments, we claim that we obtain (4.8) again.

Applying [1, Lemma A.2] with \(\varepsilon = 1\), we deduce from (4.8) that

\[
\sup_{0 \leq \delta < T} \sup_{0 \leq t \leq T} \int_{B(R)} |u^\delta - \mathcal{T}_h u^\delta|(t) \leq \|u_0 - \mathcal{T}_h u_0\|_{L^1(B(R+2LT+r))} \\
+ 2\|u_0\|_{\infty} |B(R+2LT)| \int_{\mathbb{R}^N \setminus B(r/T^{1/\lambda})} K^{[2]}(1) + \omega_T(h)T|B(R+2LT)|,
\]
holds for all $r > 0$. We conclude by a “3ε argument”: if $\varepsilon > 0$ is given
we fix $r > 1$ large enough so that $0 \leq \int_{R^N \backslash B(r/T)} K^{|2|}(1) \leq \varepsilon$; since
$u_0 \in L^\infty(R^N) \subset L^1(B(R+2LT+r))$ we have $\|u_0 - T_h u_0\|_{L^1(B(R+2LT+r))} \leq \varepsilon$
for $h$ small enough; recall also that $\omega_T(h) \leq \varepsilon$ for $h$ small enough. Therefore

$$\lim_{h \to 0} \sup_{0 < \delta < T} \sup_{0 \leq t \leq T} \int_{B(R)} |u^\delta - T_h u^\delta(t)| = 0,$$

which concludes the first step, by the Riesz-Fréchet-Kolmogorov’s theorem.

**Step 2.** Still fixing $T > 0$, we prove that, for all $Q$ compact subset of $R^N$,
\{u^\delta : 0 < \delta < T\} is equicontinuous $[0, T] \to L^1(Q)$.

From (4.3), we see that \{u^\delta(t) : 0 < \delta < T, t \in [0, T]\} is bounded in
$L^\infty(R^N)$. Since \{u^\delta : 0 < \delta < T\} is bounded in $L^\infty((0, T) \times R^N)$, in view of
Lemma 3.2 we see (5) that $\{\Pi \ast u^\delta : 0 < \delta < T\}$ and $\{\text{div}(f(u^\delta)) + g_\lambda[u^\delta] : 0 < \delta < T\}$ are bounded in $L^\infty(0, T; W^{-2, \infty}(R^N))$, where we recall that
$W^{-2, \infty}$ denotes the dual space of $W^{2,1}$.

Hence, equations (4.1) and (4.2), which are satisfied in the distributional
sense, show that \{u^\delta : 0 < \delta < T\} is uniformly Lipschitz-
continuous $[0, T] \to W^{-2, \infty}(R^N)$, and thus also $[0, T] \to (C^2_c(Q))'$
where $(C^2_c(Q))'$ is the dual space of $C^2_c(Q)$ endowed with the norm
$|||\varphi|||^2_{C^2_c(Q)} = \sup_{|\alpha| \leq 2} ||\partial^\alpha \varphi||$$\infty$.

We then need the following Lemma which can be considered as a metric-
space variant of the classical Lions “three-spaces” lemma.

**Lemma 4.2.** Let $(E, d_E)$ and $(F, d_F)$ be metric vector spaces such that $E$
is continuously embedded in $F$; let $K$ be a compact subset of $E$. Then, for
all $\varepsilon > 0$, there exists $C_{K, \varepsilon} > 0$ such that, for all $(x, y) \in K^2$,
$d_E(x, y) \leq \varepsilon + C_{K, \varepsilon} d_F(x, y)$.

**Proof.** The proof can be made by way of contradiction. Given $\varepsilon > 0$, if for
all integer $n$ we can find $(x_n, y_n) \in K^2$ such that $d_E(x_n, y_n) > \varepsilon + nd_F(x_n, y_n)$,
then — up to a subsequence — we can assume that $(x_n, y_n) \to (x, y)$ in $E$, and
thus in $F$. Letting $n \to \infty$ in $d_F(x_n, y_n) < \frac{1}{n}d_E(x_n, y_n)$ we deduce that
$d_F(x, y) = 0$ so that $x = y$. Letting then $n \to \infty$ in $\varepsilon < d_E(x_n, y_n)$ we see
that $\varepsilon \leq 0$, which is a contradiction. This concludes the proof.

Let us now conclude the proof that $\{u^\delta : 0 < \delta < T\}$ is equicontinuous
$[0, T] \to L^1(Q)$. Let $M$ be a uniform (independent on $\delta$) Lipschitz constant
of $u^\delta : [0, T] \to (C^2_c(Q))'$. If we denote by $K$ the closure of $\{u^\delta(t) : 0 < \delta < T, t \in [0, T]\}$ in $L^1(Q)$, we have from Step 1 that $K$ is compact in $L^1(Q)$.
Let $\varepsilon > 0$ and select $C_{K, \varepsilon} > 0$ as in Lemma 4.2 applied to $E = L^1(Q)$ and

$^5$It suffices to notice that, for all $\varphi \in C^\infty_c(R^N)$, we have $||\Pi \ast u^\delta(t), \varphi|| \leq ||\Pi||_{L^\infty} ||u^\delta(t)||_\infty ||\varphi||$ and $||\text{div}(f(u^\delta(t))), \varphi|| |f(u^\delta(t))| \leq ||f(u^\delta(t))||_\infty ||\nabla \varphi||$ and $||g_\lambda[u^\delta(t)], \varphi|| |g_\lambda| \leq C ||u^\delta(t)||_\infty ||\varphi||_{W^{2,1}}$.
\(F = (C^2_\varepsilon(Q))^\prime\). Then, if \((t, s) \in [0, T]^2\) are such that \(|t - s| \leq \varepsilon/(MC_{K, \varepsilon})\), we have, for all \(\delta > 0\),
\[
d_{L^1(Q)}(u^\delta(t), u^\delta(s)) \leq \varepsilon + C_{K, \varepsilon} d(C^2_\varepsilon(Q))\prime(u^\delta(t), u^\delta(s)) \leq \varepsilon + C_{K, \varepsilon} M|t - s| \leq 2\varepsilon,
\]
and the equicontinuity of \(\{u^\delta : 0 < \delta < T\}\) on \([0, T]\) with values in \(L^1(Q)\) is proved.

**Conclusion.** Gathering Steps 1 and 2, we conclude that \(\{u^\delta \rightarrow u\} \text{ is relatively compact in } C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))\) for all \(T > 0\).

### 4.2 Convergence to an entropy solution

Up to a subsequence, we can assume that, as \(\delta \to 0\), \(u^\delta\) converges to some \(u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))\) for all \(T > 0\). Obviously, \(u\) also satisfies (4.3) and thus belongs to \(L^\infty((0, T) \times \mathbb{R}^N)\) for all \(T > 0\). We now prove that \(u\) is an entropy solution to (1.4) with initial data \(u_0 \in L^\infty(\mathbb{R}^N)\).

Let \(r > 0\), \(\varphi \in C^\infty_c([0, \infty[ \times \mathbb{R}^N)\) be non-negative, \(\eta \in C^1(\mathbb{R})\) be convex and \(\Phi : \mathbb{R} \to \mathbb{R}^N\) be such that \(\nabla\Phi = \eta\nabla f\).

First, we claim that from (2.2) we can deduce an “entropy formulation with final value” for solutions to (4.1). More precisely, if \(v\) is the entropy solution to (4.1) with initial data \(v_0\) then, for all \(s > 0\),
\[
\int_0^s \int_{\mathbb{R}^N} (\eta(v)\partial_t \varphi + 2\Phi(v) \cdot \nabla \varphi) + 2\int_0^s G_{\lambda, r}[v, \eta, \varphi](t) \, dt \\
+ \int_{\mathbb{R}^N} \eta(v_0) \varphi(0, \cdot) - \int_{\mathbb{R}^N} \eta(v(s, \cdot)) \varphi(s, \cdot) \geq 0. 
\tag{4.9}
\]

Indeed, take \(\gamma_\varepsilon : [0, \infty) \to [0, 1]\) which tends to the characteristic function of \([0, s]\) as \(\varepsilon \to 0\) and such that \(-\gamma_\varepsilon'\) tends to the Dirac mass at \(t = s\), and apply the entropy formulation (2.2) with \(\varphi(t, x)\) replaced by \(\varphi(t, x)\gamma_\varepsilon(t)\); letting \(\varepsilon \to 0\), and since \(v \in C([0, \infty[; L^1_{\text{loc}}(\mathbb{R}^N))\) — see [1] — we deduce that (4.9) holds.

The definition of \(u^\delta\) then ensures that, for all \(n \geq 0\),
\[
\int_{2n\delta}^{(2n+1)\delta} \int_{\mathbb{R}^N} (\eta(u^\delta)\partial_t \varphi + 2\Phi(u^\delta) \cdot \nabla \varphi) + 2\int_{2n\delta}^{(2n+1)\delta} G_{\lambda, r}[u^\delta, \eta, \varphi](t) \, dt \\
+ \int_{\mathbb{R}^N} \eta(u^\delta(2n\delta, \cdot)) \varphi(2n\delta, \cdot) \\
- \int_{\mathbb{R}^N} \eta(u^\delta((2n + 1)\delta, \cdot)) \varphi((2n + 1)\delta, \cdot) \geq 0. 
\tag{4.10}
\]

On the other hand, multiplying (4.2) by \(\eta'(u^\delta)\varphi\) and integrating by parts.
(6), we have, for all \( n \geq 0 \),
\[
\int_{(2n+1)\delta}^{(2n+2)\delta} \int_{\mathbb{R}^N} \eta(u^\delta) \partial_t \varphi - 2\eta'(u^\delta) \varphi \left( \Pi \ast u^\delta \right) + \int_{\mathbb{R}^N} \eta(u^\delta((2n+1)\delta, \cdot)) \varphi((2n+1)\delta, \cdot) - \int_{\mathbb{R}^N} \eta(u^\delta((2n+2)\delta, \cdot)) \varphi((2n+2)\delta, \cdot) = 0. \tag{4.11}
\]

Summing (4.10) and (4.11) on all \( n \geq 0 \) (note that since \( \varphi \) is compactly supported, the sum is actually made of a finite number of terms), all the boundary terms but the first one cancel out each other and we find
\[
\int_0^\infty \int_{\mathbb{R}^N} (\eta(u^\delta) \partial_t \varphi + 2I_\delta \Phi(u^\delta) \cdot \nabla \varphi) + \int_0^\infty 2I_\delta(t) G_{\lambda,r} \big[ u^\delta, \eta, \varphi \big](t) dt - \int_0^\infty 2J_\delta(t) \int_{\mathbb{R}^N} \eta'(u^\delta) \varphi \Pi \ast u^\delta + \int_{\mathbb{R}^N} \eta(u_0) \varphi(0, \cdot) \geq 0, \tag{4.12}
\]
where \( I_\delta \) is the characteristic function of \( \cup_{n \geq 0} (2n\delta, (2n+1)\delta) \) and \( J_\delta \) is the characteristic function of \( \cup_{n \geq 0} ((2n+1)\delta, (2n+2)\delta) \).

It is classical that, as \( \delta \to 0 \), both \( I_\delta \) and \( J_\delta \) tend to the constant function \( 1/2 \) in \( L^\infty(0, \infty) \) weak-\*.
Select \( T > 0 \) large enough so that \( \text{supp} \ \varphi \subset [0,T] \times \mathbb{R}^N \). We claim that the functions \( t \mapsto \int_{\mathbb{R}^N} \Phi(u^\delta) \cdot \nabla \varphi, t \mapsto G_{\lambda,r} \big[ u^\delta, \eta, \varphi \big](t) \) and \( t \mapsto \int_{\mathbb{R}^N} \eta'(u^\delta) \varphi \Pi \ast u^\delta \) tend in \( L^1(0, \infty) \) to the same quantities with \( u^\delta \) replaced by \( u \); indeed, let \( A[u^\delta] \) be any one of these three functions: from \( u^\delta \to u \) in \( C([0,T]; L^1_{\text{loc}}(\mathbb{R}^N)) \), we deduce that \( A[u^\delta](t) \to A[u](t) \) for \( 0 \leq t \leq T \), and from \( \sup_{0 < \delta < T} \sup_{0 \leq t \leq T} | A[u^\delta](t) | \leq \infty \) and \( A[u^\delta] \equiv 0 \) on \( (T, \infty) \), we infer that \( A[u^\delta] \to A[u] \) in \( L^1(0, \infty) \).

We can therefore pass to the limit \( \delta \to 0 \) in (4.12), to conclude that \( u \) satisfies (2.3) and is an entropy solution to (1.4) with initial condition \( u_0 \).

## 5 Uniqueness of the entropy solution

The uniqueness of the entropy solution will be obtained while proving the following “finite speed propagation” property.

**Proposition 5.1** (Finite speed propagation). Let \( u \) and \( v \) be entropy solutions to (1.4) with initial conditions \( u_0 \) and \( v_0 \) in \( L^\infty \) and let \( T > 0 \). Define
\[
m_0(T) := e^{\| \Pi \|_1 T} \max\{ \| u_0 \|_\infty, \| v_0 \|_\infty \}.
\]

\(^6\)This is possible since \( \partial_x u^\delta(\cdot, x) \in C([0,T], \mathbb{R}) \). Indeed from \( u^\delta \in C([0,T]; L^1_{\text{loc}}) \) and \( \sup_{t \in [0,T]} \| u^\delta(t) \|_\infty < \infty \) we deduce that \( u^\delta \in C([0,T]; L^\infty_{\text{weak-*}}) \). Combined with the continuity of \( v \in L^\infty_{\text{weak-*}} \to \Pi \ast v(x) \in \mathbb{R} \) this shows that \( \Pi \ast u^\delta(\cdot, x) \in C([0,T], \mathbb{R}) \).
Then, for all \(x_0 \in \mathbb{R}^N\), all \(0 < t < T\) and all \(R > 0\),
\[
\int_{B(x_0, R)} |u - v|(t) \leq \int_{B(x_0, R+L_t)} K(t) \ast S_{\Pi}(t) \ast |u_0 - v_0|,
\]
where \(L\) is a Lipschitz constant of \(f\) on \([-m_0(T), m_0(T)]\).

**Proof.** The proof mainly follows [1, Section 4].

Define \(\psi(t, s, x, y) := \theta_v(s - t)\rho_\mu(y - x)\phi(t, x)\), where \(\theta_v \in C^\infty_c((0, \nu))\) and \(\rho_\mu \in C^\infty_c(B(0, \mu))\) are two approximate units and \(\phi \in C^\infty([-m(T), m(T)] \times \mathbb{R}^N)\) is non-negative. By using the so-called doubling variables technique, we see that [1, inequality (4.3)] holds true with an additional term, namely
\[
-\int_0^\infty \int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \psi(t, s, x, y) \text{sgn}(u(t, x) - v(s, y)) \times ((\Pi \ast u)(t, x) - (\Pi \ast v)(s, y)) \, dy \, dx \, ds \, dt.
\]

By bounding this term from above, we see that [1, inequality (4.6)] holds true with the additional term
\[
A_{\nu, \mu} := \int_0^\infty \int_0^\infty \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \theta_v(s - t)\rho_\mu(y - x)\phi(t, x) \times |(\Pi \ast u)(t, x) - (\Pi \ast v)(s, y)| \, dy \, dx \, ds \, dt.
\]

Since \(\Pi \ast v\) is locally integrable, it follows from classical properties of approximate units that, as \((\nu, \mu) \to (0, 0)\),
\[
A_{\nu, \mu} \to \int_0^\infty \int_{\mathbb{R}^N} \phi(t, x)|\Pi \ast (u - v)|(t, x) \, dx \, dt,
\]
which is bounded from above by
\[
\int_0^\infty \int_{\mathbb{R}^N} \phi(t, x) |\Pi| \ast |u - v| = \int_0^\infty \int_{\mathbb{R}^N} |u - v| (|\tilde{\Pi}| \ast \phi),
\]
where \(\tilde{\Pi}(x) := \Pi(-x)\). Then, we collect the analogues of [1, (4.11)] with this additional term: for all non-negative \(\phi \in C^\infty_c([0, \infty) \times \mathbb{R}^N)\) such that \(\text{Supp} \phi \subset [0, T] \times B(0, R)\), we have
\[
\int_0^\infty \int_{\mathbb{R}^N} |u - v| \left( \partial_t \phi + L|\nabla \phi| + |\tilde{\Pi}| \ast \phi - g_\lambda[\phi] \right) + \int_{\mathbb{R}^N} |u_0 - v_0|\phi(0, \cdot) \geq 0,
\]
with \(L\) a Lipschitz constant of \(f\) on \([-m(T), m(T)]\), where
\[
m(T) := \max\{||u||_{L^\infty([0,T] \times \mathbb{R}^N)}, ||v||_{L^\infty([0,T] \times \mathbb{R}^N)}\}.
\]
Let us define $\Lambda (t) := K(t) * S_{[\Pi]} (t)$, so that the solution to $\partial_t v - [\Pi] * v + g_\lambda [v] = 0$ with initial condition $v_0$ is given by $\Lambda (t) * v_0$. Now, we fix $x_0 \in \mathbb{R}^N$ and $M > LT$. Let $\gamma \in C^\infty_c(0, \infty)$ be non-negative, non-increasing and equal to 1 on $[0, M]$, and let $\Theta \in C^\infty_c([0, T])$. We define

$$\phi (t, x) := \begin{cases} \Theta (t) \left[ \Lambda(T - t) * \gamma (| \cdot | - x_0) + Lt \right] (x) & \text{if } 0 \leq t < T, \\ 0 & \text{if } t \geq T. \end{cases} \quad (5.3)$$

Note that $(t, x) \in [0, T] \times \mathbb{R}^N \mapsto \gamma(| x - x_0 | + Lt)$ belongs to $C^\infty_c([0, T] \times \mathbb{R}^N)$ (it is equal to 1 on a neighbourhood of $[0, T] \times \{ x_0 \}$, so the non-smoothness of $| \cdot |$ at 0 does not play any role). Therefore, the definition of $\Lambda$ implies that the function $\phi$ belongs to $C^\infty([0, \infty) \times \mathbb{R}^N)$, is non-negative and belongs to $L^1(0, T; W^{2,1}(\mathbb{R}^N))$. Hence, as in [1], we claim that, even if its support is not compact, $\phi$ can be used as a test function in (5.1).

We have $\partial_t (\Lambda(T - t) + [\Pi] * \Lambda(T - t) - g_\lambda [\Lambda(T - t)] = 0$ and $g_\lambda [a * b] = g_\lambda [a] * b$. Therefore we see that, for all $(t, x) \in (0, T) \times \mathbb{R}^N$,

$$\left( \partial_t \phi + [\Pi] * \phi - g_\lambda [\phi] \right) (t, x) = \Theta' (t) \left[ \Lambda(T - t) * \gamma (| \cdot | - x_0) + Lt \right] (x) + L \Theta (t) \left[ \Lambda(T - t) * \gamma' (| \cdot | - x_0) + Lt \right] (x). \quad (5.4)$$

Since $\Lambda \geq 0$ and $\gamma' \leq 0$ we also have

$$| \nabla \phi (t, x) | = \left| \Theta (t) \left[ \Lambda(T - t) * \frac{\cdot - x_0}{| \cdot - x_0 |} \gamma' (| \cdot | - x_0) + Lt \right] \right| (x) \leq - \Theta (t) \left[ \Lambda(T - t) * \gamma' (| \cdot | - x_0) + Lt \right] (x). \quad (5.5)$$

Summing (5.4) and (5.5) we obtain

$$(\partial_t \phi + L | \nabla \phi | + [\Pi] * \phi - g_\lambda [\phi] ) (t, x) \leq \Theta' (t) \left[ \Lambda(T - t) * \gamma (| \cdot | - x_0) + Lt \right] (x),$$

and, injecting this result into (5.1), we see that

$$\int_0^T \Theta' (t) \left( \int_{\mathbb{R}^N} | u - v | (t, \cdot) \left[ \Lambda(T - t) * \gamma (| \cdot | - x_0) + Lt \right] \right) dt \leq \int_{\mathbb{R}^N} \Theta (0) | u_0 - v_0 | \left[ \Lambda (T) * \gamma (| \cdot | - x_0) \right]. \quad (5.6)$$

The above estimate is enough to prove the uniqueness of the entropy solution to ((1.4),(1.2)). Indeed, assume that $u_0 \equiv v_0$. We select a non-increasing $\Theta \in C^\infty_c([0, T])$ such that $\Theta' (t) = -1$ for all $0 \leq t \leq T/2$; then (5.6) yields

$$\int_{\mathbb{R}^N} | u - v | (t, \cdot) \left[ \Lambda(T - t) * \gamma (| \cdot | - x_0) + Lt \right] = 0, \quad (5.7)$$

for all $0 \leq t \leq T/2$. We notice that, for all $s > 0$, $\Lambda(s) = K(s) + K(s) * (S_{[\Pi]} (s) - \delta_0) \geq K(s) > 0$ on $\mathbb{R}^N$. Moreover, for all $t \in [0, T]$, $\gamma (| \cdot | - x_0) + Lt$
is non-negative on $\mathbb{R}^N$ and positive on a ball around $x_0$; we deduce that, for all $t \in (0, T)$, $\Lambda(T - t) \ast [\gamma(| \cdot - x_0| + Lt)] > 0$ on $\mathbb{R}^N$. Hence, equation (5.7) shows that $u = v$ on $[0, T/2] \times \mathbb{R}^N$; this relation being valid for any $T$, this concludes the proof that the entropy solution is unique. As a by-product, we notice that this entropy solution is the one constructed in Section 4, and therefore that it belongs to $C([0, \infty) ; L^1_{loc}(\mathbb{R}^N))$ and satisfies $\|u\|_{L^\infty((0,T) \times \mathbb{R}^N)} \leq e^{\|\Pi\|_1 T} \|u_0\|_{L^\infty(\mathbb{R}^N)}$; hence, $m(T)$ defined in (5.2) is bounded from above by $m_0(T)$ defined in Proposition 5.1.

We now conclude the proof of Proposition 5.1. For $0 < \nu < T$, let $\theta_\nu \in \mathcal{C}_c^\infty((0, \nu))$ be an approximate unit. Hence, $\Theta$ given by

$$\Theta(t) := \int_t^\infty \theta_\nu(T - s) \, ds$$

belongs to $C^\infty_c([0, T))$ and satisfies $\Theta(0) = 1$. From (5.6), we infer

$$\int_0^T \theta_\nu(T - t) \left( \int_{\mathbb{R}^N} |u - v|(t, \cdot) \left[ \Lambda(T - t) \ast \gamma(| \cdot - x_0| + Lt) \right] \right) \, dt \leq \int_{\mathbb{R}^N} |u_0 - v_0| \left[ \Lambda(T) \ast \gamma(| \cdot - x_0|) \right].$$

(5.8)

The function $t \in [0, T] \rightarrow \Lambda(T - t) \ast \gamma(| \cdot - x_0| + Lt) \in L^1(\mathbb{R}^N)$ is continuous (\footnote{This function is continuous and is an approximate unit as $t \to 0$, and the function $(t, x) \in [0, \infty) \times \mathbb{R}^N \mapsto \gamma(| \cdot - x_0| + Lt)$ is continuous with compact support.}); moreover, by the continuity of the entropy solutions $u, v$ with values in $L^1_{loc}(\mathbb{R}^N)$ (proved above) and their $L^\infty$ bound, we see that $t \in [0, \infty) \mapsto |u - v|(t, \cdot)$ is continuous with values in $L^\infty(\mathbb{R}^N)$ weak-$\ast$. We can therefore pass to the limit $\nu \to 0$ in (5.8) to find

$$\int_{\mathbb{R}^N} |u - v|(T, \cdot) \gamma(| \cdot - x_0| + LT)$$

$$\leq \int_{\mathbb{R}^N} |u_0 - v_0| \left[ K(T) * S_{[\Pi]}(T) * \gamma(| \cdot - x_0|) \right]$$

$$= \int_{\mathbb{R}^N} \gamma(| \cdot - x_0|) \left[ K(T) * S_{[\Pi]}(T) * |u_0 - v_0| \right].$$

(5.9)

where we have used the fact that $K(T)$ is even. To conclude we approximate in $L^1(\mathbb{R}^N)$ the characteristic function of the ball $B(x_0, R + LT)$ by functions of the form $\gamma(| \cdot - x_0|)$, with $\gamma$ as above. Passing to such approximation limit in (5.9) we collect

$$\int_{B(x_0, R)} |u - v|(T) \leq \int_{B(x_0, R + LT)} K(T) * S_{[\Pi]}(T) * |u_0 - v_0|,$$

which concludes the proof of Proposition 5.1. \footnote{This function is continuous and is an approximate unit as $t \to 0$, and the function $(t, x) \in [0, \infty) \times \mathbb{R}^N \mapsto \gamma(| \cdot - x_0| + Lt)$ is continuous with compact support.}
6 Regularising effect for $1 < \lambda \leq 2$

In this section we assume $1 < \lambda \leq 2$ and we prove Theorem 2.6.

6.1 Duhamel’s formula for the entropy solution

Denoting by $u^\delta$ the function constructed by the splitting method in Section 4, we first obtain an integral equation on $u^\delta$ which, by letting $\delta \to 0$, shows that the entropy solution $u = \lim_{\delta \to 0} u^\delta$ satisfies the Duhamel’s formula corresponding to $\partial_t u + \mathcal{G}[u] = -\text{div}(f(u))$. More precisely the following holds.

Proposition 6.1. Let $u$ be the entropy solution to (1.4) with initial data $u_0 \in L^\infty(\mathbb{R}^N)$. Then, for all $t > 0$,

$$u(t) = (K(t) * S_{-\Pi}(t)) * u_0 - \int_0^t \nabla K(t - s) * S_{-\Pi}(t - s) * f(u(s)) \, ds, \quad (6.1)$$

where $h^{(1)}_i * h^{(2)}_j := \sum_{i=1}^N h^{(1)}_i * h^{(2)}_j$ if $h^{(j)} = (h^{(j)}_1, \ldots, h^{(j)}_N) : \mathbb{R}^N \to \mathbb{R}^N$, $j = 1, 2$.

Proof. Let us first recall that $K^{[2]}(t) := K(2t)$ and $S_{-\Pi^{[2]}}(t) := S_{-\Pi}(2t)$. Assume that $2n\delta < t \leq (2n + 1)\delta$, for some $n \geq 0$. Since $u^\delta$ is the entropy solution to (4.1) on $([2n\delta, t]$ and since $\lambda > 1$, we can write the following Duhamel’s formula (see [14])

$$u^\delta(t) = K^{[2]}(t - 2n\delta) * u^\delta(2n\delta) - 2 \int_{2n\delta}^t \nabla K^{[2]}(t - s) * f(u^\delta(s)) \, ds. \quad (6.2)$$

Now, if $n \neq 0$ we go further in the past. On $((2n - 1)\delta, 2n\delta]$, $u^\delta$ solves (4.2) so that

$$u^\delta(2n\delta) = S_{-\Pi^{[2]}}(\delta) * u^\delta((2n - 1)\delta), \quad (6.3)$$

which, combined with (6.2), yields

$$u^\delta(t) = K^{[2]}(t - 2n\delta) * S_{-\Pi^{[2]}}(\delta) * u^\delta((2n - 1)\delta) - 2 \int_{2n\delta}^t \nabla K^{[2]}(t - s) * f(u^\delta(s)) \, ds. \quad (6.4)$$

Another Duhamel’s formula for $u^\delta$ on $(2(n - 1)\delta, (2n - 1)\delta]$ yields

$$u^\delta((2n - 1)\delta) = K^{[2]}(\delta) * u^\delta((2n - 1)\delta) - 2 \int_{2(n - 1)\delta}^{(2n - 1)\delta} \nabla K^{[2]}((2n - 1)\delta - s) * f(u^\delta(s)) \, ds.$$
Case-by-case study show that the following pointwise estimates hold:

By plugging this into (6.4) and using the semi-group properties of $K$ and $S_{-\Pi}$ (see Proposition 3.1), we deduce

$$u^\delta(t) = K^{[2]}(t - 2n\delta + \delta) \ast S_{-\Pi}^{[2]}(\delta) \ast u^\delta(2(n - 1)\delta)$$

$$- 2 \int_{2n\delta}^t \nabla K^{[2]}(t - s) \ast f(u^\delta(s)) \, ds$$

$$- 2 \int_{2(n-1)\delta}^{2(n-1)\delta+\delta} \nabla K^{[2]}(t - s - \delta) \ast S_{-\Pi}^{[2]}(\delta) \ast f(u^\delta(s)) \, ds$$

(6.5)

Iterating $n - 1$ more times the process from (6.3) to (6.5), we arrive at

$$u^\delta(t) = K^{[2]}(t - n\delta) \ast S_{-\Pi}^{[2]}(n\delta) \ast u_0 - 2 \int_{2n\delta}^t \nabla K^{[2]}(t - s) \ast f(u^\delta(s)) \, ds$$

$$- \sum_{k=1}^n 2 \int_{2(n-k)\delta}^{2(n-k)\delta+\delta} \nabla K^{[2]}(t - s - k\delta) \ast S_{-\Pi}^{[2]}(k\delta) \ast f(u^\delta(s)) \, ds \,. \quad (6.6)$$

Let $a^i_\delta, \ i = 1, ..., 4$, be the functions defined, for all $n \geq 0$ and all $0 \leq k \leq n$, by

$$a^1_\delta(t) := \begin{cases} 
2(t - n\delta) & \text{if } 2n\delta \leq t < (2n + 1)\delta \\
2((2n + 1)\delta - n\delta) & \text{if } (2n + 1)\delta \leq t < 2(2n + 1)\delta 
\end{cases}$$

$$a^2_\delta(t) := \begin{cases} 
2(n\delta) & \text{if } 2n\delta \leq t < (2n + 1)\delta \\
2(n\delta + t - (2n + 1)\delta) & \text{if } (2n + 1)\delta \leq t < 2(2n + 1)\delta 
\end{cases}$$

$$a^3_\delta(t) := \begin{cases} 
2(t - s - k\delta) & \text{if } 2n\delta \leq t < (2n + 1)\delta \text{ and } 2(n-k)\delta \leq s < 2(n-k)\delta + \delta \\
(2n + 1)\delta \leq t < 2(n+1)\delta \text{ and } 2(n-k)\delta \leq s < 2(n-k)\delta + \delta \\
2(n-k)\delta + \delta \leq s < 2(n-k)\delta + 2\delta 
\end{cases}$$

$$a^4_\delta(t, s) := \begin{cases} 
2(k\delta) & \text{if } 2n\delta \leq t < (2n + 1)\delta \text{ and } 2(n-k)\delta \leq s < 2(n-k)\delta + 2\delta \\
(2n + 1)\delta \leq t < 2(n+1)\delta \text{ and } 2(n-k)\delta \leq s < 2(n-k)\delta + 2\delta \\
2(n-k)\delta + 2\delta 
\end{cases}$$

Case-by-case study show that the following pointwise estimates hold:

$$|a^1_\delta(t) - t| \leq \delta, \quad |a^2_\delta(t) - t| \leq \delta, \quad |a^3_\delta(t, s) - (t - s)| \leq 2\delta$$

and

$$|a^4_\delta(t, s) - (t - s)| \leq 2\delta.$$
Moreover (6.6) is recast as
\[ u^\delta(t) = K(a_3^\delta(t)) * S_{-\Pi}(a_3^\delta(t)) * u_0 - \int_t^\infty 2I_\delta(s) \nabla K(a_3^\delta(t, s)) * S_{-\Pi}(a_3^\delta(t, s)) * f(u^\delta(s)) \, ds, \tag{6.7} \]
with \(I_\delta\) the characteristic function of \(\cup_{n\geq 0}[2n\delta, (2n+1)\delta)\) \(^8\).

If \((2n+1)\delta < t \leq 2(n+1)\delta\) for some \(n \geq 0\) then, writing \(u^\delta(t) = S_{-\Pi}[2](t-(2n+1)\delta) * u^\delta((2n+1)\delta)\) and using (6.7) for \(t = (2n+1)\delta\), we see — by our choice of the functions \(a_3^\delta\) — that (6.7) remains valid.

We aim at letting \(\delta \to 0\) in (6.7). From our pointwise estimates on the functions \(a_3^\delta\) and item (viii) in Proposition 3.1, we see that, for all \(t > 0\),
\[ K(a_3^\delta(t)) * S_{-\Pi}(a_3^\delta(t)) \to K(t) * S_{-\Pi}(t) \quad \text{in} \quad L^1(\mathbb{R}^N), \]
and that, for all \(0 < s < t\),
\[ \nabla K(a_3^\delta(t, s)) * S_{-\Pi}(a_3^\delta(t, s)) \to \nabla K(t-s) * S_{-\Pi}(t-s) \quad \text{in} \quad L^1(\mathbb{R}^N)^N. \]

Recalling that \(u^\delta \to u\) in \(C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))\) and that \(u^\delta\) remains bounded in \(L^\infty((0, T) \times \mathbb{R}^N)\) we also get that, for all \(s > 0\), \(f(u_\delta(s)) \to f(u(s))\) in \(L^\infty(\mathbb{R}^N)\) weak*-.

Combining this with the above limit yields that, for all \(0 < s < t\),
\[ Z_\delta(t, s) := \nabla K(a_3^\delta(t, s)) * S_{-\Pi}(a_3^\delta(t, s)) * f(u^\delta(s)) \to \nabla K(t-s) * S_{-\Pi}(t-s) * f(u(s)). \tag{6.8} \]

Moreover, by Young’s inequality for the convolution and the integrability property of \(\nabla K\) (see item (ii) in Proposition 3.1), we see that
\[ \|Z_\delta(t, s)\|_{C_b(\mathbb{R}^N)} \leq C a_3^\delta(t, s)^{-1/\lambda}, \]
where, here and in the following, \(C\) does not depend on \(\delta\), \(t\) or \(s\) and may change from place to place. Studying separately the case \(k = 1\) in the first line defining \(a_3^\delta\), the case \(k = 0\) in the second line defining \(a_3^\delta\) and the other cases \((k \neq 1\) in the first line, \(k \neq 0\) in the second, \(k \geq 0\) in the third\), one can find a lower bound on \(a_3^\delta\) which shows that
\[ a_3^\delta(t, s)^{-1/\lambda} \leq \frac{C1_{[2(n-1)\delta, 2n\delta+\delta)}(s)}{(t-s)^{1/\lambda}} + \frac{C1_{[2n\delta, 2n\delta+\delta)}(s)}{((2n+1)\delta - s)^{1/\lambda}} + C (t-s)^{1/\lambda}, \tag{6.9} \]
where \(n\) is taken such that \(2n\delta \leq t < 2(n+1)\delta\). The integral for \(s \in (0, t)\) of the two first functions in the right-hand side member of (6.9) is bounded by

\(^8\)Note that the definition of \(a_3^\delta(t, s)\) for \(2(n-k)\delta + \delta \leq s < 2(n-k)\delta + 2\delta\) does not play any role in (6.7), and the choice \(a_3^\delta(t, s) = t-s\) in these cases is made by convenience.
and thus tends to 0 as $\delta \to 0$. The estimate (6.9) therefore shows that the sequence $(a^3_{\delta}(t,\cdot)^{-1/\lambda})_{\delta \to 0}$ is equi-integrable on $(0,t)$ and, using Vitali’s Theorem, we conclude that the convergence in (6.8) also holds in $L^1(0,t)$, pointwise on $\mathbb{R}^N$.

Since $2I_\delta \to 1$ in $L^\infty(0,\infty)$ weak-$\ast$, the above considerations allow us to pass to the limit $\delta \to 0$ in (6.7). Hence, the entropy solution $u$ to (1.4) satisfies the Duhamel’s formula (6.1).

6.2 Regularity of the entropy solution: proof of Theorem 2.6

Let us recall that, in the case where $\Pi \equiv 0$, a regularising effect is proved for $1 < \lambda \leq 2$ in [14]. The authors take advantage of the Duhamel’s formula involving $K$ rather than $K \ast S_{-\Pi}$. Since the regularity and integrability properties of $K \ast S_{-\Pi}$ and $\nabla (K \ast S_{-\Pi})$ are similar to the properties of $K$ and $\nabla K$ (see Proposition 3.1), we can reproduce the techniques used in the proof of [14, Proposition 5.1, Theorem 5.2]. Therefore the entropy solution $u$ to (1.4) is indefinitely derivable with respect to $x$ on $(0,\infty) \times \mathbb{R}^N$. Moreover, for all $0 < a < T$ and all $(i_1,\ldots,i_N) \in \mathbb{N}^N$, we have $\partial_{x_1}^{i_1} \cdots \partial_{x_N}^{i_N} u \in C_b((a,T) \times \mathbb{R}^N)$. Finally, the entropy formulation (2.3) with $\eta(s) = \pm s$ shows that $u$ satisfies (1.4) in the distributional sense; hence the spatial regularity of $u$ ensures, by a bootstrap argument, that it is also regular in time.

Theorem 2.6 is proved.

7 Generalizations

Here we handle two generalisations of (1.4) by the preceding methods.

7.1 Dirac masses in $\Pi$

Our results remain true if Assumption 1 is replaced by Assumption 2, i.e. if there exists $c \in \mathbb{R}$ such that $\Pi := \mathcal{F}^{-1}(|\cdot|^\lambda (H(\cdot) - 1)) \in c\delta_0 + L^1(\mathbb{R}^N)$. This allows to consider the cases where $|\xi|^\lambda (H(\xi) - 1) \to c$ quickly enough as $|\xi| \to \infty$: for example, it is satisfied if $|\cdot|^\lambda (H(\cdot) - 1) - c \in W^{N+1,1}(\mathbb{R}^N)$ (see also the appendix for a less demanding property on $H$, which implies Assumption 2).

Defining $\Pi_1 := \Pi - c\delta_0 \in L^1(\mathbb{R}^N)$, equation (1.4) then becomes

$$\partial_t u + \text{div}(f(u)) + g_\lambda[u] + \Pi_1 \ast u + cu = 0.$$ 

Thus Assumption 2 consists in adding a linear reaction term $cu$ into the considered equation.

In terms of mathematical study, the replacement of Assumption 1 by Assumption 2 brings minor changes (some of which are listed below) and all the preceding theorems remain valid.
(i) the term $\Pi * u$ is changed into $\Pi_1 * u + cu$, 

(ii) the estimate (4.3) becomes $||u^\delta(t)||_\infty \leq e^{-ct}e^{||\Pi_1||_1 t}||u_0||_\infty$ (and thus the multiplicative term $e^{-ct}$ must be applied to all the estimates derived from (4.3)), 

(iii) on $((2n-1)\delta, 2n\delta]$ we have $\partial_t u^\delta + 2\Pi_1 * u^\delta + 2cu^\delta = 0$ so that, if $v^\delta := e^{2ct}u^\delta$, equality $\partial_t(v^\delta - \mathcal{T}_h v^\delta) + 2(\Pi_1 - \mathcal{T}_h \Pi_1) * v^\delta = 0$ holds. Hence, if $w_T(h) := 2||\Pi_1 - \mathcal{T}_h \Pi_1||_1 e^{||\Pi_1||_1 t} ||u_0||_\infty$, we see that (4.5) holds true for $v^\delta$ in place of $u^\delta$. Coming back to $u^\delta$ the estimate (4.5) is changed into 

$$||u^\delta - \mathcal{T}_h u^\delta||_{(2n\delta, x)} \leq e^{-2c2n\delta} \omega_T(h)\delta + e^{-2c\delta} ||u^\delta - \mathcal{T}_h u^\delta||_{((2n-1)\delta, x)} \leq e^{2c||T||} \omega_T(h)\delta + e^{2c||\delta||} ||u^\delta - \mathcal{T}_h u^\delta||_{((2n-1)\delta, x)}.$$ 

Therefore (4.6) is valid with $\omega_T(h)$ multiplied by $e^{2c||T||}$ and $K^{(2)}(t-2n\delta)$ by $e^{2c||\delta||}$; after having cumulated all the time steps, the final inequality (4.8) is valid with $\omega_T(h)$ and $K^{(2)}(s)$ multiplied by $e^{2c||T||}$ and the end of the translation estimates follows, 

(iv) the semi-groups $S_{-\Pi}(t)$, $S_{[\Pi]}(t)$ and $S_{[\Pi]}(t)$ are replaced by $e^{ct}S_{-\Pi}(t)$, $e^{c|t|S_{[\Pi]}(t)}$ and $e^{c|t|S_{[\Pi]}(t)}$.

### 7.2 Time-dependent $\Pi$

It is also possible to handle the case where $\Pi$ depends on $t$, for example $\Pi \in C([0, \infty); L^1(\mathbb{R}^N))$. In this case, the solution to $\partial_t u(t) + \Pi(t) * u(t) = 0$ with initial data $u(t_0) = u_0$ is no longer given by a semi-group but by the flow $S_{-\Pi}(t; t_0) * u_0$ with 

$$S_{-\Pi}(t; t_0) := \delta_0 + \sum_{n \geq 1} \frac{1}{n!} \left( \int_{t_0}^{t} -\Pi(s) \, ds \right)^{(n)}.$$ 

Here again the adaptation of the techniques and estimates are quite straightforward; for example, the estimate (4.3) becomes 

$$||u^\delta(t)||_\infty \leq e^{2f_{[0, t] \cap J_\delta} ||\Pi(s)||_1 \, ds ||u_0||_\infty.$$ 

The existence and uniqueness of the entropy solution (Theorem 2.3) are valid under the assumption $\Pi \in C([0, \infty); L^1(\mathbb{R}^N))$, and the regularising effect (Theorem 2.6) under the assumption $\Pi \in C^\infty([0, \infty); L^1(\mathbb{R}^N))$. 

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Appendix: the mathematical assumptions in the physical context

We come back here to the physical model presented in Section 1. As seen in [10] and [12], the function $W$ has the integral representation

$$W(is) = \int_0^\infty w_1(\xi)e^{-is\xi}d\xi + \int_0^\infty (1+is\xi)w_2(\xi)e^{-is\xi}d\xi,$$

with $w_1$ and $w_2$ regular functions such that $w_1(0) + w_2(0) = ib$. The numerical approximations [10] of $w_1$ and $w_2$ exhibit rapid convergence to 0 at infinity. Hence, integrating-by-part, one can find asymptotic expansions of $W$ and its derivatives which show that

$$\lim_{s \to \infty} s(sW(is)-b)$$

exists, is finite and, for $k = 1, 2$,

$$\left| \frac{d^k}{ds^k}(sW(is)) \right| + \left| \frac{d^k}{ds^k}(s(sW(is)-b)) \right| = O\left( \frac{1}{s} \right) \text{ as } s \to \infty. \quad (A.1)$$

We prove here that, thanks to this property of $W$, the function $H(\xi) = \sqrt{1 + W(is)}$ is such that

$$F^{-1}(|\cdot|(H(\cdot)-1)) \in \frac{b}{2}\delta_0 + L^1(\mathbb{R}). \quad (A.2)$$

In other words, $H$ satisfies Assumption 2 with $\lambda = 1$, and thus our preceding study in Sections 4 and 5 covers the physical model under consideration.

We take a cut-off function $\chi \in C_c^\infty(\mathbb{R})$, equal to 1 on $[-1, 1]$, and we write

$$|\xi|(H(\xi) - 1) = \frac{|\xi| - \frac{W(is)}{\sqrt{1 + W(is)}}}{\sqrt{1 + W(is)} + 1}$$

$$= \frac{|\xi|\chi(\xi)}{\sqrt{1 + W(is)}} - \frac{W(is)}{\sqrt{1 + W(is)} + 1}$$

$$+ |\xi|(1 - \chi(\xi)) - \frac{W(is)}{\sqrt{1 + W(is)} + 1}$$

$$=: T_1(\xi) + T_2(\xi). \quad (A.3)$$

We are first concerned with $T_1$. By regularity of $W$, an asymptotic expansion of $\frac{W(is)}{\sqrt{1 + W(is)} + 1}$ around $s = 0$ shows that

$$T_1(\xi) = d|\xi|\chi(\xi) + \xi^2\chi(\xi)\gamma(|\xi|),$$

---

9In [10], [11], $W$ is actually a complex-valued function and we should take the real part of $\sqrt{1 + W}$ when defining $H$. However, in order to simplify the presentation, we will omit this and study the “full” $H = \sqrt{1 + W}$ (the real part of this expression cannot have a worst behaviour than the expression itself). Note also that, in the physical context, $W$ seems to be small enough to ensure that a smooth determination of the complex square root can be chosen, so that $H$ can be considered smooth outside $\xi = 0$. 

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Lemma A.1 below thus ensures that $A \to \infty$.

The limits at $f$ dominate convergence theorem then ensures that $\mu \in S$ and $\alpha$ with $|\alpha|$ as $\xi$ goes to $\infty$. Hence, the Fourier transform is therefore integrable. Hence,

$$F^{-1}(T_1) \in L^1(\mathbb{R}).$$  \tag{A.4}

We now handle $T_2$. Since $W(is) \sim b/s$ as $s \to \infty$, we see that $T_2(\xi) \to b/2$ as $|\xi| \to \infty$. Moreover, for $|\xi|$ large enough (such that $\chi(\xi) = 0$), we have

$$T_2(\xi) - \frac{b}{2} = \frac{2(|\xi|W(i\xi)| - b) - b(\sqrt{1 + W(\xi)}) - 1}{2(\sqrt{1 + W(\xi)}) + 1}.$$  \tag{A.5}

From this relation we understand that $T_2(\xi) - \frac{b}{2}$ behaves “at worst” at $\infty$ as $W(i\xi)| - b$ or $W(\xi)$. More precisely, since $T_2 - \frac{b}{2}$ is regular at $\xi = 0$, we can write

$$T_2(\xi) - \frac{b}{2} = \frac{\mu(\xi)}{|\xi|} + \alpha(\xi),$$

with $\alpha \in C^\infty_c(\mathbb{R})$ and $\mu$ regular, vanishing on a neighbourhood of 0, having limits at $\pm \infty$ and satisfying $|\mu'(\xi)| + |\mu''(\xi)| = O(1/|\xi|)$ at infinity. Lemma A.1 below thus ensures that $F^{-1}(T_2 - \frac{b}{2}) \in L^1(\mathbb{R})$, i.e. that

$$F^{-1}(T_2) \in \frac{b}{2} \delta_0 + L^1(\mathbb{R}). \tag{A.5}$$

Gathering (A.4), (A.5) and (A.3), we infer that (A.2) holds true, thus concluding the proof that, in the considered framework, Assumption 2 holds.

Lemma A.1. Let $\mu \in C^1_c(\mathbb{R})$ be such that $\mu = 0$ on a neighbourhood of 0 and $\mu'(\xi) = O(1/|\xi|)$ as $|\xi| \to \infty$. Then $F^{-1}(\frac{\mu(\xi)}{|\xi|}) = L^1_{loc}(\mathbb{R})$.

Moreover, if $\mu \in C^2_b(\mathbb{R})$ and if $F'\frac{\mu(\xi)}{|\xi|} \in L^1(\mathbb{R})$, then $F^{-1}\frac{\mu(\xi)}{|\xi|} \in L^1(\mathbb{R})$.

Proof. Let $A > 0$ and $f_A := F^{-1}\left(\frac{\mu(\xi)}{|\xi|}1_{[-A,A]}(\xi)\right)$. Then $f_A \in L^\infty(\mathbb{R})$ and, since $\frac{\mu(\xi)}{|\xi|}1_{[-A,A]}(\xi) \to \frac{\mu(\xi)}{|\xi|}$ in $S'(\mathbb{R})$ as $A \to \infty$, we have $f_A \to f := F^{-1}\left(\frac{\mu(\xi)}{|\xi|}\right)$ in $S'(\mathbb{R})$ and thus also in $D'(\mathbb{R})$. We prove below that $f_A$ converges a.e. as $A \to \infty$ and that $(f_A)_{A>0}$ stays bounded by a function $g \in L^1_{loc}(\mathbb{R})$: the dominated convergence theorem then ensures that $f_A$ converges in $L^1_{loc}(\mathbb{R})$ and thus that $f \in L^1_{loc}(\mathbb{R})$.

\footnote{This is where (A.1) is used: $\mu(\xi)$ and its derivatives behave at infinity “at worst” like $|\xi|(W(i\xi)| - b)$ or $|\xi|W(i\xi)$ and their derivatives.}
To prove the convergence and boundedness of \( f_A \), we take \( a > 0 \) such that \( \mu = 0 \) on \([-a, a]\) and we write, for \( x \neq 0 \),

\[
 f_A(x) = \int_{|\xi| \leq A} \frac{\mu(\xi)}{|\xi|} e^{2\pi i x \xi} d\xi = \int_{a \leq |\xi| \leq \text{min}(A, 1/|x|)} \frac{\mu(\xi)}{|\xi|} e^{2\pi i x \xi} d\xi + 1_{\{|x| \geq 1\}} \int_{1/|x| \leq |\xi| \leq A} \frac{\mu(\xi)}{|\xi|} e^{2\pi i x \xi} d\xi.
\]

Using, in the second integral sign, the change of variable \( z = x \xi \) and an integration by parts, we find

\[
 f_A(x) = \int_{a \\leq |\xi| \leq \text{min}(A, 1/|x|)} \frac{\mu(\xi)}{|\xi|} e^{2\pi i x \xi} d\xi + 1_{\{|x| \geq 1\}} \int_{1/|x| \leq |\xi| \leq A} \frac{\mu(z/x)}{|z|} e^{2\pi i z} dz
\]

\[
 = \int_{a \\leq |\xi| \leq \text{min}(A, 1/|x|)} \frac{\mu(\xi)}{|\xi|} e^{2\pi i x \xi} d\xi + 1_{\{|x| \geq 1\}} \left\{ \frac{\mu(|x|A/x)}{|x|A} e^{2\pi i x} - \frac{\mu(-|x|A/x)}{|x|A} e^{-2\pi i x} \right\} - 1_{\{|x| \geq 1\}} \left( \frac{\mu(\xi)}{z} - \frac{\mu(\xi)}{z^2} \right) e^{2\pi i z} dz.
\]

Since \( \mu \) is bounded and \( \mu' = O(1/|\xi|) \) as \( |\xi| \to \infty \), the integrand in the last integral sign is bounded by \( C/\xi^2 \), with \( C \) not depending on \( x \) or \( A \). Therefore the above expression of \( f_A(x) \) shows that it converges, for all \( x \neq 0 \), as \( A \to \infty \). Moreover, using again the above expression, we find \( C > 0 \), still not depending on \( x \) or \( A \), such that

\[
 |f_A(x)| \leq \int_{a \\leq |\xi| \leq 1/|x|} \frac{C}{|\xi|} d\xi + C 1_{\{|x| \geq 1\}} + 1_{\{|x| \geq 1\}} \int_{1/|x|}^{C/\xi^2} \frac{C}{\xi^2} dz
\]

\[
 \leq 2 C \ln \left( \frac{1}{a|x|} \right) + C + 2C =: g(x).
\]

Since \( g \in L^1_{\text{loc}}(\mathbb{R}) \), the proof that \( f \in L^1_{\text{loc}}(\mathbb{R}) \) is complete.

We now assume that \( \mu \in C^2_b(\mathbb{R}) \) and that \( \frac{\mu''(\xi)}{|\xi|} \in L^1(\mathbb{R}) \). Then, noticing that

\[
 \nu(\xi) := \frac{d^2}{d\xi^2} \frac{\mu(\xi)}{|\xi|} = \frac{\mu''(\xi)}{|\xi|} - 2\text{sgn}(\xi) \frac{\mu'(\xi)}{\xi^2} + 2\text{sgn}(\xi) \frac{\mu(\xi)}{\xi^3} = \frac{\mu''(\xi)}{|\xi|} + O\left( \frac{1}{\xi^2} \right)
\]

\[
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\]
as $\xi \to \infty$, we see that $\nu \in L^1(\mathbb{R})$ and thus that $\mathcal{F}^{-1}(\nu) \in L^\infty(\mathbb{R})$. Since $f(x) = \mathcal{F}^{-1}(\omega_1(x)) = \frac{1}{(2\pi x)^2} \mathcal{F}^{-1}(\nu)(x)$, we infer that $f(x) = \mathcal{O}(1/x^2)$ at infinity so that $f \in L^1(\mathbb{R})$.

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**References**


